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THERMODYNAMIC BETHE ANSATZ IN RELATIVISTIC MODELS: SCALING 3-STATE POTTS AND LEE–YANG MODELS

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Two integrable 2D relativistic field theory models are studied by the thermodynamic Bethe ansatz method. One of them describes the scaling limit $T \to T_c$ of the 3-state Potts model and the other corresponds to the scaling region near the Lee–Yang singularity of the 2D Ising model. The finite volume ground state energy of these two theories is calculated numerically using the integral equations of the temperature Bethe ansatz approach. Numerical results are compared with the perturbations near the corresponding conformal theories. This allows us to relate the mass scales of the theories to their dimensional coupling constants.

1. Introduction

The critical point of the 3-state Potts model (for definition see e.g. ref. [1]) is described by the minimal model $\mathcal{M}(5/6)$ of conformal field theory (CFT) [2,3]. This conformal model corresponds to the Virasoro central charge

$$c = 4/5 \tag{1.1}$$

and the following table of primary field dimensions

$$\begin{array}{ccccc} 3 & 13/8 & 2/3 & 1/8 & 0 \\ 7/5 & 21/40 & 1/15 & 1/40 & 2/5 \\ 2/5 & 1/40 & 1/15 & 21/40 & 7/5 \\ 0 & 1/8 & 2/3 & 13/8 & 3 \end{array} . \tag{1.2}$$

The scalar field $\Phi = \Phi_{(2,1)}$ with dimensions $(2/5, 2/5)$ is relevant and corresponds to the temperature deformation of the critical theory [3]. This means that the perturbation of the critical fixed point action results in a field theory that describes the scaling limit $T \to T_c$ of the Potts model. It seems natural to denote this field theory as $[\mathcal{M}(5/6)]_{(2,1)}$, where the content of the square brackets refers to the short-distance CFT and the index indicates the relevant scalar perturbation. To avoid this somewhat overequipped symbol in this paper we shall call this theory the scaling Potts model (SPM). Conventionally SPM may be defined through the action

$$A_{\mathrm{SPM}} = A_{\mathscr{M}(5/6)} + \lambda \int \Phi(x)\, \mathrm{d}^2 x . \tag{1.3}$$

The coupling constant λ here has dimension $\lambda \sim (\text{mass})^{6/5}$.

It was shown in ref. [4] that SPM is an integrable field theory, i.e. it possesses an infinite series of commuting integrals of motion. SPM develops a finite correlation length R_c and therefore its spectrum is massive. In a theory with massive excitations the integrability implies the factorization of their scattering (see e.g. ref. [5]). The structure of integrals of motion discovered in ref. [4] strongly supports the following scattering theory, suggested earlier in [6]. The particle spectrum consists of a Z_3 doublet of massive particles $\mathrm{A}, \bar{\mathrm{A}}$ of the same mass m_{A}. The antiparticle $\bar{\mathrm{A}}$ can be viewed as a bound state of two particles A (since it shows up as a pole in the corresponding scattering amplitude) and vice versa. All the scattering amplitudes are expressed in terms of two-particle amplitudes

$$\begin{aligned} |\mathrm{A}(\beta_1)\mathrm{A}(\beta_2)\rangle_{\mathrm{in}} &= S_{\mathrm{AA}}(\beta_1-\beta_2)\,|\mathrm{A}(\beta_1)\mathrm{A}(\beta_2)\rangle_{\mathrm{out}}, \\ |\mathrm{A}(\beta_1)\bar{\mathrm{A}}(\beta_2)\rangle_{\mathrm{in}} &= S_{\mathrm{A\bar{A}}}(\beta_1-\beta_2)\,|\mathrm{A}(\beta_1)\bar{\mathrm{A}}(\beta_2)\rangle_{\mathrm{out}}, \end{aligned} \tag{1.4}$$

where β_1 and β_2 are rapidities of colliding particles. Note the absence of reflection in $\mathrm{A\bar{A}}$ scattering. The factorized scattering theory without reflection is called purely elastic. Multiparticle amplitudes in this case are simply products of pairwise transition ones [4, 6]

$$S_{\mathrm{AA}}(\beta) = \frac{\sinh(\beta/2 + i\pi/3)}{\sinh(\beta/2 - i\pi/3)}, \qquad S_{\mathrm{A\bar{A}}}(\beta) = \frac{\sinh(\beta/2 + i\pi/6)}{\sinh(\beta/2 - i\pi/6)}. \tag{1.5}$$

The scattering theory described contains all the on-mass-shell information about SPM. It seems interesting to study this relativistic field theory off-mass-shell, in particular to make a connection between the mass-shell region and the ultraviolet (UV) limit governed by the CFT $\mathscr{M}(5/6)$. In this paper the finite size effects in SPM are considered by means of the thermodynamic Bethe ansatz (TBA) method [7–12].

The relativistic version of this approach gives information about the ground state energy of the relativistic integrable field theory placed in a box of finite length R with periodic boundary conditions, i.e. on a circle. The ground state energy $E(R)$ is a function of the circle circumference R. From dimensional arguments it is clear that $RE(R)$ is a function of the scaling length $r = R/R_c$. We define the ground state scaling function $F(r)$ as $E(R) = 2\pi F(R/R_c)/R$. Off-mass-shell effects become important if R is comparable or less than the correlation length $R_c \sim 1/m$, where m is the mass scale of the theory. In the following considerations we shall always specify this quantity as the inverse mass m_0 of the lightest particle in the theory. Thus the scaling length is $r = m_0 R$ and

$$E(R) = \frac{2\pi}{R} F(m_0 R) . \tag{1.6}$$

The limit $R \to 0$ corresponds to the UV limit and is described by CFT. In particular $E(R)$ behaves as $E(R) \sim \pi c/6R$ in this limit [13], where c is the CFT central charge.

From the space-time point of view this finite box theory lives on an infinite cylinder based on a circle of circumference R. In euclidean relativistic field theory one can alternatively treat this geometry as the theory in an infinite space volume but at finite temperature $1/R$. The infinite volume thermodynamic effects can be treated completely in terms of on-mass-shell data. In the case of factorized scattering theory the finite-temperature states are described by the Bethe wave function. This permits one to reduce the problem to a system of nonlinear integral equations (the TBA equations), the content of the scattering theory being encoded in their form. Equations of this type were first considered by Yang and Yang [7] and successfully used later in thermodynamics of integrable lattice systems [8–12].

In sect. 3 the TBA equations are specified for the case of scattering theory of SPM. It turns out that in the charge symmetric case these equations are exactly the same as the TBA equations, corresponding to the S-matrix of the field theory describing the scaling region near the Lee–Yang singularity of the 2D Ising model [14, 15].

Critical behavior in the Lee–Yang singularity is described by the minimal model $\mathscr{M}(2/5)$ [16]. This model is nonunitary and corresponds to the central charge

$$c = -22/5 . \tag{1.7}$$

The table of primary field dimensions

$$0 \quad -1/5 \quad -1/5 \quad 0 \tag{1.8}$$

contains only two primary fields: the unity operator I and the scalar field φ with dimensions $(-1/5, -1/5)$. According to the conventions suggested above the field theory corresponding to perturbation of the conformal theory by operator φ is denoted as $[\mathscr{M}_{(2/5)}]_{(1,2)}$ or $[\mathscr{M}_{(2/5)}]_{(1,3)}$. In the present case this general notation is redundant because φ is the unique relevant perturbation of $\mathscr{M}(2/5)$. We shall call this field theory the scaling Lee–Yang model (SLYM). The corresponding conventional action is

$$A_{\mathrm{SLYM}} = A_{\mathcal{M}(2/5)} + g\int \varphi(x)\, \mathrm{d}^2x\,, \tag{1.9}$$

where the coupling constant g has dimension $g \sim (\text{mass})^{12/5}$.

In ref. [17] it was shown that this field theory is integrable and its particle spectrum consists of a single massive particle B with mass m_{B}. The two-particle amplitude exhibits a pole corresponding to fusion BB → B → BB and has the form

$$S_{\mathrm{BB}}(\beta) = \frac{\sinh(\beta) + i\sin(\pi/3)}{\sinh(\beta) - i\sin(\pi/3)}\,. \tag{1.10}$$

The scattering theory is nonunitary and the pole has the wrong residue.

In sect. 2 we present briefly the TBA method for relativistic systems in the simplest case of purely elastic scattering. In sect. 3 the TBA equations are specified for the scattering theories of SPM and SLYM. In the charge symmetric case this leads to the same single integral equation. This equation is studied in the low- and high-temperature limits. In particular, the nonperturbative bulk vacuum energy is extracted exactly from this equation. In sect. 4 the equation is investigated numerically. The ground state scaling function is calculated with high accuracy. Also we estimate several coefficients in its high-temperature perturbative expansion. In sect. 5 these coefficients are compared with perturbations of SPM and SLYM in coupling constants λ and g, respectively. This provides us with the numerical relation between the dimensional coupling constants and mass scales in these two theories. With the conventional CFT normalization of fields in (1.3) and (1.9) we obtain

$$m_{\mathrm{A}} = (4.504307863\ldots)\,\lambda^{5/6}\,,$$

$$m_{\mathrm{B}} = (2.642944662\ldots)(-ig)^{5/12}\,. \tag{1.11}$$

2. TBA for relativistic purely elastic S-matrix

We start by considering a relativistic field theory in toroidal geometry, having in mind to pass to a cylinder as a limiting case of the torus. We take a flat torus generated by two orthogonal geodesic circles C and B of circumference R and L, respectively (fig. 1) and follow a cartesian coordinate system with the x-axis along the contour C and the y-direction parallel to B. There are two topologically different ways to develop the hamiltonian approach to this situation. On the one

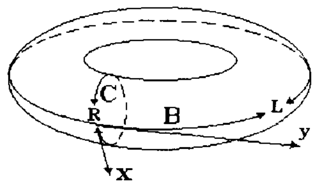

Fig. 1. Flat torus generated by two orthogonal geodesic circles C and B of circumference R and L, respectively.

hand one can consider the field theory states on circle C. Denote the corresponding space of states as $\mathscr{C}$. The y axis plays the role of time direction and states are evolved by the hamiltonian

$$H_C = P_y^{(C)} = \frac{1}{2\pi}\int_C T_{yy}\,dx\,, \tag{2.1}$$

where $T_{\mu\nu}$ is the stress tensor of the theory. The momentum

$$P_C = P_x^{(C)} = \frac{1}{2\pi}\int T_{xy}\,dx \tag{2.2}$$

is quantized and its eigenvalues are $2\pi n/R$ with n integer. Alternatively, the contour B can be chosen as a quantization surface and the evolution of the corresponding space of states $\mathscr{B}$ in the $-x$ direction (we take $-x$ as the time direction to save the frame orientation) is described by hamiltonian

$$H_B = -P_x^{(B)} = \frac{1}{2\pi}\int_C T_{xx}\,dy\,. \tag{2.3}$$

The space momentum

$$P_B = P_y^{(B)} = -\frac{1}{2\pi}\int T_{xy}\,dy \tag{2.4}$$

has eigenvalues $2\pi n/L$, $n \in \mathbb{Z}$ in space $\mathscr{B}$.

In the following we consider the limit $L \to \infty$, $L \gg R$. In this limit the partition function of the field theory $Z(R,L)$ is dominated by the ground state of H_C with the ground state energy $E(R)$

$$Z(R,L) \sim e^{-E(R)L}\,. \tag{2.5}$$

On the other hand

$$Z(R,L)=\operatorname*{tr}_{\mathscr{B}}[e^{-RH_B}]. \tag{2.6}$$

In the thermodynamic limit $L\to\infty$ one has

$$\log Z(R,L)\sim -LRf(R), \tag{2.7}$$

where $f(R)$ is the bulk free energy of the system on B at temperature $1/R$. This gives the relation

$$E(R)=Rf(R). \tag{2.8}$$

Due to the translational invariance of the torus the one-point functions are coordinate independent. In particular

$$\langle T_{yy}\rangle=2\pi\frac{E(R)}{R}, \qquad \langle T_{xx}\rangle=2\pi\frac{dE(R)}{dR} \tag{2.9}$$

and the trace of the stress tensor is

$$\langle T^{\mu}_{\mu}\rangle=\langle T_{xx}+T_{yy}\rangle=2\pi\frac{d}{R\,dR}[RE(R)]. \tag{2.10}$$

In the space parity invariant theory with nondegenerate ground state we also have

$$\langle T_{xy}\rangle=0. \tag{2.11}$$

Consider now more closely the structure of the space $\mathscr{B}$ in a theory with factorized purely elastic scattering. The TBA equations arise in the limit $L\gg R_c$, where the off-mass-shell effects can be neglected. To avoid irrelevant complications we consider first the simplest scattering theory with a single neutral particle of mass m and a pair scattering amplitude $S(\beta_1-\beta_2)$. The rapidities β_1 and β_2 of particles parameterize their on-shell energies and momenta

$$e_i(\beta)=m\cosh\beta_i, \qquad p_i(\beta)=m\sinh\beta_i. \tag{2.12}$$

Amplitude $S(\beta)$ satisfies unitarity

$$S(\beta)S(-\beta)=1 \tag{2.13}$$

and crossing symmetry

$$S(\beta)=S(i\pi-\beta). \tag{2.14}$$

The Bethe wave function. In relativistic theory the wave function formalism is inappropriate to describe a system of relativistic particles (this is due to virtual and real particle creation). In the configuration space, however, there are regions where we have a set of relativistic particles strongly separated in their space positions x_i. More specifically we take $|x_i - x_j| \gg R_c$. With a short range interaction, in these regions the particles move as free ones and off-mass-shell effects can be neglected. In these regions, which we call free regions, we can talk about the space coordinates x_i and momenta p_i of particles and introduce the wave function $\Psi(x_1, \dots, x_N)$.

In the factorized picture the number of particles N is the same in all the free regions. The set of their momenta p_i, $i = 1, 2, \dots, N$, is also the same and the wave function is the Bethe wave function. We denote a free region as $\{i_1, i_2, \dots, i_N\}$ if $x_{i_1} < x_{i_2} < \dots < x_{i_N}$.

The transition between two adjacent free regions passes through configurations where two or more particles are close to each other. In these configurations, of course, the relativistic effects are essential and we cannot use the wave function. The scattering theory, however, provides conditions to match wave functions in adjacent free regions. In the purely elastic case every transition, say $\{i_1, \dots, i_p, i_{p+1}, \dots, i_N\} \to \{i_1, \dots, i_{p+1}, i_p, \dots, i_N\}$, results in multiplication of the wave function by the corresponding scattering amplitude, $S(\beta_{i_p} - \beta_{i_{p+1}})$ in this case. Note that for β real this amplitude is an unimodular number

$$S(\beta) = e^{i\delta(\beta)} \tag{2.15}$$

with real phase $\delta(\beta)$.

These matching conditions lead in particular to the quantization equations for the momenta p_i, $i = 1, 2, \dots, N$, of N particles in a periodic box of length $L \gg R_c$

$$e^{ip_iL} \prod_{j \neq i} S(\beta_i - \beta_j) = 1; \quad i = 1, \dots, N \tag{2.16}$$

or

$$mL \sinh \beta_i + \sum_{j \neq i} \delta(\beta_i - \beta_j) = 2\pi n_i \tag{2.17}$$

with N integer numbers n_i. This system of transcendental equations selects admissible sets of rapidities $(\beta_1, \dots, \beta_N)$ in free regions $\{i_1, \dots, i_N\}$. The energy and momentum of the state $(\beta_1, \dots, \beta_N)$ are

$$H_B = \sum_{i=1}^{N} m \cosh \beta_i, \qquad P_B = \sum_{i=1}^{N} m \sinh \beta_i. \tag{2.18}$$

From eqs. (2.17) and (2.13) it is readily seen that $2\pi P_B L$ is an integer number, as it should be in a periodic box.

Additional selection rules on rapidity sets $(\beta_1, \ldots, \beta_N)$ are to be taken into account if the particles are identical. The Bethe wave function should be symmetrized or antisymmetrized depending on their statistics. The unitarity condition (2.13) specifies that $S^2(0) = 1$. Two different cases are possible:

$$\text{(a)} \quad S(0) = -1. \tag{2.19}$$

If there are two identical particles with the same rapidities, the wave function is antisymmetric in their coordinates. This is incompatible with Bose statistics. Therefore in the bosonic case such states should be excluded. In this sense bosons behave like fermions: each value of rapidity can be occupied by at most one particle. It implies in particular that all the integers n_i in (2.17) are different. We shall denote this situation in the Bethe ansatz equations as "fermionic". On the other hand, if identical particles are fermions, the states with coinciding rapidities are allowed and fermions can occupy each rapidity value in any number. This case will be referred to as "bosonic".

$$\text{(b)} \quad S(0) = 1. \tag{2.20}$$

In this case the situation is inverted. Bose particles occupy each rapidity value in any number and fermions behave like fermions.

Eq. (2.17) is a complicated system of transcendental equations. The situation yields to analyses in the thermodynamic limit $L \to \infty$. The number of particles in thermodynamic states grows $\sim L$ and as $L \to \infty$ becomes very large. The spectrum of rapidities, determined by eq. (2.17), condenses and the distance between adjacent levels behaves as $\beta_i - \beta_{i+1} \sim 1/mL$. It makes sense in this limit to introduce a continuous rapidity density of particles $\rho_1(\beta)$. Taking a small rapidity interval $\Delta\beta$ with n particles inside, one defines

$$\rho_1(\beta) = n/\Delta\beta. \tag{2.21}$$

This smooth function of β is independent of a choice of interval $\Delta\beta$ while $1/mL \ll \Delta\beta \ll 1$. The phase sum in eq. (2.17) is nearly constant when varying from one β_i to the next β_{i+1} and can be estimated as an integral. Eq. (2.17) acquires the form

$$mL \sinh\beta_i + \int \delta(\beta_i - \beta')\rho_1(\beta')\,\mathrm{d}\beta' = 2\pi n_i$$

and can be considered as the equation for rapidity levels, defined as solutions to this equation for all integer numbers n on the r.h.s. but not only n_i corresponding to actual state. The situation is analogous to that for a system of free particles, where the set of allowed levels is determined by the one-particle quantization condition and one talks about occupied and free levels. The only difference from the free case is that now the set of levels is organized self-consistently with the particle distribution. Introducing the level density $\rho(\beta)$ we arrive at the integral equation

$$2\pi\rho(\beta) = mL\cosh\beta + \int\varphi(\beta-\beta')\rho_1(\beta')\,\mathrm{d}\beta', \tag{2.22}$$

where

$$\varphi(\beta) = \partial\delta(\beta)/\partial\beta. \tag{2.23}$$

The energy (2.18) of the system now reads

$$H_{\mathrm{B}} = \int m\cosh\beta\,\rho_1(\beta)\,\mathrm{d}\beta. \tag{2.24}$$

It should be realized that in the thermodynamic limit a large number of quantum states correspond to every consistent pair of densities ρ and ρ_1. Consider a partition of the rapidity axis in small intervals $\Delta\beta_\alpha \ll 1$. If also $\Delta\beta_\alpha \gg 1/mL$, there is a large number $N_\alpha \sim \rho(\beta_\alpha)\Delta\beta_\alpha$ of levels (note that $\rho, \rho_1 \sim L$) in each interval and about $n_\alpha \sim \rho_1(\beta_\alpha)\Delta\beta_\alpha$ particles are distributed between them. The averaged densities are not sensitive to local redistributions of particles among levels. The number of different distributions in the interval $\Delta\beta_\alpha$ amounts to

$$\frac{(N_\alpha)!}{(n_\alpha)!(N_\alpha-n_\alpha)!} \tag{2.25}$$

in the "fermionic" case and

$$\frac{(N_\alpha+n_\alpha-1)!}{(N_\alpha-1)!(n_\alpha)!} \tag{2.26}$$

in the "bosonic" case. The limiting behavior as $L\to\infty$ of the number of states $\mathcal{N}(\rho,\rho_1)$, corresponding to given consistent densities ρ and ρ_1, is estimated by the entropy $\mathcal{S}(\rho,\rho_1) = \log\mathcal{N}(\rho,\rho_1)$. While $1/mL \ll \Delta\beta_\alpha \ll 1$ the number $\mathcal{N}(\rho,\rho_1)$ is correctly estimated as the product of numbers (2.25) or (2.26) over intervals $\Delta\beta_\alpha$ and the entropy is

$$\mathcal{S}_{\mathrm{Fermi}} = \int\mathrm{d}\beta\left[\rho\log\rho - \rho_1\log\rho_1 - (\rho-\rho_1)\log(\rho-\rho_1)\right] \tag{2.27}$$

in the "fermionic" case and

$$\mathscr{S}_{\text{Bose}} = \int d\beta[(\rho+\rho_1)\log(\rho+\rho_1) - \rho\log\rho - \rho_1\log\rho_1] \tag{2.28}$$

in the "bosonic" one.

With the entropy taken into account the summation over states in (2.6) reduces to minimization of the free energy

$$-RLf(\rho,\rho_1) = -RH_{\text{B}}(\rho_1) + \mathscr{S}(\rho,\rho_1) \tag{2.29}$$

in the macroscopic characteristics ρ and ρ_1, constrained by the dynamical relation (2.22). It is convenient to introduce "pseudoenergy" $\varepsilon(\beta)$ as

$$\frac{\rho_1}{\rho} = \frac{e^{-\varepsilon}}{1+e^{-\varepsilon}}; \qquad e^{-\varepsilon} = \frac{\rho_1}{\rho-\rho_1} \quad \text{(fermionic case)}, \tag{2.30a}$$

$$\frac{\rho_1}{\rho} = \frac{e^{-\varepsilon}}{1-e^{-\varepsilon}}; \qquad e^{-\varepsilon} = \frac{\rho_1}{\rho+\rho_1} \quad \text{(bosonic case)}. \tag{2.30b}$$

In this notation the extremum condition takes the form

$$-Rm\cosh\beta + \varepsilon(\beta) + \int\varphi(\beta-\beta')\log(1+e^{-\varepsilon(\beta')})\frac{d\beta'}{2\pi} = 0 \quad \text{(fermionic case)}, \tag{2.31a}$$

$$-Rm\cosh\beta + \varepsilon(\beta) - \int\varphi(\beta-\beta')\log(1-e^{-\varepsilon(\beta')})\frac{d\beta'}{2\pi} = 0 \quad \text{(bosonic case)}. \tag{2.31b}$$

It turns out that densities ρ and ρ_1 enter the expression for the extremal free energy f in the ratio ρ_1/ρ only for

$$Rf(R) = \mp m\int\cosh\beta\log(1\pm e^{-\varepsilon(\beta)})\frac{d\beta}{2\pi}, \tag{2.32}$$

where the upper sign refers to the fermionic case and the lower sign to the bosonic case.

In the more general case there are several types of particles A_a, $a = 1,2,\dots,M$ with masses m_a. The purely elastic scattering theory is described by a symmetric $M\times M$ matrix of pair transition amplitudes $S_{ab}(\beta)$, each satisfying the unitarity condition (2.13) (if there are charged particles, the crossing symmetry relations may have a slightly more complicated form than eq. (2.24)). In the TBA approach one considers M level densities $\rho^{(a)}(\beta)$ and M particle densities $\rho_1^{(a)}$. Eq. (2.22) turns into a system of integral equations

$$\rho^{(a)} = \frac{m_a L}{2\pi} \cosh \beta + \varphi_{ab} * \rho_1^{(b)}, \tag{2.33}$$

where φ_{ab} is the symmetric matrix kernel

$$\varphi_{ab}(\beta) = -i \frac{\mathrm{d}}{\mathrm{d}\beta} \log S_{ab}(\beta) \tag{2.34}$$

and $*$ in (2.33) denotes the convolution

$$\varphi * \rho_1 = \int \varphi(\beta - \beta')\rho_1(\beta') \frac{\mathrm{d}\beta'}{2\pi}. \tag{2.35}$$

The extremum equation (2.31) becomes a system of nonlinear integral equations for M pseudo-energies $\varepsilon_a(\beta)$

$$\rho_1^{(a)}(\beta) = \frac{\mathrm{e}^{-\varepsilon_a(\beta)}}{1 \pm \mathrm{e}^{-\varepsilon_a(\beta)}} \rho^{(a)}(\beta), \tag{2.36}$$

where the upper sign in the denominator is chosen for particles A_a of "fermionic" type (this depends both of their statistics and the sign of the amplitude $S_{aa}(0)$) and the lower sign refers to particles of "bosonic" type. Introducing quantities

$$L_a(\beta) = \pm \log(1 \pm \mathrm{e}^{-\varepsilon_a(\beta)}), \tag{2.37}$$

where again the upper and lower signs correspond to particles of fermionic and bosonic types respectively, we can write down the TBA equations in unified form

$$-m_a R \cosh \beta + \varepsilon_a + \sum_{b=1}^{M} \varphi_{ab} * L_b = 0. \tag{2.38}$$

The bulk free energy f, and therefore the ground state energy in space $\mathscr{E}$, are given by the formula

$$E(R) = Rf(R) = -\sum_{a=1}^{M} m_a \int L_a(\beta) \cosh \beta \frac{\mathrm{d}\beta}{2\pi}. \tag{2.39}$$

Note that comparison of eqs. (2.33) and (2.39) leads to the useful relation

$$\rho^{(a)}(\beta) = \frac{L}{2\pi} \frac{\partial \varepsilon_a(\beta)}{\partial R}. \tag{2.40}$$

The TBA expressions for the stress tensor expectation values have the form

$$\langle T_{xx} \rangle = \frac{2\pi}{L} \sum_{a=1}^{M} m_a \int \rho_1^{(a)}(\beta) \cosh\beta \, d\beta \tag{2.41}$$

and

$$\langle T_\mu^\mu \rangle = \sum_{a=1}^{M} m_a \frac{e^{-\varepsilon_a(\beta)}}{1 \pm e^{-\varepsilon_a(\beta)}} \left(\frac{\partial \varepsilon_a(\beta)}{\partial R} \cosh\beta - \frac{1}{R} \frac{\partial \varepsilon_a(\beta)}{\partial \beta} \sinh\beta \right) d\beta. \tag{2.42}$$

Introducing the solutions $\psi_+^{(a)}$ and $\psi_-^{(a)}$ of the linear integral equations

$$\psi_\pm^{(a)} = m_a e^{\pm\beta} + \sum_{b=1}^{M} \varphi_{ab} * \frac{e^{-\varepsilon_b}}{1 \pm e^{-\varepsilon_b}} \psi_\pm^{(b)} \tag{2.43}$$

we get expression (2.42) in the form

$$\langle T_\mu^\mu \rangle = \frac{1}{2} \sum_{a=1}^{M} m_a \frac{e^{-\varepsilon_a(\beta)}}{1 \pm e^{-\varepsilon_a(\beta)}} \left(\psi_+^{(a)}(\beta) e^{-\beta} + \psi_-^{(a)}(\beta) e^{\beta} \right) d\beta. \tag{2.44}$$

These formulas are useful in the high-temperature limit of TBA equations.

3. TBA for SPM and SLYM scattering theories

Consider first the SPM scattering theory with the doublet of charged bosons A and $\bar{\mathrm{A}}$. We introduce the corresponding pseudoenergies ε_{A} and $\varepsilon_{\bar{\mathrm{A}}}$. The matrix kernel in the integral (2.38) in this case has the following entries

$$\varphi_{\mathrm{AA}}(\beta) = \varphi_{\bar{\mathrm{A}}\bar{\mathrm{A}}}(\beta) = -\frac{\sqrt{3}}{2\cosh\beta + 1},$$

$$\varphi_{\mathrm{A}\bar{\mathrm{A}}}(\beta) = \varphi_{\mathrm{A}\bar{\mathrm{A}}}(\beta) = -\frac{\sqrt{3}}{2\cosh\beta - 1}. \tag{3.1}$$

Note that

$$\int \varphi_{\mathrm{AA}}(\beta) \frac{d\beta}{2\pi} = \frac{1}{3}; \qquad \int \varphi_{\mathrm{A}\bar{\mathrm{A}}}(\beta) \frac{d\beta}{2\pi} = \frac{2}{3}. \tag{3.2}$$

For the scattering theory under consideration the Bethe ansatz is of the fermionic type and TBA equations read as

$$-m_{\mathrm{A}}R\cosh\beta+\varepsilon_{\mathrm{A}}+\varphi_{\mathrm{AA}}*L_{\mathrm{A}}+\varphi_{\mathrm{A\bar{A}}}*L_{\bar{\mathrm{A}}}=0\,,$$

$$-m_{\mathrm{A}}R\cosh\beta+\varepsilon_{\bar{\mathrm{A}}}+\varphi_{\bar{\mathrm{A}}\mathrm{A}}*L_{\mathrm{A}}+\varphi_{\bar{\mathrm{A}}\bar{\mathrm{A}}}*L_{\bar{\mathrm{A}}}=0 \tag{3.3}$$

where

$$L_{\mathrm{A}}=\log(1+\mathrm{e}^{-\varepsilon_{\mathrm{A}}});\qquad L_{\bar{\mathrm{A}}}=\log(1+\mathrm{e}^{-\varepsilon_{\bar{\mathrm{A}}}})\,. \tag{3.4}$$

The ground state energy is

$$E(R)=-m_{\mathrm{A}}\int\cosh\beta(L_{\mathrm{A}}+L_{\bar{\mathrm{A}}})\frac{\mathrm{d}\beta}{2\pi}\,. \tag{3.5}$$

Here we shall consider only the charge symmetric thermodynamic state and set $\varepsilon_{\mathrm{A}}(\beta)=\varepsilon_{\bar{\mathrm{A}}}(\beta)$. It makes sense however to study thermodynamic states with the charge symmetry broken by, say, insertion of a charge asymmetric operator, commuting with H_{B} into the trace in eq. (2.6). An example is the Z_3 charge Ω: $\Omega\mathrm{A}=\mathrm{e}^{2i\pi/3}\mathrm{A}$, $\Omega\bar{\mathrm{A}}=\mathrm{e}^{-2i\pi/3}\bar{\mathrm{A}}$. From the point of view of the theory quantized on C this corresponds to the study of the ground state in sectors with nonperiodic (twisted) boundary conditions. At present we are not concerned with this interesting problem.

In the charge symmetric case the two equations (3.3) are the same and we are left with the single integral equation for one pseudoenergy $\varepsilon=\varepsilon_{\mathrm{A}}=\varepsilon_{\bar{\mathrm{A}}}$

$$-Rm_{\mathrm{A}}\cosh\beta+\varepsilon+\varphi*\log(1+\mathrm{e}^{-\varepsilon})=0 \tag{3.6}$$

where

$$\varphi=\varphi_{\mathrm{AA}}+\varphi_{\mathrm{A\bar{A}}}=-\frac{2\sqrt{3}\,\sinh(2\beta)}{\sinh(3\beta)}\,. \tag{3.7}$$

Eq. (3.6) has exactly the same form as the TBA equation for the SLYM scattering theory with pair amplitude (1.10). This observation can be readily traced to the fact that

$$S_{\mathrm{BB}}(\beta)=S_{\mathrm{AA}}(\beta)S_{\mathrm{A\bar{A}}}(\beta)\,. \tag{3.8}$$

It follows that the ground state scaling functions (1.6) in these two field theories are the same except for factor of 2 which is due to two species of particles contributing to (3.5)

$$F_{\mathrm{SPM}}(r)=2F_{\mathrm{SLYM}}(r)=2F(r)\,, \tag{3.9}$$

where the scaling length $r = m_A R$ in SPM and $r = m_B R$ in SLYM. By definition

$$F(r) = \frac{-r}{2\pi}\int \cosh\beta\, L(\beta)\frac{d\beta}{2\pi}$$

$$L(\beta) = \log(1 + e^{-\varepsilon(\beta)}) \tag{3.10}$$

where $\varepsilon(\beta)$ is the solution to the integral equation

$$-r\cosh\beta + \varepsilon + \varphi * L = 0. \tag{3.11}$$

Before turning to a numerical study of eq. (3.11) consider a few limiting cases. Iterations of eq. (3.11) produce the asymptotic low-temperature expansion

$$F(r) = \frac{r}{2\pi}(C_1(r) + C_2(r) + \dots), \tag{3.12}$$

where $C_n(r) \sim \exp(-nr)$ as $r \to \infty$ up to a power-like factor. Separate terms $C_n(r)$ correspond to n-particle clusters in the trace (2.6). We have

$$C_1(r) = \frac{-1}{2\pi}\int \cosh\beta\, e^{-r\cosh\beta}\, d\beta,$$

$$C_2(r) = \frac{1}{4\pi}\int \cosh\beta\, e^{-2r\cosh\beta}\, d\beta - \int \frac{d\beta_1}{2\pi}\frac{d\beta_2}{2\pi}\, e^{-r(\cosh\beta_1 + \cosh\beta_2)}\varphi(\beta_1 - \beta_2). \tag{3.13}$$

These integrals can be expressed in terms of modified Bessel functions

$$C_1(r) = -\frac{1}{\pi}K_1(r),$$

$$C_2(r) = \frac{1}{2\pi}K_1(2r) + \frac{\sqrt{3}}{2\pi^2}\left[K_1(r)\int_r^\infty I_0(t)K_0^2(t)t\,dt + I_1(r)\int_r^\infty K_0^3(t)t\,dt\right]$$
$$+ \frac{3}{2\pi^2}\left[K_1(\sqrt{3}r)\int_r^\infty I_0(\sqrt{3}t)K_0^2(t)t\,dt + I_1(\sqrt{3}r)\int_r^\infty K_0(\sqrt{3}t)K_0^2(t)t\,dt\right]. \tag{3.14}$$

For any r and for large $|\beta|$ the solution $\varepsilon(\beta)$ to eq. (3.11) follows asymptotically the "free" energy $r\cosh\beta$. Corrections are given by the following $\beta\to\infty$ asymptotic expansion

$$\varepsilon(\beta)-\tfrac{1}{2}r\,e^{\beta}=-2\sqrt{3}\Big((2\pi/r)s_1(r)e^{-\beta}-(2\pi/r)^5 s_5(r)e^{-5\beta}+(2\pi/r)^7 s_7(r)e^{-7\beta}-(2\pi/r)^{11}s_{11}(r)e^{-11\beta}+\dots\Big),\tag{3.15}$$

where

$$s_1(r)=F(r)-\frac{\sqrt{3}r^2}{24\pi},$$
$$(2\pi/r)^n s_n(r)=-\int L(\beta)e^{n\beta}\frac{d\beta}{2\pi},\quad n=5,7,11,13,\dots.\tag{3.16}$$

Note that exponents in the r.h.s. of expansion (3.15) are exactly the numbers of integrals of motion in SLYM [17] and neutral integrals of motion in SPM [4]. Functions $s_n(r)$ in (3.16) are chosen as to have a finite limit as $r\to 0$.

In the high-temperature limit $r\to 0$ the solution $\varepsilon(\beta)$ flattens in the central region $-\log(2/r)\ll\beta\ll\log(2/r)$, tending to the limiting value there

$$\varepsilon_0=\log\left[\frac{\sqrt{5}+1}{2}\right]=0.4812118251\dots.\tag{3.17}$$

Therefore $L(\beta)$ looks like a plateau in the central region with the same limiting height ε_0 and double exponential falloff outside this region (see fig. 2, where some numerical examples are plotted). As $r\to 0$ the plateau widens and the form of its right and left edges tends to some universal pattern. The r-dependence of the function $L(\beta)$ reduces to shifts of the edges. The limiting form of, say, the right

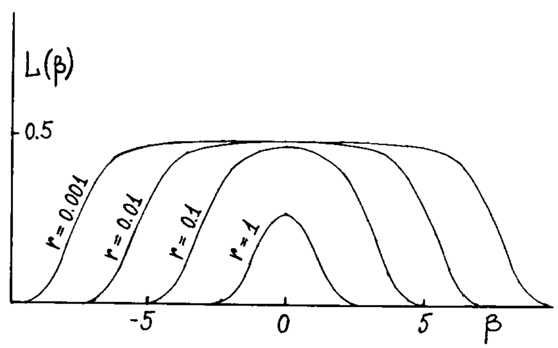

Fig. 2. Few patterns of $L(\beta)$ for different values of r.

edge is determined by the solution $L_{\text{kink}}(\beta) = \log(1 + e^{-\varepsilon_{\text{kink}}(\beta)})$ of the equation

$$-e^{\beta} + \varepsilon_{\text{kink}} + \varphi * L_{\text{kink}} = 0 \tag{3.18}$$

shifted to the right in β by $\log(2/r)$. We call the r-independent function $L_{\text{kink}}(\beta)$ the kink solution as it interpolates between limiting values ε_0 and 0.

In particular, the kink solution governs the $r \to 0$ asymptotics of the ground state scaling function $F(r)$

$$F(0) = -\frac{1}{2\pi^2} \int L_{\text{kink}}(\beta) e^{\beta} \, d\beta . \tag{3.19}$$

A known [8–10], but somewhat mysterious fact is that this integral can be calculated explicitly in terms of the Rogers dilogarithm function. Differentiating (3.18) w.r.t. β one substitutes for e^{β} in (3.19)

$$e^{\beta} = \frac{\partial \varepsilon_{\text{kink}}}{\partial \beta} - \varphi * \frac{e^{-\varepsilon}_{\text{kink}}}{1 + e^{-\varepsilon_{\text{kink}}}} \frac{\partial \varepsilon_{\text{kink}}}{\partial \beta} . \tag{3.20}$$

This gives

$$2F(0) = -\frac{1}{2\pi^2} \int_{\varepsilon_0}^{\infty} d\varepsilon \left[\frac{e^{-\varepsilon}}{1 + e^{-\varepsilon}} + \log(1 + e^{-\varepsilon}) \right] \tag{3.21}$$

and

$$F(0) = -1/30 . \tag{3.22}$$

This value matches well with the ground state energy asymptotics predicted by conformal invariance. In the UV limit one has [13]

$$E(R) = \frac{2\pi}{R} \left(\Delta + \bar{\Delta} - \frac{c}{12} \right), \tag{3.23}$$

where Δ and $\bar{\Delta}$ are right and left dimensions of the ground state. In SPM the ground state corresponds to the vacuum state of $\mathcal{M}(5/6)$ with $\Delta = \bar{\Delta} = 0$ and

$$E_{\text{SPM}} \sim -2\pi/15R . \tag{3.24}$$

This corresponds precisely to (3.22). In SLYM it is the negative dimension state φ of $\mathcal{M}(2/5)$ with $\Delta = \bar{\Delta} = -1/5$ that plays the role of the ground state. Therefore

$$E_{\text{SLYM}} \sim -\pi/15R , \tag{3.25}$$

again in accordance with (3.22).

Turn now to next-to-leading high-temperature corrections. To get rid of the dominant "conformal" contribution (3.22) to the scaling function it is convenient to consider the stress tensor trace expectation value (2.42), which is zero in the conformal limit. Using eq. (2.42) we have

$$m_{\mathrm{B}}^{-2}\langle T_{\mu}^{\mu}\rangle_{\mathrm{SLYM}} = \tfrac{1}{2}m_{\mathrm{A}}^{-2}\langle T_{\mu}^{\mu}\rangle_{\mathrm{SPM}} = \frac{(2\pi)^2}{r}\frac{\mathrm{d}F(r)}{\mathrm{d}r} = \int \frac{\mathrm{e}^{-\varepsilon(\beta)}}{1+\mathrm{e}^{-\varepsilon(\beta)}}\psi_{+}(\beta)\mathrm{e}^{-\beta}\,\mathrm{d}\beta, \tag{3.26}$$

where $\psi_{+}(\beta)$ is the solution to the linear equation

$$\psi_{+} = \mathrm{e}^{\beta} + \varphi * \frac{\mathrm{e}^{-\varepsilon}}{1+\mathrm{e}^{-\varepsilon}}\psi_{+}\,. \tag{3.27}$$

In the limit $r \to 0$ the integrand in eq. (3.26) is localized near the right edge of the central region. Its shape there is determined by the kink function $L_{\mathrm{kink}}(\beta)$. It follows that the limiting value is

$$T_0 = \frac{(2\pi)^2}{r}\frac{\mathrm{d}F(r)}{\mathrm{d}r}\bigg|_{r=0} = -\int \frac{\mathrm{d}L_{\mathrm{kink}}(\beta)}{\mathrm{d}\beta}\mathrm{e}^{-\beta}\,\mathrm{d}\beta\,. \tag{3.28}$$

Note that this is just the quantity that governs the $\beta \to -\infty$ asymptotics of the convolution

$$\varphi * L_{\mathrm{kink}} = -\varepsilon_0 + \frac{\sqrt{3}}{\pi}T_0\mathrm{e}^{\beta} + \dots\,. \tag{3.29}$$

On the other hand, one can argue that $\varepsilon_{\mathrm{kink}}(\beta)$ at $t=0$ is a regular function of $t = \exp(6\beta/5)$. The cancellation of e^{β} terms in eq. (3.18) therefore requires

$$T_0 = \pi/\sqrt{3}\,. \tag{3.30}$$

The number determines the next-to-leading term in the scaling function

$$F(r) = -\tfrac{1}{30} + \frac{\sqrt{3}\,r^2}{24\pi} + \dots\,. \tag{3.31}$$

It seems true that the remainder part of $F(r)$ near $G=0$ is a regular function of $G = r^{12/5}$

$$F(r) - \frac{\sqrt{3}\,r^2}{24\pi} = -\tfrac{1}{30} + \sum_{n=1}^{\infty} F_n G^n \tag{3.32}$$

with a finite radius of convergence. We shall see in sect. 5 that the series on the r.h.s. of eq. (3.32) corresponds to the perturbative expansions in coupling constants λ in SPM and g in SLYM. This UV structure of the scaling function is also supported by numerical analyses of sect. 4, where several first coefficients F_n and the radius of convergence are estimated. Note in this connection that the coefficient $s_1(r)$ in the asymptotic expansion (3.15) is regular in G. Remainder coefficients $s_n(r)$, $n = 5, 7, 11, \ldots$ are also regular.

4. Numerical work

Eq. (3.11) was solved by iterative numerical integration of the convolution $\varphi * L$

$$\varepsilon_{n+1} = r \cosh \beta + \varphi * L_n \tag{4.1}$$

starting from the initial value $\varepsilon_0(\beta) = r \cosh \beta$. The integration was replaced by summation with the β-step $\Delta\beta = 0.1$. The iterative process is well convergent and after 3–70 iterations (dependent on the value of r) one finishes with an $\varepsilon(\beta)$ with 14 significant digits accuracy. It was verified that the final $\varepsilon(\beta)$ is unchanged under the integration step decrease. Several patterns of $L(\beta)$ are plotted in fig. 2.

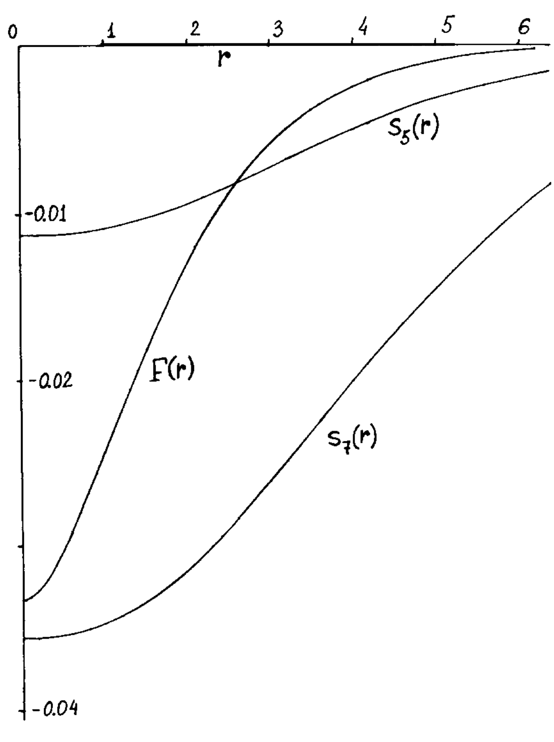

Fig. 3. Scaling function $F(r)$ and expansion coefficients $s_5(r)$ and $s_7(r)$ versus scaling length r.

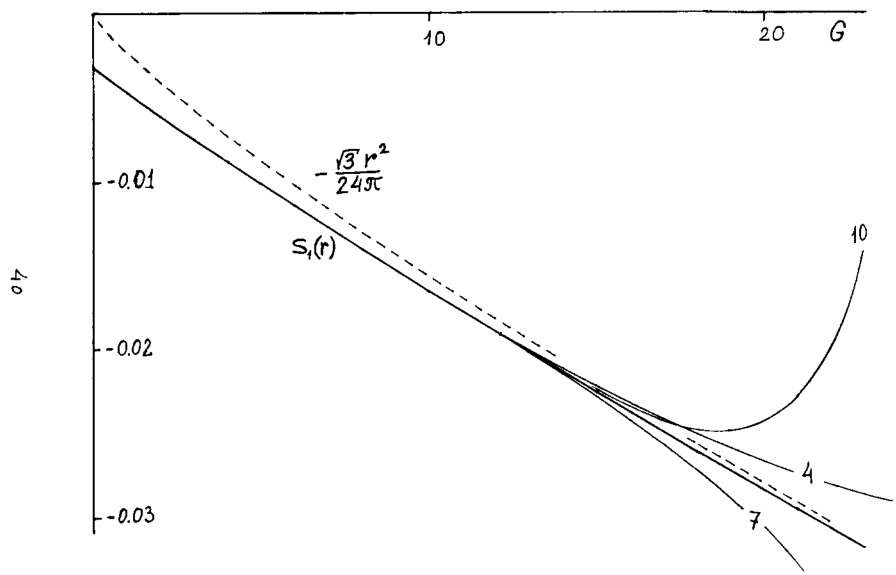

Fig. 4. Perturbative part of scaling functions $s_1(r)$ versus $G = r^{12/5}$.

The scaling function is plotted in fig. 3. The first two expansion coefficients in eq. (3.15), $s_5(r)$ and $s_7(r)$, are also presented there. The $r \to 0$ limiting values are estimated as

$$s_5(0) = -1.13476515585\ldots \times 10^{-2},$$

$$s_7(0) = -3.56664913023\ldots \times 10^{-2}. \tag{4.2}$$

In fig. 4 the "perturbative part" of the scaling function $s_1(r) = F(r) - \sqrt{3}r^2/24\pi$ is plotted against $G = r^{12/5}$. Supposing this to be expandable in power series (3.32) in G one can estimate several first expansion coefficients F_n from numerical data on $F(r)$. The result for the first ten F_n's is presented in table 1. The accuracy of these extracted numbers fall rapidly with the order number n. A few partial sums of the perturbative series (3.32) are plotted in fig. 4 and show fast convergence for $|G| \lessapprox 14$ (note that this corresponds to ~ 3 correlation lengths).

To estimate the convergence region of the series (3.32) the values of F_n listed in table 1 for $n > 1$ were fitted by the formula

$$\frac{F_n}{F_{n-1}} = \frac{1}{G_0}\left(1 + \alpha/n + O(1/n^2)\right) \tag{4.3}$$

TABLE 1

Coefficients in the perturbative expansion (3.32) of the ground state scaling function (numerical estimations)

n	F_n
1	$-1.415365357153 \times 10^{-2}$
2	$1.3587274489 \times 10^{-4}$
3	$-4.75827523 \times 10^{-6}$
4	2.130038×10^{-7}
5	-1.07141×10^{-8}
6	$5.7788 \pm 0.001 \times 10^{-10}$
7	$-3.264 \pm 0.003 \times 10^{-11}$
8	$1.90 \pm 0.01 \times 10^{-12}$
9	$-1.11 \pm 0.08 \times 10^{-13}$
10	$7 \pm 3 \times 10^{-15}$

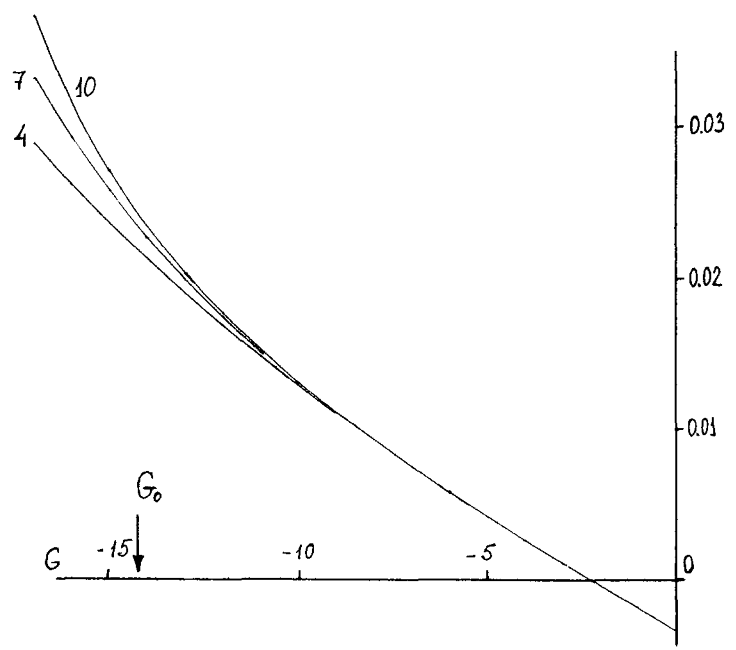

Fig. 5. Truncated series (3.32) for negative G. G_0 is the estimated position of singularity.

corresponding to a singularity of the type

$$F \sim (G - G_0)^{-1-\alpha}. \tag{4.4}$$

Supposing the coefficients near the terms omitted in the $1/n$ expansion (4.3) to be of order 1 one finds

$$G_0 = -14.3 \pm 0.4; \qquad \alpha = -1.3 \pm 0.3. \tag{4.5}$$

The value of G_0 gives the position of singularity and determines the convergence region of the series (3.32). In fig. 5, the truncated series (3.32) is presented for negative G.

5. Perturbative calculations

Here we consider the scaling function from the point of view of perturbations in the coupling constants λ in SPM and g in SLYM. The perturbation theory for the finite size effects in CFT perturbed by a relevant operator was studied in ref. [18]. In unperturbed CFT the circle hamiltonian and momentum are

$$H_c = \frac{2\pi}{R}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right), \qquad P_c = \frac{2\pi}{R}\left(L_0 - \bar{L}_0\right). \tag{5.1}$$

The perturbative corrections to the ground state energy of SPM are given by the series

$$E_{\text{SPM}}^{(\text{pert})} = -\frac{2\pi}{15R} - R\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int\langle\Phi(X_1)\dots\Phi(X_n)\rangle_c \mathrm{d}^2X_2\dots\mathrm{d}^2X_n, \tag{5.2}$$

where $X_i = (x_i, y_i)$ are points on the cylinder we considered above and the connected correlation functions $\langle\dots\rangle_c$, defined via the usual combinatorial prescription, are calculated in CFT (at $\lambda = 0$). We adopt the usual CFT normalization of field Φ (and hence the coupling constant λ) by the relation

$$\langle\Phi(X)\Phi(0)\rangle \sim |X|^{-4\Delta}, \tag{5.3}$$

where $\Delta = 2/5$ in SPM. Using the conformal mapping $z = \exp(-2\pi i\zeta/R)$, where $\zeta = x + iy$ is the complex coordinate on the cylinder, one can express integrals in eq. (5.2) in terms of connected CFT correlation functions $\langle\langle\dots\rangle\rangle_c$ on the infinite plane z

$$E^{(\mathrm{pert})} = -\frac{\pi c}{6R} - R\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\left(\frac{2\pi}{R}\right)^{2(\Delta-1)n+2}$$
$$\times\int\langle\langle V(0)\Phi(z_1,\bar z_1)\dots\Phi(z_n,\bar z_n)V(\infty)\rangle\rangle_c\prod_{i=1}^{n}(z_i\bar z_i)^{\Delta-1}\,\mathrm{d}^2z_2\dots\mathrm{d}^2z_n, \tag{5.4}$$

where V is the CFT field corresponding to the cylinder unperturbed ground state ($V=I$ in the case of SPM).

In fact, in SPM due to CFT selection rules the series in (5.2) is over even powers of the coupling λ only. Since the perturbation dimension $\Delta<1/2$, all the integrals in eq. (5.2) are UV convergent. On the cylinder they are also infrared convergent. It follows from dimensional arguments that the expansion is in fact in the dimensionless parameter $\lambda^2R^{12/5}$, i.e.

$$E^{(\mathrm{pert})}_{\mathrm{SPM}} = \frac{2\pi}{R}\left(-1/15 + \mathscr{E}_1\lambda^2R^{12/5} + \mathscr{E}_2\lambda^4R^{24/5} + \dots\right). \tag{5.5}$$

We suppose that this series converges in some finite region near $\lambda^2R^{12/5}=0$. Thermodynamic arguments require that as $R\to\infty$

$$E^{(\mathrm{pert})}_{\mathrm{SPM}} \sim \mathscr{E}_0(\lambda)R, \tag{5.6}$$

the asymptotic corrections being exponentially small in the massive theory. This quantity

$$\mathscr{E}_0(\lambda) = f_0\lambda^{5/3} \tag{5.7}$$

is the singular part of the infinite volume bulk free energy caused by the long-range fluctuations of the Potts model near criticality. In field theory the vacuum energy is conventionally normalized to zero. Subtracting this from the perturbative series (5.5) one has

$$E_{\mathrm{SPM}}(R) = E^{(\mathrm{pert})}_{\mathrm{SPM}} - \mathscr{E}_0R. \tag{5.8}$$

This leads to the following short-distance structure

$$E_{\mathrm{SPM}}(R) = -f_0\lambda^{5/3}R + \frac{2\pi}{R}\left(-1/15 + \mathscr{E}_1\lambda^2R^{12/5} + \mathscr{E}_2\lambda^4R^{24/5} + \dots\right). \tag{5.9}$$

This is exactly the structure we have found from the analyses of the TBA equation.

Therefore, from the perturbative point of view the term $\sqrt{3}r^2/24\pi$ in the scaling function is due to infrared divergent subtractions and should be attributed to the singular infinite volume vacuum energy $\mathscr{E}_0$

$$\mathscr{E}_0^{(\mathrm{SPM})} = -\frac{\sqrt{3}}{6}m_{\mathrm{A}}^2 = \frac{\sqrt{3}}{6R_{\mathrm{c}}^2}\,. \tag{5.10}$$

The behavior of $\mathscr{E}_0 \sim R_{\mathrm{c}}^{-2}$ is predicted by scaling. It is surprising that the numerical factor can be so easily extracted from the TBA approach.

The perturbative series in SLYM is

$$E_{\mathrm{SLYM}}^{(\mathrm{pert})} = -\frac{\pi}{15R} - R\sum_{n=1}^{\infty}\frac{(-g)^n}{n!}\int\langle\varphi(X_1)\ldots\varphi(X_n)\rangle_{\mathrm{c}}\,\mathrm{d}^2X_2\ldots\mathrm{d}^2X_n\,. \tag{5.11}$$

With the same arguments as used above we find

$$E_{\mathrm{SLYM}}(R) = -\mathscr{E}_0^{(\mathrm{SLYM})}\lambda^{5/3}R + \frac{2\pi}{R}\left(-1/30 + \mathscr{E}_1^{(\mathrm{SLYM})}gR^{12/5} + \mathscr{E}_2^{(\mathrm{SLYM})}g^2R^{24/5} + \ldots\right). \tag{5.12}$$

As we know from the TBA analyses, when suitably normalized, this is the same function as E_{SPM}. In particular,

$$\mathscr{E}_0^{(\mathrm{SLYM})} = -\frac{\sqrt{3}}{12}m_{\mathrm{B}}^2 = \frac{\sqrt{3}}{12R_{\mathrm{c}}^2}\,. \tag{5.13}$$

Also, the perturbative coefficients are the same up to the rescaling factor

$$\left(\frac{\lambda}{m_{\mathrm{A}}^{6/5}}\right)^{2n}\mathscr{E}_n^{(\mathrm{SPM})} = 2\left(\frac{g}{m_{\mathrm{B}}^{12/5}}\right)^{n}\mathscr{E}_n^{(\mathrm{SLYM})} = 2F_n\,, \tag{5.14}$$

where F_n are expansion coefficients of the TBA ground state scaling function (3.32). This permits us to estimate the mass scales of these two models in terms of couplings by comparing the numerical results of sect. 4 with perturbative calculations.

The simplest is the first-order perturbation in SLYM. It is given by the ground state expectation of the field φ in $\mathscr{M}(2/5)$

$$\mathscr{E}_1^{(\mathrm{SLYM})} = \frac{R^{-2/5}}{2\pi}\langle\varphi\rangle_{g=0}\,. \tag{5.15}$$

As the ground state in this case corresponds to the primary state φ, this expectation value is determined by the $\mathcal{M}(2/5)$ triple-φ structure constant [16]

$$\langle\varphi\rangle = \left(\frac{2\pi}{R}\right)^{-2/5} \mathbb{C}_{\varphi\varphi\varphi}\,. \tag{5.16}$$

The latter can be extracted from the general expressions found in ref. [19]

$$\mathbb{C}_{\varphi\varphi\varphi} = \frac{i}{5}\gamma(1/5)^{3/2}\gamma(2/5)^{1/2} \approx (1.911312699\ldots)i \tag{5.17}$$

with $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Therefore

$$\mathscr{E}_1^{(\text{SLYM})} = (2\pi)^{-7/5}\mathbb{C}_{\varphi\varphi\varphi} \approx (0.1458410994\ldots)i\,.$$

Comparing this with the value of F_1 quoted in table 1 we find

$$g = (0.09704845636\ldots)im_{\text{B}}^{12/5}\,. \tag{5.18}$$

In particular, the coupling constant g in SLYM is imaginary with $\operatorname{Im} g > 0$. Note that the singularity discussed at the end of sect. 4 is located at negative imaginary g.

In SPM the first correction is given by the following integral

$$\mathscr{E}_1^{(\text{SPM})} = -\frac{R^{-2/5}}{4\pi}\int\langle\Phi(0)\Phi(X)\rangle\,\mathrm{d}^2X\,. \tag{5.19}$$

The ground state now corresponds to the identity operator and

$$\langle\Phi(X)\Phi(0)\rangle = \left[\left(\frac{\pi}{R}\right)^2 \frac{1}{\sin(\pi\zeta/R)\sin(\pi\bar{\zeta}/R)}\right]^{4/5}. \tag{5.20}$$

The integral in (5.19) can be calculated explicitly

$$\mathscr{E}_1^{(\text{SPM})} = -\frac{\gamma(2/5)^2\gamma(1/5)}{4(2\pi)^{2/5}} \approx -1.048590494\ldots\,. \tag{5.21}$$

By comparing with the relation (5.14) this leads to

$$\lambda = (0.1643033129\ldots)m_{\text{A}}^{6/5}\,. \tag{5.22}$$

For confidence it is worth verifying relation (5.14) in the next order of the SLYM perturbation theory. To order g^2 we find using the form (5.4) of the perturbation expansion

$$\mathscr{E}_2^{(\mathrm{SLYM})} = -\frac{1}{2(2\pi)^{19/5}} \int \mathscr{G}(z,\bar{z})(z\bar{z})^{-6/5}\,\mathrm{d}^2 z, \tag{5.23}$$

where $\mathscr{G}$ is the connected four-point correlation function of fields φ on infinite plane

$$\mathscr{G}(z,\bar{z}) = \langle\langle \varphi(\infty)\varphi(1)\varphi(z,\bar{z})\varphi(0)\rangle\rangle_{\mathrm{c}}. \tag{5.24}$$

It can be found explicitly in terms of hypergeometric functions [2, 16]

$$\mathscr{G}(z,\bar{z}) = (z\bar{z})^{2/5}[(1-z)(1-\bar{z})]^{1/5}{}_2F_1\left(\frac{2}{5},\frac{3}{5},\frac{6}{5},z\right){}_2F_1\left(\frac{2}{5},\frac{3}{5},\frac{6}{5},\bar{z}\right)$$
$$+\,\mathbb{C}^2_{\varphi\varphi\varphi}(z\bar{z})^{1/5}[(1-z)(1-\bar{z})]^{1/5}{}_2F_1\left(\frac{1}{5},\frac{2}{5},\frac{4}{5},z\right){}_2F_1\left(\frac{1}{5},\frac{2}{5},\frac{4}{5},\bar{z}\right). \tag{5.25}$$

Integral (5.23) was evaluated numerically with the result

$$\int \mathscr{G}(z,\bar{z})(z\bar{z})^{-6/5}\,\mathrm{d}^2 z = 2\pi(4.955511876\ldots). \tag{5.26}$$

This gives

$$\mathscr{E}_2^{(\mathrm{SLYM})} = -1.442629784\ldots \times 10^{-2} \tag{5.27}$$

in good agreement with the value of F_2 quoted in table 1.

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