

CURRENT ALGEBRA, BARYONS, AND QUARK CONFINEMENT

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It is shown that ordinary baryons can be understood as solitons in current algebra effective lagrangians. The formation of color flux tubes can also be seen in current algebra, under certain conditions.

The idea that in some sense the ordinary proton and neutron might be solitons in a non-linear sigma model has a long history. The first suggestion was made by Skyrme more than twenty years ago [1]. Finkelstein and Rubinstein showed that such objects could in principle be fermions [2], in a paper that probably represented the first use of what would now be called θ vacua in quantum field theory. A gauge invariant version was attempted by Faddeev [3]. Some relevant miracles are known to occur in two space-time dimensions [4]; there also exists a different mechanism by which solitons can be fermions [4].

It is known that in the large- N limit of quantum chromodynamics [5] meson interactions are governed by the tree approximation to an effective local field theory of mesons. Several years ago, it was pointed out [6] that baryons behave as if they were solitons in the effective large- N meson field theory. However, it was not clear in exactly what sense the baryons actually *are* solitons.

The first relevant papers mainly motivated by attempts to understand implications of QCD current algebra were recent papers by Balachandran et al. [7] and by Boguta [8].

We will always denote the number of colors as N and the number of light flavors as n . For definiteness we first consider the usual case $n = 3$. Nothing changes for $n > 3$. Some modifications for $n < 3$ are pointed out later. Except where stated otherwise, we discuss standard current algebra with global $SU(n) \times SU(n)$ spontaneously broken to diagonal $SU(n)$, presumably as a result of an underlying $SU(N)$ gauge interaction.

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Standard current algebra can be described by a field $U(x)$ which (for each space-time point x) is a point in the $SU(3)$ manifold. Ignoring quark bare masses, this field is governed by an effective action of the form

$$I = -\frac{1}{16}F_\pi^2 \int d^4x \text{Tr} \partial_\mu U \partial_\mu U^{-1} + N\Gamma + \text{higher order terms.} \quad (1)$$

Here Γ is the Wess-Zumino term [9] which cannot be written as the integral of a manifestly $SU(3) \times SU(3)$ invariant density, and $F_\pi \simeq 190$ Mev. In quantum field theory the coefficient of Γ must a priori be an integer [10], and indeed we will see that the quantization of the soliton excitations of (1) is inconsistent (they obey neither bose nor fermi statistics) unless N is an integer.

Any finite energy configuration $U(x, y, z)$ must approach a constant at spatial infinity. This being so, any such configuration represents an element in the third homotopy group $\pi_3(SU(3))$. Since $\pi_3(SU(3)) \simeq \mathbb{Z}$, there are soliton excitations, and they obey an additive conservation law. Actually, higher-order terms in (1) are needed to stabilize the soliton solutions and prevent them from shrinking to zero size. We will see that such higher-order terms (which could be measured in principle by studying meson processes) must be present in the large- N limit of QCD and are related to the bag radius. Our remarks will not depend on the details of the higher-order terms.

A technical remark is in order. To study solitons, it is convenient to work with a euclidean space-time M of topology $S^3 \times S^1$. Here S^3 represents the spatial variables, and S^1 is a compactified euclidean time coordinate. A given non-linear sigma model field $U(x)$ defines a mapping of M into $SU(3)$. We may think of M as the boundary of a five-dimensional manifold Q with topology $S^3 \times D$, D being a two-dimensional disc. Using the fact that $\pi_1(SU(3)) = \pi_4(SU(3)) = 0$, it can be shown that the mapping of M into $SU(3)$ defined by $U(x)$ can be extended to a mapping from Q into $SU(3)$. Then as in ref. [10] the functional Γ is defined by $\Gamma = \int_Q \omega$, where ω is the fifth-rank antisymmetric tensor on the $SU(3)$ manifold defined in ref. [10]. By analogy with the discussion in ref. [10], Γ is well-defined modulo 2π . (It is essential here that because $\pi_2(SU(3)) = 0$, the five-dimensional homology classes in $H_5(SU(3))$ that can be represented by cycles with topology $S^3 \times S^2$ are precisely those that can be represented by cycles with topology S^5 . There are closed five-surfaces S in $SU(3)$ such that $\int_S \omega$ is an *odd* multiple of π , but they do not arise if space-time has topology $S^3 \times S^1$ and Q is taken to be $S^3 \times D$.)

Now let us discuss the quantum numbers of the current algebra soliton. First, let us calculate its baryon number (which was first demonstrated to be non-zero in ref. [7], where, however, different assumptions were made from those we will follow). In previous work [10] it was shown that the baryon-number current has an anomalous piece, related to the $N\Gamma$ term in eq. (1). If the baryon number of a quark is $1/N$, so that an ordinary baryon made from N quarks has baryon number 1, then the

anomalous piece in the baryon number current B_μ was shown to be

$$B_\mu = \frac{\epsilon_{\mu\nu\alpha\beta}}{24\pi^2} \text{Tr}(U^{-1} \partial_\nu U)(U^{-1} \partial_\alpha U)(U^{-1} \partial_\beta U). \tag{2}$$

So the baryon number of a configuration is

$$B = \int d^3x B_0 = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr}(U^{-1} \partial_i U)(U^{-1} \partial_j U)(U^{-1} \partial_k U). \tag{3}$$

The right-hand side of eq. (24) can be recognized as the properly normalized integral expression for the winding number in $\pi_3(\text{SU}(3))$. In a soliton field the right-hand side of (3) equals one, so the soliton has baryon number one; it is a baryon. (In ref. [7] the baryon number of the soliton was computed using methods of Goldstone and Wilczek [11]. The result that the soliton has baryon number one would emerge in this framework if the elementary fermions are taken to be quarks.)

Now let us determine whether the soliton is a boson or a fermion. To this end, we compare the amplitude for two processes. In one process, a soliton sits at rest for a long time T . The amplitude is $\exp(-iMT)$, M being the soliton energy. In the second process, the soliton is adiabatically rotated through a 2π angle in the course of a long time T . The usual term in the lagrangian $L_0 = \frac{1}{16} F_\pi^2 \text{Tr} \partial_\mu U \partial_\mu U^{-1}$ does not distinguish between the two processes, because the only piece in L_0 that contains time derivatives is quadratic in time derivatives, and the integral $\int dt \text{Tr}(\partial U / \partial t)(\partial U^{-1} / \partial t)$ vanishes in the limit of an adiabatic process. However, the anomalous term Γ is linear in time derivatives, and distinguishes between a soliton that sits at rest and a soliton that is adiabatically rotated. For a soliton at rest, $\Gamma = 0$. For a soliton that is adiabatically rotated through a 2π angle, a slightly laborious calculation explained at the end of this paper shows that $\Gamma = \pi$. So for a soliton that is adiabatically rotated by a 2π angle, the amplitude is not $\exp(-iMT)$ but $\exp(-iMT)\exp(iN\pi) = (-1)^N \exp(-iMT)$.

The factor $(-1)^N$ means that for odd N the soliton is a fermion; for even N it is a boson. This is uncannily reminiscent of the fact that an ordinary baryon contains N quarks and is a boson or a fermion depending on whether N is even or odd.

These results are unchanged if there are more than three light flavors of quarks. How do they hold up if there are only two light flavors? The field $U(x)$ is then an element of $\text{SU}(2)$. Because $\pi_3(\text{SU}(2)) = \mathbb{Z}$, there are still solitons. The baryon-number current has the same anomalous piece, and the soliton still has baryon number one. But in $\text{SU}(2)$ current algebra, there is no Γ term, so how can we see that the soliton can be a fermion?

The answer was given long ago [2]. Although $\pi_4(\text{SU}(3)) = 0$, $\pi_4(\text{SU}(2)) = \mathbb{Z}_2$. With suitably compactified space-time, there are thus two topological classes of maps from space-time to $\text{SU}(2)$. In the $\text{SU}(2)$ non-linear sigma model, there are hence two

“ θ -vacua”: fields that represent the non-trivial class in $\pi_4(\text{SU}(2))$ may be weighted with a sign $+1$ or -1 . An explicit field $U(x, y, z, t)$ which goes to 1 at space-time infinity and represents the non-trivial class in $\pi_4(\text{SU}(2))$ can (fig. 1) be described as follows (a variant of this description figures in recent work by Goldstone [12]). Start at $t \rightarrow -\infty$ with a constant, $U = 1$; moving forward in time, gradually create a soliton-anti-soliton pair and separate them; rotate the soliton through a 2π angle without touching the anti-soliton; bring together the soliton and anti-soliton and annihilate them. Weighting this field with a factor of -1 , while a configuration without the 2π rotation of the soliton is homotopically trivial and gets a factor $+1$, corresponds to quantizing the soliton as a fermion. Thus, internally to $\text{SU}(2) \times \text{SU}(2)$ current algebra, one sees that the soliton can be a fermion. In $\text{SU}(3) \times \text{SU}(3)$ current algebra one finds the stronger result that the soliton *must* be a fermion if and only if N is odd.

Our results so far are consistent with the idea that quantization of the current algebra soliton describes ordinary nucleons. However, we have not established this. Perhaps there are ordinary baryons and exotic, topologically excited solitonic baryons. However, certain results will now be described which seem to directly show that the ordinary nucleons are the ground state of the soliton.

For simplicity, we will focus now on the case of only two flavors. Soliton states can be labeled by their spin and isospin quantum numbers, which we will call J and I , respectively. We will determine semiclassically what values of I and J are expected for solitons. A semiclassical description of current algebra solitons will be accurate quantitatively only in the limit of large N . (Since F_π^2 is proportional to N , N enters the effective lagrangian (1) as an overall multiplicative factor. Hence, N plays the role usually played by $1/\hbar$.) So we will check the results we find for solitons by comparing to the expected quantum numbers of baryons in the large- N limit.

Let us first determine the expected baryon quantum numbers. We make the usual assumption that the multi-quark wave function is symmetric in space and antisymmetric in color, and hence must have complete symmetry in spin and isospin. The spin-isospin group is $\text{SU}(2) \times \text{SU}(2) \sim \text{O}(4)$. A quark transforms as $(\frac{1}{2}, \frac{1}{2})$; this is the

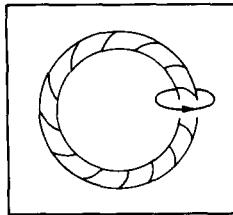


Fig. 1. A soliton-antisoliton pair is created from the vacuum; the soliton is rotated by a 2π angle; the pair is then annihilated. This represents the non-trivial homotopy class in $\pi_4(\text{SU}(2))$.

vector representation of $O(4)$. We may represent a quark as ϕ_i , where $i = 1 \dots 4$ is a combined spin-isospin index labeling the $O(4)$ four-vector.

We must form symmetric combinations of N vectors ϕ_i . As is well known, there is a quadratic invariant $\phi^2 = \sum_{i=1}^4 \phi_i^2$. One can also form symmetric traceless tensors of any rank $A_{i_1 \dots i_p}^{(p)} = (\phi_{i_1} \phi_{i_2} \dots \phi_{i_p} - \text{trace terms})$; this transforms as $(\frac{1}{2}p, \frac{1}{2}p)$ under $SU(2) \times SU(2)$. The general symmetric expression that we can make from N quarks is $(\phi^2)^k A_{i_1 \dots i_{N-2k}}^{(N-2k)}$, where $0 \leq k \leq \frac{1}{2}N$. So the values of I and J that are possible are the following:

$$\begin{aligned} N \text{ even,} \quad I = J = 0, 1, 2, 3, \dots, \\ N \text{ odd,} \quad I = J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots \end{aligned} \tag{4}$$

For instance, in nature we have $N = 3$. The first two terms in the sequence indicated above are the nucleon, of $I = J = \frac{1}{2}$, and the delta, of $I = J = \frac{3}{2}$. If the number of colors were five or more, we would expect to see more terms in this series. Moreover, simple considerations involving color magnetic forces suggest that, as for $N = 3$, the mass of the baryons in this sequence is always an increasing function of I or J .

Now let us compare to what is expected in the soliton picture. (This question has been treated previously in ref. [7].) We do not know the effective action of which the soliton is a minimum, because we do not know what non-minimal terms must be added to eq. (1). We will make the simple assumption that the soliton field has the maximum possible symmetry. The soliton field cannot be invariant under I or J (or any component thereof), but it can be invariant under a diagonal subgroup $I + J$. This corresponds to an ansatz $U(x) = \exp[iF(r)]T \cdot x$, where $F(r) = 0$ at $r = 0$ and $F(r) \rightarrow \pi$ as $r \rightarrow \infty$.

Quantization of such a soliton is very similar to quantization of an isotropic rigid rotor. The hamiltonian of an isotropic rotor is invariant under an $SU(2) \times SU(2)$ group consisting of the rotations of body fixed and space fixed coordinates, respectively. We will refer to these symmetries as I and J , respectively. A given configuration of the rotor is invariant under a diagonal subgroup of $SU(2) \times SU(2)$. This is just analogous to our solitons, assuming the classical soliton solution is invariant under $I + J$.

The quantization of the isotropic rigid rotor is well known. If the rotor is quantized as a boson, it has $I = J = 0, 1, 2, \dots$. If it is quantized as a fermion, it has $I = J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$. The agreement of these results with eq. (4) is hardly likely to be fortuitous.

In the case of three or more flavors, it may still be shown that the quantization of collective coordinates gives the expected flavor quantum numbers of baryons. The analysis is more complicated; the Wess-Zumino interaction plays a crucial role.

So far, we have assumed that the color gauge group is $SU(N)$. Now let us discuss what would happen if the color group were $O(N)$ or $Sp(N)$. (By $Sp(N)$ we will

mean the group of $N \times N$ unitary matrices of quaternions; thus $\text{Sp}(1) \approx \text{SU}(2)$.) We will see that also for these gauge groups, the topological properties of the current algebra theory correctly reproduce properties of the underlying gauge theory.

In an $\text{O}(N)$ gauge theory, we assume that we have n multiplets of left-handed (Weyl) spinors in the fundamental N -dimensional representation of $\text{O}(N)$. There is no distinction between quarks and antiquarks, because this representation is real. (If n is even, the theory is equivalent to a theory of $\frac{1}{2}n$ Dirac multiplets.) The anomaly free flavor symmetry group is $\text{SU}(n)$. Simple considerations based on the most attractive channel idea suggest that the flavor symmetry will be spontaneously broken down to $\text{O}(n)$, which is the maximal subgroup of $\text{SU}(n)$ that permits all fermions to acquire mass. In this case the current algebra theory is based on a field that takes values in the quotient space $\text{SU}(n)/\text{O}(n)$.

In an $\text{Sp}(N)$ gauge theory, we assume the fermion multiplets to be in the fundamental $2N$ -dimensional representation of $\text{Sp}(N)$. Since this representation is pseudoreal, there is again no distinction between quarks and antiquarks. In this theory the number of fermion multiplets must be even; otherwise, the $\text{Sp}(N)$ gauge theory is inconsistent because of a non-perturbative anomaly [2] involving $\pi_4(\text{Sp}(N))$. If there are $2n$ multiplets, the flavor symmetry is $\text{SU}(2n)$. Simple arguments suggest that the $\text{SU}(2n)$ flavor group is spontaneously broken to $\text{Sp}(n)$, so that the current algebra theory is based on the quotient space $\text{Su}(2n)/\text{Sp}(n)$. This corresponds to symmetry breaking in the most attractive channel; $\text{Sp}(n)$ is the largest unbroken symmetry that lets all quarks get mass.

In $\text{O}(N)$, since there is no distinction between quarks and antiquarks, there is also no distinction between baryons and anti-baryons. A baryon can be formed from an antisymmetric combination of N quarks; $B = \epsilon_{i_1 i_2 \dots i_N} q^{i_1} q^{i_2} \dots q^{i_N}$. But in $\text{O}(N)$, a product of two epsilon symbols can be rewritten as a sum of products of N Kronecker deltas:

$$\epsilon_{i_1 i_2 \dots i_N} \epsilon_{j_1 j_2 \dots j_N} = \left(\delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_N j_N} \pm \text{permutations} \right).$$

This means that in an $\text{O}(N)$ gauge theory, two baryons can annihilate into N mesons.

On the other hand, in an $\text{Sp}(N)$ gauge theory there are no baryons at all. The group $\text{Sp}(N)$ can be defined as the subgroup of $\text{SU}(2N)$ that leaves fixed an antisymmetric second rank tensor γ_{ij} . A meson made from two quarks of the same chirality can be described by the two quark operator $\gamma_{ij} q^i q^j$. In $\text{Sp}(N)$ the epsilon symbol can be written as a sum of products of N γ 's:

$$\epsilon_{i_1 i_2 \dots i_{2N}} = \left(\gamma_{i_1 i_2} \gamma_{i_3 i_4} \dots \gamma_{i_{2N-1} i_{2N}} \pm \text{permutations} \right).$$

So in an $\text{Sp}(N)$ gauge theory, a single would-be baryon can decay to N mesons.

Now let us discuss the physical phenomena that are related to the topological properties of our current algebra spaces $SU(n)/O(n)$ and $SU(n)/Sp(n)$. We recall from ref. (10) that the existence in QCD current algebra with at least three flavors of the Wess-Zumino interaction, with its a priori quantization law, is closely related to the fact that $\pi_5(SU(n)) = \mathbb{Z}$, $n \geq 3$. The analogue is that

$$\begin{aligned} \pi_5(SU(n)/O(n)) &= \mathbb{Z}, & n \geq 3, \\ \pi_5(SU(2n)/Sp(n)) &= \mathbb{Z}, & n \geq 2. \end{aligned} \tag{5}$$

So also the $O(N)$ and $Sp(N)$ gauge theories possess at the current algebra level an interaction like the Wess-Zumino term, provided the number of flavors is large enough. Built into the current algebra theories is the fact that in the underlying theory there is a parameter (the number of colors) which a priori must be an integer.

Now we come to the question of the existence of solitons. These are classified by the third homotopy group of the configuration space, and we have

$$\begin{aligned} \pi_3(SU(n)/O(n)) &= \mathbb{Z}_2, & n \geq 4, \\ \pi_3(SU(2n)/Sp(n)) &= 0, & \text{any } n. \end{aligned} \tag{6}$$

Thus, in the case of an $O(N)$ gauge theory with at least four flavors, the current algebra theory admits solitons, but the number of solitons is conserved only modulo two. This agrees with the fact that in the $O(N)$ gauge theory there are baryons which can annihilate in pairs. In current algebra corresponding to $Sp(N)$ gauge theory there are no solitons, just as the $Sp(N)$ gauge theory has no baryons.

For $O(N)$ gauge theories with less than four light flavors we have

$$\begin{aligned} \pi_3(SU(3)/O(3)) &= \mathbb{Z}_4, \\ \pi_3(SU(2)/O(2)) &= \mathbb{Z}. \end{aligned} \tag{7}$$

Thus, the spectrum of current algebra solitons seems richer than the expected spectrum of baryons in the underlying gauge theory. The following remark seems

TABLE I
Some homotopy groups of certain homogeneous spaces

	$SU(n)$	$SU(n)/O(n)$	$SU(2n)/Sp(n)$
π_2	0	$\mathbb{Z}_2, n \geq 3$	0
π_3	\mathbb{Z} , all n	$\mathbb{Z}_2, n \geq 4$	0
π_5	$\mathbb{Z}, n \geq 3$	$\mathbb{Z}, n \geq 3$	$\mathbb{Z}, n \geq 3$

appropriate in this connection. It is only in the multi-color, large- N limit that a semiclassical description of current algebra solitons becomes accurate. Actually, large- N gauge theories are described by weakly interacting theories of mesons, but it is not only Goldstone bosons that enter; one has an infinite meson spectrum. Corresponding to the rich meson spectrum is an unknown and perhaps topologically complicated configuration space P of the large- N theory. Plausibly, baryons can always be realized as solitons in the large- N theory, even if all or almost all quark flavors are heavy. Perhaps $\pi_3(P)$ is always \mathbb{Z} , \mathbb{Z}_2 , or \mathbb{O} for $SU(N)$, $O(N)$, and $Sp(N)$ gauge theories. The Goldstone boson space is only a small subspace of P and would not necessarily reflect the topology of P properly. Our results suggest that as the number of flavors increases, the Goldstone boson space becomes an increasingly good *topological* approximation to P . In this view, the extra solitons suggested by eq. (7) for $O(N)$ gauge theories with two or three flavors become unstable when $SU(2)/O(2)$ or $SU(3)/O(3)$ is embedded in P .

One further physical question will be addressed here. Is color confinement implicit in current algebra?

Do current algebra theories in which the field U labels a point in $SU(n)$, $SU(n)/O(n)$, or $SU(2n)/Sp(n)$ admit flux tubes? By a flux tube we mean a configuration $U(x, y, z)$ which is independent of z and possesses a non-trivial topology in the x - y plane. To ensure that the energy per unit length is finite, U must approach a constant as $x, y \rightarrow \infty$. The proper topological classification involves therefore the *second* homotopy group of the space in which U takes its values. In fact, we have

$$\begin{aligned}\pi_2(SU(n)) &= 0, \\ \pi_2(SU(n)/O(n)) &= \mathbb{Z}_2, \quad n \geq 3, \\ \pi_2(SU(2n)/Sp(n)) &= 0.\end{aligned}\tag{8}$$

Thus, current algebra theories corresponding to underlying $SU(N)$ and $Sp(N)$ gauge theories do not admit flux tubes. The theories based on underlying $O(N)$ gauge groups do admit flux tubes, but two such flux tubes can annihilate.

These facts have the following natural interpretation. Our current algebra theories correspond to underlying gauge theories with quarks in the fundamental representation of $SU(N)$, $O(N)$, or $Sp(N)$. $SU(N)$ or $Sp(N)$ gauge theories with dynamical quarks cannot support flux tubes because arbitrary external sources can be screened by sources in the fundamental representation of the group. For $O(N)$ gauge theories it is different. An external source in the spinor representation of $O(N)$ cannot be screened by charges in the fundamental representation. But two spinors make a tensor, which can be screened. So the $O(N)$ gauge theory with dynamical quarks supports only one type of color flux tube: the response to an external source in the

spinor representation of $O(N)$. It is very plausible that this color flux tube should be identified with the excitation that appears in current algebra because $\pi_2(SU(n)/O(n)) = Z_2$.

The following fact supports this identification. The interaction between two sources in the spinor representation of $O(N)$ is, in perturbation theory, N times as big as the interaction between two quarks. Defining the large- N limit in such a way that the interaction between two quarks is of order one, the interaction between two spinor charges is therefore of order N . This strongly suggests that the energy per unit length in the flux tube connecting two spinor charges is of order N . This is consistent with our current algebra identification; the whole current algebra effective lagrangian is of order N (since $F_\pi^2 \sim N$), so the energy per unit length of a current algebra flux tube is certainly of order N .

In conclusion, it still remains for us to establish the contention made earlier that the value of the Wess-Zumino functional Γ for a process consisting of a 2π rotation of a soliton is $\Gamma = \pi$.

First of all, the soliton field can be chosen to be of the form

$$V(x_i) = \left(\begin{array}{c|c} W(x_i) & 0 \\ \hline 0 & 1 \end{array} \right), \tag{9}$$

where the $SU(2)$ matrix W is chosen to be invariant under a combined isospin rotation plus rotation of the spatial coordinate x_i . This being so, a 2π rotation of V in space is equivalent to a 2π rotation of V in isospin. Introducing a periodic time coordinate t which runs from 0 to 2π , the desired field in which a soliton is rotated by a 2π angle can be chosen to be

$$U(x_i, t) = \left(\begin{array}{c|c} e^{it/2} & \\ \hline & e^{-it/2} \\ & & 1 \end{array} \right) V(x_i) \left(\begin{array}{c|c} e^{-it/2} & \\ \hline & e^{it/2} \\ & & 1 \end{array} \right). \tag{10}$$

Note that $U(x_i, t)$ is periodic in t with period 2π even though the individual exponentials $\exp(\pm \frac{1}{2}it)$ have period 4π . Because of the special form of V , we can equivalently write U in the much more convenient form

$$U(x_i, t) = \left(\begin{array}{c|c} 1 & \\ \hline & e^{-it} \\ & & e^{it} \end{array} \right) V(x_i) \left(\begin{array}{c|c} 1 & \\ \hline & e^{it} \\ & & e^{-it} \end{array} \right). \tag{11}$$

This field $U(x_i, t)$ describes a soliton that is rotated by a 2π angle as t ranges from 0 to 2π . We wish to evaluate $\Gamma(U)$.

To this end we introduce a fifth parameter ρ ($0 \leq \rho \leq 1$) so as to form a five-manifold of which space-time is the boundary; this five-manifold will have the topology of three-space times a disc. A convenient choice is to write

$$\tilde{U}(x_i, t, \rho) = A^{-1}(t, \rho)U(x_i, t)A(t, \rho), \tag{12}$$

where

$$A(t, \rho) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho e^{it} & \sqrt{1 - \rho^2} \\ 0 & -\sqrt{1 - \rho^2} & \rho e^{-it} \end{pmatrix}. \tag{13}$$

Note that at $\rho = 0$, $A(t, \rho)$ is independent of t . So we can think of ρ and t as polar coordinates for the plane, ρ being the radius and t the usual angular variable. Also $\tilde{U}(x_i, t, 1) = U(x_i, t)$ so the product of three space with the unit circle in the ρ - t plane can be identified with the original space-time. According to eq. (14) of ref. (10), what we must calculate is

$$\Gamma(U) = -\frac{i}{240\pi^2} \int_0^1 d\rho \int_0^{2\pi} dt \int d^3x \varepsilon^{ijklm} \times [\text{Tr } \tilde{U}^{-1} \partial_i \tilde{U} \tilde{U}^{-1} \partial_j \tilde{U} \tilde{U}^{-1} \partial_k \tilde{U} \tilde{U}^{-1} \partial_l \tilde{U} \tilde{U}^{-1} \partial_m \tilde{U}], \tag{14}$$

where i, j, k, l , and m may be ρ, t, x_1, x_2 , or x_3 . The integral can be done without undue difficulty (the fact that W is invariant under spatial rotations plus isospin is very useful), and one finds $\Gamma(U) = \pi$.

This calculation can also be used to fill in a gap in the discussion of ref. (10). In that paper, the following remark was made. Let $A(x, y, z, t)$ be a mapping from space-time into $SU(2)$ that is in the non-trivial homotopy class in $\pi_4(SU(2))$. Embed A in $SU(3)$ in the trivial form

$$\hat{A} = \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array} \right).$$

Then $\Gamma(\hat{A}) = \pi$. In fact, as we have noted above, the non-trivial homotopy class in $\pi_4(SU(2))$ differs from the trivial class by a 2π rotation of a soliton (which may be one member of a soliton-antisoliton pair). The fact that $\Gamma = \pi$ for a 2π rotation of soliton means that $\Gamma = \pi$ for the non-trivial homotopy class in $\pi_4(SU(2))$.

The following important fact deserves to be demonstrated explicitly. As before, let A be a mapping of space-time into $SU(2)$ and let \hat{A} be its embedding in $SU(3)$. Then $\Gamma(\hat{A})$ depends only on the homotopy class of A in $\pi_4(SU(2))$. In fact, suppose \hat{A} is

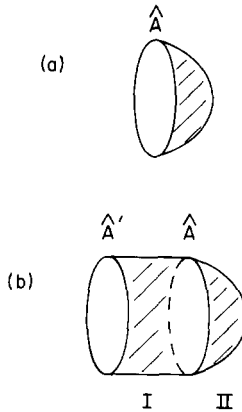


Fig. 2. A demonstration that Γ is a homotopy invariant for SU(2) mappings.

homotopic to \hat{A}' : Let us prove that $\Gamma(\hat{A}) = \Gamma(\hat{A}')$. To compute $\Gamma(\hat{A})$ we realize space-time as the boundary of a disc, extend \hat{A} to be defined over that disc, and evaluate an appropriate integral (fig. 2a). To evaluate $\Gamma(\hat{A}')$ we again must extend \hat{A}' to a disc. This can be done in a very convenient way (fig. 2b). Since \hat{A} is homotopic to \hat{A}' , we first deform \hat{A}' into \hat{A} via matrices of the form $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$ (matrices that are really SU(2) matrices embedded in SU(3)) and then we extend \hat{A} over a disc as before. The integral contribution to $\Gamma(\hat{A}')$ from part I of fig. 2b vanishes because the fifth rank antisymmetric tensor that enters in defining Γ vanishes when restricted to any SU(2) subgroup of SU(3). The integral in part II of fig. 2b is the same as the integral in fig. 2a, so $\Gamma(\hat{A}) = \Gamma(\hat{A}')$.

The fact that Γ is a homotopy invariant for SU(2) mappings also means that Γ can be used to prove that $\pi_4(\text{SU}(2))$ is non-trivial. Since Γ obviously is 0 for the trivial homotopy class in $\pi_4(\text{SU}(2))$, while $\Gamma = \pi$ for a process containing a 2π rotation of a soliton, the latter process must represent a non-trivial element in $\pi_4(\text{SU}(2))$. What cannot be proved so easily is that this is the only non-trivial element.

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Note added in proof

Many physicists have asked how the soliton quantum numbers can be calculated if there are three flavors. Following is a sketch of how this question can be answered.

We assume that for $\text{SU}(3) \times \text{SU}(3)$ current algebra, the soliton solution is simply an SU(2) solution embedded in SU(3). Such a solution is invariant under combined spin-isospin transformations; and it is also invariant under hypercharge rotations.

There are now seven collective coordinates instead of three. They parametrize the coset space $X = \text{SU}(3)/\text{U}(1)$, where $\text{U}(1)$ refers to right multiplication by hypercharge. Thus a point in X is an element U of $\text{SU}(3)$ defined up to multiplication on the right by a hypercharge transformation. The space X has flavor $\text{SU}(3)$ symmetry (left multiplication of U by an $\text{SU}(3)$ matrix) and rotation $\text{SU}(2)$ symmetry (right multiplication of U by an $\text{SU}(3)$ matrix that commutes with hypercharge).

The crucial novelty of the three-flavor problem is that even when restricted to the space of collective coordinates, the Wess-Zumino term does not vanish. As usual, the quantization of collective coordinates involves the quantum mechanics of a particle moving on the manifold X , but in this case, the effect of the Wess-Zumino term is that the particle is moving under the influence of a simulated “magnetic field” on the X manifold. Moreover, this magnetic field is of the Dirac monopole type; it has string singularities which are unobservable if the Wess-Zumino coupling is properly quantized.

The wave functions of the collective coordinates are “monopole harmonics” on the X manifold with quantum numbers that depend on the “magnetic charge.” For charge three (three colors) the lowest monopole harmonic is an $\text{SU}(3)$ octet of spin $\frac{1}{2}$, and the next one is an $\text{SU}(3)$ decuplet of spin $\frac{3}{2}$.

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