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INSTANTONS, THE QUARK MODEL, AND THE 1/*N* EXPANSION

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An attempt is made to resolve certain discrepancies between instantons, the quark model and the $1/N$ expansion. It is argued that the most attractive resolution of these discrepancies is the possibility that quantum corrections cause the instanton gas to disappear in QCD. A two-dimensional model is described in which it can be seen explicitly that such a disappearance takes place. (This model has been investigated independently by D'Adda, Di Vecchia, and Lüscher.)

1. Introduction

Instantons were originally introduced in physics by Polyakov [1], who described an interesting field-theory model in which instantons are responsible for some rather surprising effects, and by Belavin, Polyakov, Schwartz and Tyupkin [2], who introduced the first instanton solution of four-dimensional gauge theories.

In the last few years instantons have been associated with some of the most interesting developments in strong interaction theory. They have led to a resolution [3] of the long-standing U(1) problem [4], and also pointed to the existence in QCD [5,6] of vacuum tunneling and of a previously unrecognized parameter, the vacuum angle θ (for a review see ref. [7]).

At the same time, instanton physics, as it now stands, is, in some ways, in conflict with some of the most successful ideas about the strong interactions.

For instance, our present field theoretic understanding is that the η mass (in the chiral limit, $m_\pi = 0$) is an instanton effect. (In discussing the U(1) problem in this paper I will consider the limit of chiral SU(2), $m_u = m_d = 0$, and I will refer to the missing isosinglet Goldstone boson as the "η".) However, in the naive quark model, which is known for its successes, there is a simple and natural explanation [8] of what splits the η from the π: in the η, which is an isosinglet, the quark-antiquark pair can annihilate into gluons; this annihilation channel is absent for the isovector

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π. From the point of view of the quark model, it is very hard to imagine why quark-antiquark annihilation, a simple tree-approximation process, can proceed only with the help of an instanton.

To make this point more quantitative, let us consider QCD with N colors and an SU(N) gauge group [9], and ask how the η mass depends on N. According to the quark model, where the η mass is determined by quark-antiquark annihilation into gluons, an effect which [9] is of order $1/N, m_\eta^2 \sim 1/N$. However, in the instanton picture, m_η^2 should vanish like e^{-cN}, c being some constant, because instanton effects vanish exponentially for large N. (The reason for this is that instanton effects are of order e^{-1/g^2}, and for large N, g^2 is of order $1/N$ [9], so e^{-1/g^2} is of order e^{-N}. This point will be discussed more fully below.) In this paper I hope to convince the reader that m_η^2 is in fact of order $1/N$, and that the quark model picture, according to which this mass comes from annihilation into gluons, is perfectly correct.

If one believes that an instanton gas of some kind (dilute or dense) plays a significant role in the strong interactions, then there is a more general conflict between instantons and other successful physical ideas. It is very attractive to believe [9] that QCD has a smooth limit for large SU(N) gauge group and that the SU(3) theory is close to that limit. This is likely to be a key element in any field theoretic derivation of the quark-model spectroscopy. It is the only field theoretic basis for the general success of Regge pole phenomenology. It also accounts for the narrowness of resonances, and it is the only sufficiently general explanation of Zweig's rule (for a recent review, see ref. [10]). Moreover, the $1/N$ expansion is certainly one of the most promising suggestions that has been made concerning a possible non-perturbative approach to the strong interactions. In particular, $1/N$ is the only expansion parameter that is known to exist in this theory.

But instantons are in direct conflict with the large N expansion, because, as noted above, instanton effects vanish exponentially for large N. Insofar as an instanton gas plays a significant role in the strong interactions, the large N expansion must be bad. It is necessary to choose between the two.

(Perhaps I should clarify why I claim that instanton effects vanish exponentially with N. This can be expected on general grounds. Green functions, in QCD and in the model discussed later in this paper, can be calculated to any finite order in $1/N$ by summing Feynman diagrams; therefore, instantons, or anything else not included in the Feynman diagrams, must be smaller than any power of $1/N$. This point will be discussed again in sect. 5. The same conclusion can be reached by considering explicit instanton calculations. A one-loop calculation [11] shows that the effects of instantons of size ρ are proportional to $\{(1/\bar{g}^2)\exp[-8\pi^2/\bar{g}^2]\}^N$ where $\bar{g}^2$ is an effective coupling constant for scale ρ, normalized so as to be independent of N. For weak coupling, where $(1/\bar{g}^2)\exp[-8\pi^2/\bar{g}^2]$ is much less than one, this vanishes exponentially with N. One might try to avoid this conclusion by arguing that for large instantons $(1/\bar{g}^2)\exp[-8\pi^2/\bar{g}^2]$, or whatever function replaces it when higher-order corrections are taken into account, becomes greater than one. Then one reaches the disastrous conclusion that instanton effects *grow* exponentially with N.

And it is hard to imagine that $(1/\bar{g}^2)\exp[-8\pi^2/\bar{g}^2]$, or the function that replaces it when one includes higher-order effects, becomes exactly equal to one but not larger. Although it might be mathematically possible, it is not very plausible that an instanton gas could have for large N a smooth limit other then zero. This is why I say that the instanton gas disappears in the large N limit.)

The purpose of this paper is to attempt to resolve the conflicts between the quark model and the $1/N$ expansion, on the one side, and instanton physics, on the other side.

2. Quantized topological charge and the instanton gas

Not in every theory that possesses a Euclidean space topological charge is it reasonable to think about an instanton gas. To illustrate this point, let us consider U(1) gauge theories in two dimensions. As we know, these theories come in two sorts, theories with unbroken gauge symmetry and Higgs theories. For example:

$$L = D_\mu\phi^* D_\mu\phi - M^2\phi^*\phi - \tfrac{1}{4}F_{\mu\nu}^2\,, \tag{1}$$

$$L = D_\mu\phi^* D_\mu\phi - \lambda(\phi^*\phi - a^2)^2 - \tfrac{1}{4}F_{\mu\nu}^2\,. \tag{2}$$

(A technical remark should perhaps be inserted here. The particle spectrum of model (2) is the characteristic Higgs spectrum, a real physical Higgs scalar and a massive gauge meson. On the other hand, because in one space dimension the Coulomb potential is confining, the particle spectrum in (1) is the characteristic spectrum of a confining theory, with neutral $\phi\phi^*$ bound states. Because of the spectrum, we will refer to (2) as a Higgs theory, despite the fact that, as is known, because of instantons, some of its behavior is anomalous for a Higgs theory. In an appendix it will be argued that there is in fact a phase transition separating (1) from (2), and the theory (2), which is on the negative M^2 side of this phase transition, will be referred to here as a Higgs theory.)

Both (1) and (2) possesses the topological charge

$$Q = \frac{e}{2\pi}\int d^2x\,\epsilon^{\mu\nu}F_{\mu\nu}\,.$$

However, in the second theory, the Higgs theory, Q is quantized and (in finite-action field configurations) it takes only integer values. In the first theory, the one with unbroken U(1) symmetry, Q is not quantized.

To see that Q is not quantized in the theory (1) with unbroken U(1) symmetry, note that in this model, because the mass term has the right sign, ϕ has no vacuum expectation value, and the ϕ field should be taken to vanish as $|x| \to \infty$. Therefore, the first two terms in the Lagrangian vanish as $|x| \to \infty$; the only dangerous term is the last term, and finite action requires only that $F_{\mu\nu} \to 0$ as $|x| \to \infty$. This condition permits a rescaling $A_\mu \to cA_\mu$, with c arbitrary, and under this rescaling $Q \to cQ$, so Q is not quantized.

On the other hand, in the Higgs theory, ϕ does not vanish at infinity, but rather approaches $a\mathrm{e}^{i\sigma(x)}$, for some phase factor $\mathrm{e}^{i\sigma(x)}$. In this case, the term $\int \mathrm{d}^2x D_\mu\phi^* D_\mu\phi$ in the action will converge only if $A_\mu = -(1/e)\,\partial_\mu\sigma$ at infinity. Only then will $D_\mu\phi$ vanish at infinity. This condition does not permit a rescaling of A_μ, and in fact, it is fairly well-known that in the Higgs theory, Q is quantized to take integer values.

One may also add charged fermions, with or without mass terms, to the Lagrangians (1) and (2). In this case also, Q is quantized in (2) but not in (1).

Now, let us ask: under what conditions is it reasonable to think about an "instanton gas"?

If the topological charge is quantized, it is reasonable to think in terms of a gas of lumps or instantons each carrying the minimum topological charge ± 1. However, if the charge is not quantized, we should envisage a smooth distribution of topological charge rather than a gas of discrete objects. In fact, although in the Higgs model (2) there is an instanton gas which plays an important role, the model (1) with unbroken gauge symmetry and unquantized topological charge is quite different; no one would try to describe this model in terms of a gas of instantons or other semiclassical objects.

Does QCD resemble more the second of these models, or the first?

At the classical level [2], QCD has a quantized topological charge, and thus seems to resemble more the Higgs theory (2). It is this that has motivated attempts to describe QCD in terms of an instanton gas. However, in this paper it will be claimed that because of quantum effects, the quantization of the topological charge, and with it the instanton gas, is an illusion in QCD; at the quantum level, QCD resembles more the theory (1) with unbroken gauge symmetry. As we will see, this will make possible a resolution of the conflicts noted in sect. 1 between instantons and the rest of physics.

It may be helpful to discuss further the similarities and differences between the two types of theories (1) and (2).

Certain aspects of instanton physics, which depend only on the existence of a topological charge and of the axial anomaly, are common to the two types of theory. These include:

Common properties (3)

(i) The existence of a vacuum angle θ;

(ii) the physics depends on θ if there are no massless fermions but not if there are massless fermions;

(iii) the resolution of the U(1) problem: there is no observable axial U(1) symmetry and no massless Goldstone boson;

(iv) the axial U(1) symmetry is broken down to a discrete k-fold symmetry, k being the number of Fermi flavors (more generally, k is the strength of the anomaly).

Because these properties are valid in both types of theory, we can expect that they will be valid in QCD regardless of which type of theory QCD turns out to be at the quantum level. They were discovered through the study of instantons, and their discovery has certainly been a significant advance in our understanding of the strong interactions.

It is fairly well-known that these properties are valid in theories like (2), but the fact that they are valid also in theories like (1) with unquantized topological charge and no instantons perhaps should be explained here. As discussed by Coleman [12], in theories like (1), θ corresponds to a background electric field, of strength $\theta e/2\pi$. (In both (1) and (2), θ can be interpreted as coming from a fractional electric charge at spatial infinity.) The Schwinger model is an example of a theory of type (1) with unquantized topological charge, and this example illustrates [12,13] that in these theories, exactly as in the Higgs theories which have instantons, the physics depends on θ if and only if all charged fermions have bare masses, and there is no observable U(1) symmetry. Finally, the last point, the existence of a discrete chiral symmetry, will be discussed in sect. 4.

There also are some general properties that distinguish between theories of the two types, (1) and (2).

For instance, in theories of type (2) with quantized topological charge and an instanton gas, the θ dependence of the physics and the η mass are non-perturbatively small; they cannot be seen in perturbation theory, because they are instanton effects. In models like (2) with an adjustable coupling constant, these effects are, for weak coupling, of order e^{-1/e^2}, the exponential of minus the one-instanton action. In QCD, it would not be quite right to say that such effects are of order e^{-1/g^2}, because the coupling constant g is, in view of the renormalization group, not really a free parameter. But QCD does have a free "coupling constant", the N of SU(N), and we can say that, based on the quantized charge-instanton gas picture, these effects should be of order e^{-N} for large N.

Thus, in theories of the second type, the θ dependence and the η mass are exponentially small and are invisible in perturbation theory. This is not so in theories of type (1). For instance, it is clear from the work of Coleman that in (1) a non-zero θ, corresponding to a background electric field, influences the spectrum even in the leading, perturbative approximation. From Coleman's work, it is clear that the θ dependence in this theory can be seen at the level of Feynman diagrams (for non-zero θ, there are extra diagrams that must be included, corresponding to interaction with the background field).

As for the η mass, it too can be seen in Feynman diagrams in models of type (1). For instance, the massless Schwinger model is the simplest model with unquantized topological charge and a U(1) problem that has to be solved. In this case, it is known [13] that the "massive photon" can be interpreted as the η; the η mass can be seen at the one-loop level (it comes from the one-loop vacuum polarization) and equals $e/\sqrt{\pi}$.

Additional contrasts between (1) and (2) involve the behavior when coupled to massless fermions. (The following comments are somewhat technical, and on first reading one might prefer to skip them.) In (2) the breaking by instantons of the axial U(1) symmetry can be represented [3,5] by an effective interaction that is very roughly $(\bar{\psi}\psi)^k$, k being the number of Fermi flavors. This arises from the following. The instanton number is quantized, the minimum instanton number being ± 1. A one-instanton configuration violates chirality by $2k$ units. $(\bar{\psi}\psi)^k$ is the lowest-dimension operator that changes the chirality by $2k$, and so it is the effective Lagrangian.

In contrast, the U(1) breaking in theories like (1) with unquantized topological charge can in no sense be described by an effective Lagrangian $(\bar{\psi}\psi)^k$.

An important consequence is that (2), but not (1), has a phase transition as a function of k. When k becomes large, the operator $(\bar{\psi}\psi)^k$ has large dimension, and eventually it becomes an irrelevant operator in the infrared. At that point, (2) has a phase transition. (In two dimensions, the transition is approximately at $k = 2$, since $(\bar{\psi}\psi)^2$ has canonical dimension two.)

More heuristically, the transition occurs because massless fermions in a background instanton field have zero-energy modes which tend to suppress the instanton gas. When the number of fermion flavors and hence of zero modes is too large, there is a phase transition to a phase in which the instanton gas is suppressed.

On the other hand, it will be argued in sect. 4 that (1) has no phase transition as a function of k. One might have guessed this from the fact that the $(\bar{\psi}\psi)^k$ interaction plays no role in this theory, or from an intuitive feeling that it would be much harder for the fermions to suppress a continuous distribution of topological charge than to suppress a gas of discrete objects, instantons.

The theory of this transition has been worked out in detail by Callan, Dashen and Gross [14], and by Raby and Ukawa [15]. In particular, the former authors showed that the phase transition can be interpreted in terms of the Z_k chiral symmetry that is present despite the anomaly. For small k, the Z_k symmetry is spontaneously broken; for large k, above the transition, it is restored.

In contrast, it will be argued in sect. 4 that in (1) the Z_k symmetry is spontaneously broken for all k.

It seems reasonable to expect that in theories like (2) there will always be a phase transition as a function of k, at the point at which the effective Lagrangian becomes an irrelevant operator. There is no reason to expect such a transition in models like (1) (and in (1) itself it will be argued that such a transition does not occur).

In summary, we can make a list of properties of instanton physics that discriminate between theories of the two types (1) and (2).

Properties that depend on quantized topological charge and an instanton gas (4)

(i) The η mass, the θ dependence of the physics, and the breaking of the U(1) symmetry are exponentially small, of order e^{-1/e^2} or e^{-N} according to the theory;

(ii) they cannot be seen by summing Feynman diagrams, not even the infinite number of Feynman diagrams of the $1/N$ expansion, because they are instanton effects;

(iii) the breaking of the U(1) symmetry can be described by an effective Lagrangian $(\bar{\psi}\psi)^k$;

(iv) there is a phase transition as a function of k; the number of Fermi flavors. (The last of these points may not be completely general.) These properties are valid in, and only in, theories of the second type (2).

At the classical level, Yang-Mills theory appears to be a theory of this type, with quantized topological charge, and therefore one might expect these properties to be valid in Yang-Mills theory. However, in this paper it will be argued that the predictions in list (4) are not valid in Yang-Mills theory at the quantum level, although those in list (3) are valid.

It is easy to see why the predictions in the second list (4) are more dangerous than those in the first list (3). Suppose, as a thought experiment, that in the Higgs theory (2), the one-loop corrections to the effective potential, or some other quantum effects, were to restore the U(1) symmetry, that is, to give an effective scalar field potential of the unbroken symmetry type. Then (2) would actually be, at the quantum level, a theory like (1); the instanton gas would evaporate; and the predictions in list (4) would be completely wrong.

Of course, we do not believe that this happens in the two-dimensional Abelian Higgs model. But I will claim that an analogous phenomenon does occur in four-dimensional QCD.

In sects. 3 and 4 I will demonstrate that such a thing is possible by consideration of a two-dimensional model. (Possible analogies between this model and QCD have also been studied recently by D'Adda, DiVecchia, and Lüscher [16]). The model is, like QCD, asymptotically free, and, at the classical level, it has no mass parameter. It possesses a global SU(N) symmetry, and there are instantons for all N. At the classical level the topological charge is quantized.

Moreover, the model is soluble in the $1/N$ expansion. When one solves it, one finds that the quantization of the topological charge disappears. The predictions in the second list above, which in perturbation theory seem quite reliable, are actually wrong. The η mass and the θ dependence of the physics are present in the leading $1/N$ approximation.

As stressed above, the η mass, the θ dependence, and the breaking of chiral U(1) down to a discrete symmetry, are, according to canonical lore, instanton effects. It should be impossible to see these effects by summing Feynman diagrams. However, in the model that we will discuss, we will be able to see these archetypal instanton effects by summing the bubble diagrams of the $1/N$ expansion.

Finally, in sects. 5 and 6 it will be argued that such a phenomenon occurs in four-dimensional QCD.

3. A two-dimensional model

The model that will be discussed here was introduced by Eichenherr [17], Golo and Perelomov [18], and Cremmer and Scherk [19]. The $1/N$ expansion in this model has been discussed, independently of the present work, by D'Adda, Di Vecchia and Luscher [16], and some of the results that will be described here, including some of the most interesting ones, have been presented by these authors (especially, the "dynamically generated" long-range force).

The model involves N complex fields n^i, $i = 1, ..., N$, satisfying a constraint $n_i^* n^i = 1$. In addition, we impose a local U(1) invariance $n^i(x) \to e^{ia(x)} n^i(x)$, for arbitrary space-time dependent $a(x)$. A Lagrangian with this invariance is

$$L = \partial_\mu n_i^* \partial_\mu n^i + (n_i^* \partial_\mu n^i)(n_j^* \partial_\mu n^j) . \tag{5}$$

Using the constraint $n_i^* n^i = 1$, one can check that this Lagrangian is indeed invariant under $n^i(x) \to e^{ia(x)} n^i(x)$.

The local U(1) invariance can be made more obvious by introducing an Abelian gauge field A_μ and writing

$$L = (\partial_\mu - iA_\mu)\, n_i^* (\partial_\mu + iA_\mu)\, n^i . \tag{6}$$

We introduce no kinetic energy for A_μ, which is therefore just a dummy field. To see that (6) is equivalent to (5), we write the Lagrangian in (6) explicitly as

$$\partial_\mu n_i^* \partial_\mu n^i - iA_\mu (n_i^* \overleftrightarrow{\partial}_\mu n^i) + A_\mu^2 n_i n^i .$$

Because $n_i^* n^i = 1$, this is

$$\partial_\mu n_i^* \partial_\mu n^i - iA_\mu (n_i^* \overleftrightarrow{\partial}_\mu n^i) + A_\mu^2 .$$

Since there are no derivatives of A_μ in the Lagrangian, A_μ can be eliminated explicitly by using its equations of motion, and one arrives at (5).

This theory has the obvious symmetry group SU(N), corresponding to rotations of the n^i. It will be referred to here as the SU(N) sigma model, in contrast to the usual O(N) sigma model. (Some authors have termed it the CP^{N-1} sigma model.)

For $N = 2$, the symmetry group SU(2) coincides with O(3), and there is a very nice way to rewrite this as an O(3) invariant theory. We introduce the vector $\boldsymbol{b} = n^* \boldsymbol{\sigma} n$ ($\boldsymbol{\sigma}$ are the usual two by two Pauli matrices). In terms of $\boldsymbol{b}$, the constraint $n_i^* n^i = 1$ becomes $\boldsymbol{b}^2 = 1$, and the Lagrangian (5) becomes

$$L = (\partial_\mu \boldsymbol{b})^2 . \tag{7}$$

This is just the Lagrangian for the usual O(3) non-linear sigma model. The fact the SU(N) sigma model coincides at $N = 2$ with the O(3) sigma model, which has been exactly solved [20], will provide some useful checks on our results below.

If one chooses, one can include a kinetic energy term for the gauge field in (6). This would not alter the main arguments and conclusions in this paper. The gauge field kinetic energy is omitted here because the theory is more beautiful in this case; for instance, only in this case is the theory scale invariant at the classical level.

(One might feel that the absence of a gauge field kinetic energy in (6), or the phase invariance in (5), could be associated with some pathology, but this is not so. The phase invariance in (5) means only that the parametrization in terms of the N fields n^i is redundant, the phase degree of freedom being superfluous. One can give a non-redundant parametrization – this is done in eq. (7) for the case $N = 2$ – and this makes it obvious that there is no pathology. However, the redundant parametrization in (5) and (6) is the most useful one for actually solving the model.)

Noting that a gauge field kinetic energy, if present, would have the form $(1/e^2)$ $F_{\mu\nu}^2$, we see that omitting it can be regarded as taking a limit $e^2 \to \infty$, just as the constraint $n_i^* n^i = 1$ can be regarded as a limit $\lambda \to \infty$ of a coupling $\lambda(n_i^* n^i - 1)^2$.

There is one more formal property of the model that we must discuss: the fact that classically, the theory has instantons.

In fact, the topological charge is

$$Q = \frac{1}{2\pi i} \int d^2x \partial_\mu (n_i^* \epsilon_{\mu\nu} \partial_\nu n^i) , \tag{8}$$

Q, being the integral of a total divergence, is obviously a topological charge. To see that Q, classically, takes only integer values, let us note the following. Finite action in (6) requires that $D_\mu n^i = (\partial_\mu + iA_\mu) n^i$ vanishes as $|x| \to \infty$ since the action is $|D_\mu n^i|^2$. This, in turn, requires $n^i = n_0^i e^{i\sigma(x)}$ as $|x| \to \infty$, where n_0^i is some constant and $e^{i\sigma(x)}$ is a phase factor, because $D_\mu n^i = 0$ means that n^i must be constant up to an overall phase factor. Q, being the integral of a total divergence, can be written as a surface integral, and using the fact that $n_{0i}^* n_0^i$ must equal one, we find

$$Q = \frac{1}{2\pi} \oint dx^\mu \frac{\partial \sigma}{\partial x^\mu} ,$$

where the integral is over a large circle at space-time infinity. The integral $\oint dx^\mu$ $(\partial\sigma/\partial x^\mu)$ is just $\Delta\sigma$, the change in σ on going around a big circle, so $Q = (1/2\pi)\, \Delta\sigma$.

If σ were a single-valued function, $\Delta\sigma$ would vanish. But σ is defined only by the relation $n^i = n_0^i e^{i\sigma}$, which actually determines σ only up to an additive multiple of 2π. It is perfectly possible that, when we go around a big circle, σ will change by an integer multiple of 2π. This, on the other hand, is the only possible change in σ. Thus, we conclude that Q, classically, takes integer values.

In fact, for any SU(N) group, instanton solutions exist for arbitrary integer values of Q. These solutions have been worked out in detail by Golo and Perelomov, and are very reminiscent of the instanton solutions of four-dimensional QCD.

Above we saw that the SU(N) sigma model for $N = 2$ is equivalent to the O(3) non-linear sigma model. It was precisely in the O(3) sigma model that Polyakov and Belavin [21] found instantons, and the topological charge Q defined above is, as one might guess, equivalent for SU(2) to the topological charge found for O(3) by those authors.

Polyakov and Belavin showed that the instantons of the O(3) σ model become unstable when O(3) is embedded in a larger O(N) group. This is related to the fact that the topological charge they found exists only in the case of O(3). By contrast, in the SU(N) model, because Q exists for every group and an object with non-zero Q is always topologically stable, the instantons of one group remain stable after embedding in a larger group. This resembles the situation in four-dimensional QCD.

The importance of the SU(N) model is that it is a generalization of the Polyakov-Belavin O(3) model which, classically, has instantons for all N and which can be solved for large N, to see what role the instantons play.

Like the O(3) sigma model and four-dimensional QCD, the SU(N) model has an inequality between the action and the topological charge. With the action defined as in (5), this inequality is $I \geqslant 2\pi\,|Q|$. To derive it, let

$$C_\mu{}^i = \partial_\mu n^i - n^i(n_j^* \partial_\mu n^j)\,.$$

Then

$$|C_\mu{}^i \pm i\epsilon_{\mu\nu} C_\nu{}^i|^2 \geqslant 0\,,$$

which gives

$$|\partial_\mu n^i|^2 + (n_i^* \partial_\mu n^i)^2 \geqslant \pm\, i\epsilon_{\mu\nu}\partial_\mu(n_i^* \partial_\nu n^i)\,.$$

After integration over x, this gives $I \geqslant 2\pi\,|Q|$.

However, so far we have omitted the coupling constant. In defining the quantum theory, we wish to include the coupling constant as a multiplicative factor in front of the action; we will take this coupling constant to be N/g^2, where the factor of N is needed so as to have a smooth limit for large N. With the action normalized in the form appropriate for the quantum theory, our inequality can be stated as

$$I \geqslant \frac{2\pi N}{g^2}\,|Q|\,. \tag{9}$$

Now we must turn to actually solving this theory in the $1/N$ expansion. We will see that because of quantum corrections, the instanton gas disappears. We will also see that the θ dependence of the physics is present in the leading large N approximation, despite the fact that if it could be understood in terms of instantons, the θ dependence would be of order e^{-N} (since in (9) we see that the action in sectors of non-zero Q is of order N).

The $1/N$ expansion for this theory can be treated by standard methods. The constraint $n_i^* n^i = 1$ can be incorporated by introducing a Lagrange multiplier field λ with a term in the Lagrangian $\lambda(n_i^* n^i - 1)$. Since we wish to discuss the θ dependence, we must include also a term proportional to the topological charge density. We could simply add a term

$$\frac{\theta}{2\pi i}\,\partial_\mu(n_i^* \epsilon_{\mu\nu}\partial_\nu n^i)$$

to the Lagrangian to represent the θ dependence. However, it is equivalent to write $(\theta/2\pi)\, \epsilon_{\mu\nu}\partial_\mu A_\nu$, where A_ν is the auxiliary gauge field that appears in (6) (the two forms for introducing θ are equivalent because after eliminating θ by means of its equations of motion, they yield the same result).

In sum, we are led to consider the following version of the action:

$$I = \int d^2x \left[\frac{N}{g^2} (\partial_\mu - iA_\mu)\, n_i^* (\partial_\mu + iA_\mu)\, n^i - \lambda(n_i^* n^i - 1) + \frac{\theta}{2\pi} \epsilon_{\mu\nu}\partial_\mu A_\nu \right] . \tag{10}$$

We will solve this theory by means of a (Minkowski space) path integral,

$$Z = \int dn\, dn^*\, d\lambda\, dA_\mu e^{iI} . \tag{11}$$

We first rescale n and n^* and carry out the Gaussian integration over them, yielding

$$Z = \int d\lambda\, dA_\mu \exp\left[-N \operatorname{Tr} \ln\left(-(\partial_\mu + iA_\mu)^2 - \frac{\lambda g^2}{N} \right) + i \int d^2x \lambda + \frac{i\theta}{2\pi} \int d^2x\, \epsilon_{\mu\nu}\partial_\mu A_\nu \right] . \tag{12}$$

The integration over λ and A_μ cannot be done exactly, so we must consider a stationary phase approximation. Lorentz invariance suggests that we look for a stationary point with $A_\mu = 0$, λ = constant.

To actually determine the stationary point, we vary with respect to the constant value of λ. The resulting equation is

$$i + g^2 \int \frac{d^2k}{4\pi^2} \frac{1}{k^2 - (g^2\lambda/N) + i\epsilon} = 0 . \tag{13}$$

(The same equation arises in the non-linear sigma model.) The integral in (13) is ultraviolet divergent, so one must introduce a cutoff Λ. The integral can be done explicitly, and the answer turns out to be

$$\frac{g^2\lambda}{N} = \Lambda^2\, e^{-4\pi/g^2} . \tag{14}$$

(The "g" in this equation is an unrenormalized coupling constant. If one wishes, the right-hand side of (14) can be rewritten in terms of a scale parameter μ and a renormalized coupling constant g_R as $\mu^2 \exp(-4\pi/g_R^2)$.)

For our purposes, the important feature of (14) is that the stationary point value of λ is non-zero and positive. Looking back to the original Lagrangian (10), one sees that a positive vacuum expectation value of λ is a mass of the n and n* particles, which we will call M.

We must now go on to consider fluctuations of λ and A_μ around their vacuum expectation values. One can easily check that, as in the $1/N$ expansion in similar models [22], terms in the expansion of the effective action (12) around the stationary point which are cubic or higher order in λ and A_μ are suppressed by powers of $1/\sqrt{N}$. The stationary point has, of course, been chosen so that the linear terms vanish, so that we need only consider the quadratic terms in an expansion of (12) aroun around the stationary point.

It turns out that the term quadratic in λ is unimportant for the questions of interest in this paper. (It leads to a weak, short-range interaction among the n and n* particles, which has no qualitative effect on the discussion.) Therefore, we will simply replace $\lambda g^2/N$ in (12) by its vacuum expectation value, the physical mass squared M^2 of the n and n* particles.

Also, the λ-A_μ mixing term turns out to vanish. Therefore, we need only consider the terms quadratic in A_μ.

It is well-known that the expansion of functional determinants like the one in (12) can be understood in terms of Feynman diagrams. In this case, the relevant diagrams (fig. 1) are the one-loop vacuum polarization due to a massive charged scalar (the mass coming from the vacuum expectation value of λ).

Although these diagrams can, of course, be calculated explicitly, the main points can be understood without calculation. By gauge invariance the answer is $(-g_{\mu\nu}k^2 + k_\mu k_\nu)$ times some $f(k^2)$. There is an overall factor of N because there are N particles n^i in the diagrams of fig. 1, and $f(k^2)$ is non-singular at $k^2 = 0$ because these particles have mass. Thus, the answer has the form

$$N(-g_{\mu\nu}k^2 + k_\mu k_\nu)(c + O(k^2)) , \tag{15}$$

where c is a constant. The infrared behavior and particle structure are determined by the small k behavior, and for the questions we wish to discuss we may ignore the $O(k^2)$ corrections. So, only the numerical value of c has to be computed. It turns out $c = 1/12\pi M^2$, M being the n and n* mass.

Now we realize that $(-g_{\mu\nu}k^2 + k_\mu k_\nu)$ is simply the usual gauge field kinetic energy $-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ written in momentum space. So, the one-loop corrections have generated a kinetic energy for A_μ, even though A_μ was first introduced as a dummy field.

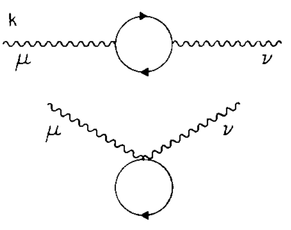

Fig. 1. The one-loop vacuum polarization due to a massive, charged scalar.

In summary, there are two important quantum effects in the large N limit. A mass is generated for the scalar particles, and a kinetic energy is generated for the gauge field.

Although we could proceed with a further systematic analysis of the exact formula (12), it is much simpler at this point to write an effective Lagrangian which includes the two important quantum effects and which describes well the physics for large N. The effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = (\partial_\mu - iA_\mu)\, n_i^* (\partial_\mu + iA_\mu)\, n^i - M^2 n_i^* n^i$$

$$- \frac{N}{48\pi M^2} F_{\mu\nu}^2 + \frac{\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu \,. \qquad (16)$$

However, it is useful to rescale the gauge field so as to give the standard normalization to the kinetic energy. After a rescaling

$$\sqrt{\frac{N}{12\pi M^2}}\, A_\mu \to A_\mu \,,$$

we have

$$\mathcal{L}_{\text{eff}} = \left(\partial_\mu - iA_\mu M \sqrt{\frac{12\pi}{N}}\right) n_i^* \left(\partial_\mu + iA_\mu\, M \sqrt{\frac{12\pi}{N}}\right) n^i$$

$$- M^2 n_i^* n^i - \tfrac{1}{4} F_{\mu\nu}^2$$

$$+ \frac{\theta}{2\pi} M \sqrt{\frac{12\pi}{N}}\, \epsilon_{\mu\nu} \partial_\mu A_\nu \,. \qquad (17)$$

The first thing to note about (17) is that the n and n* particles have, effectively, charges $\sqrt{12\pi/N}\, M$ (recall that in two space-time dimensions the electric charge has dimensions of mass). They are thus weakly charged for large N, and superficially one would think that for large N the gauge coupling would be a small effect.

But here we encounter a surprise. In one space dimension the Coulomb potential is a linear potential. Even if the coefficient is small, a linear potential has a dramatic effect – it confines the charges.

Referring henceforth to the n and n* particles of this theory as "quarks" and "antiquarks," we see that between a quark at x and an antiquark at y there is a linear potential,

$$V(x, y) = (12\pi M^2/N)|x - y| \,.$$

This "dynamically generated" confining potential is certainly a surprising result. However, for the purposes of this paper, we are more interested in the following questions: what role, if any, do instantons play in this theory? And can the θ dependence of the physics be seen in the $1/N$ expansion, or is it of order e^{-N}, as instanton lore indicates?

What is the role played by instantons? The effective field theory (17) has no instantons. It is a theory of unbroken U(1) gauge symmetry, and like other such theories, (17) has no instantons. In fact, apart from the fact that (17) has N charged scalar fields instead of one, (17) coincides with the model (1) discussed in sect. 2 as an archetype of a model with unbroken U(1) symmetry and no instantons.

Although the instantons have disappeared, it is easy to see that the physics is θ dependent; the θ dependence corresponds, as in (1), to a possible background electric field. Moreover, in contradiction to instanton lore, the θ dependence is present in perturbation theory in $1/N$. This can be seen by analogy with Coleman's argument that in (1), the θ dependence is present in perturbation theory in e.

The situation is particularly simple in the gauge $A_1 = 0$. As Coleman showed, in the two-particle (quark-antiquark) sector, one can incorporate the θ dependence by using a modified photon propagator

$$D(x, t; y, t') = \delta(t - t')\left(|x - y| + \frac{\theta}{2\pi}(x - y)\right). \tag{18}$$

The θ-dependent term corresponds to the effects of an electric field of strength $\theta/2\pi$ times the electric charge $e = 12\pi M^2/N$. Obviously, since the effective photon propagator is θ dependent, all physical quantities will have a non-trivial θ dependence in perturbation theory.

It is especially interesting to consider the mass spectrum. The finite-energy states will be electrically neutral bound states of an n "quark" and an n* "antiquark." Since the n and n^* transform as N and $\bar{N}$ under SU(N), the bound states will transform according to the singlet and adjoint representations of SU(N).

For large N, these bound states become non-relativistic and can be described by a Schrödinger equation. In this Schrödinger equation we must use a θ-dependent potential

$$V(x, y) = \frac{12\pi M^2}{N}\left(|x - y| + \frac{\theta}{2\pi}(x - y)\right), \tag{19}$$

where the θ dependence corresponds to the θ dependence in (18), and represents the interaction between the dipole moment of the quark-antiquark pair and the background electric field.

The non-relativistic Hamiltonian for the quark-antiquark system, including the quark masses, is then

$$H = 2M + \frac{1}{2M}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right) + \frac{12\pi M^2}{N}\left(|x - y| + \frac{\theta}{2\pi}(x - y)\right). \tag{20}$$

The N dependence of the bound-state masses can easily be found by a rescaling. Let $X = N^{1/3}Mx$, $Y = N^{1/3}My$. Then, $H = 2M + N^{-2/3}MH'$, where

$$H' = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}X^2} - \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}Y^2} + \left(|X - Y| + \frac{\theta}{2\pi}(X - Y)\right). \tag{21}$$

This makes it clear that the mass of the kth bound state has the form $M_k = M(2 + N^{-2/3} f_k(\theta))$ where the $f_k(\theta)$ are non-trivial functions of θ that could be found by solving the Schrödinger equation (21).

In particular, the θ dependence of the masses appears in order $N^{-2/3}$, while the θ dependence of the mass splittings is present in leading order.

The Schrödinger equation (21) is a good approximation to all of the states in which the motion is non-relativistic. For large N, the number of such states is of order $N^{2/3}$.

In short, the θ dependence of this theory can be calculated in perturbation theory in $1/N$, without considering instantons. It is very unlikely that any method of calculation based on instantons would give the correct answer for the θ dependence, since the actual θ dependence comes from a certain sum of Feynman diagrams, the Feynman diagrams of the $1/N$ expansion.

More generally, the instanton gas in this theory has disappeared. The effective action (17) was derived without considering instantons, by summing the bubble diagrams of the $1/N$ expansion, and (17), as mentioned above, is a theory of a type that has no instantons.

Some suggestions as to why the instanton gas in this model has disappeared will be made in sect. 6, along with some suggestions as to why this might happen in four-dimensional QCD.

Before concluding this section, I should mention an interesting check on the validity of the $1/N$ expansion in this model. This check suggests that the $1/N$ expansion is qualitatively correct not just for large N but also for the smallest physical value $N = 2$.

If there were no confinement, the physical states would transform as N and $\bar{N}$ under SU(N). However, the $1/N$ expansion predicts confinement, as a result of which the physical particles are singlets or in the adjoint representation.

For SU(2), without confinement, the spectrum would consist of doublets. With confinement, the possible states are singlets and triplets. However, as noted above, the SU(2) model is equivalent to the O(3) sigma model, and thus the exact spectrum is known [20]. It consists of one triplet; there are no doublets. This indicates that the confinement, found above for large N, persists down to $N = 2$.

Perhaps the absence of singlets at $N = 2$, and the fact that there is only one triplet, require comment. For large N the confining potential is shallow and the number of stable states large, of order $N^{2/3}$. Moreover, the adjoint and singlet states are degenerate in this limit. As N is reduced, the potential becomes steeper, the number of stable states becomes less, and the degeneracy is lifted. It is perfectly plausible that by the time we reach $N = 2$, the singlets have all become unstable, and only one triplet remains.

We will see in sect. 4 that also with massless fermions included, the $1/N$ expansion seems to be qualitatively correct down to $N = 2$.

We also will see that in the presence of massless fermions, the confinement mechanism found in this section disappears. This is one reason for doubting that this mechanism is relevant to four dimensions. It is only the mechanism for the disappearance of the instanton gas that I will argue is relevant to four dimensions.

4. Incorporating fermions

4.1. Some general properties of fermions

In sect. 3 we have seen that the SU(N) sigma model, although apparently a model like (2) with instantons, is actually, at the quantum level, a model similar to (1). Before discussing the coupling of the SU(N) sigma model to fermions, it is useful to first discuss the general properties of models like (1) when coupled to massless fermions.

Thus, let us include in (1) a charged Fermi field:

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\,\phi^*(\partial_\mu + ieA_\mu)\,\phi - M^2\phi^*\phi$$
$$- \tfrac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(i\partial\!\!\!/ - ke A\!\!\!/)\,\psi - m\,\bar{\psi}\psi$$
$$+ \frac{\theta e}{2\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu \,. \tag{22}$$

We have included a fermion of charge ke. In principle we would like to study the behavior as a function of the number of Fermi flavors. However, in two space-time dimensions it is exactly equivalent to introduce one Fermi flavor of charge ke or k flavors each of charge e; the equivalence of the two can easily be seen by bosonization. It is slightly more convenient to discuss (22).

We wish to see that (22) has the following properties:

(i) The physics depends on θ if and only if $m \neq 0$; if $m = 0$, the resolution of the U(1) problem (that is, the absence of a physical U(1) symmetry) can be seen in perturbation theory in e;

(ii) There is then a residual k-fold chiral symmetry, which is spontaneously broken for any k. Later we will see the same properties in the SU(N) sigma model.

Except for the comments about the discrete chiral symmetry, the discussion that follows is not novel. It is included for completeness.

The easiest way to analyze (22) is by bosonization of fermions. We introduce, in the standard way [23], a new canonical boson field σ satisfying

$$\bar{\psi} i\partial\!\!\!/\,\psi = \tfrac{1}{2}(\partial_\mu\sigma)^2 \,,$$
$$\bar{\psi}\gamma^\mu\psi = \frac{1}{\sqrt{\pi}}\,\epsilon^{\mu\nu}\partial_\nu\sigma \,,$$
$$\bar{\psi}\psi = \cos\sqrt{4\pi}\,\sigma \,. \tag{23}$$

With these substitutions, and after one integration by parts, we can write a Lagrangian equivalent to (22).

$$\mathcal{L} = (\partial_\mu - ie\,A_\mu)\,\phi^*(\partial_\mu + ie\,A_\mu)\,\phi - M^2\phi^*\phi$$

$$- \tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - m\cos\sqrt{4\pi}\,\sigma$$
$$+\frac{ke}{\sqrt{\pi}}\sigma\epsilon^{\mu\nu}\partial_\mu A_\nu + \frac{\theta e}{2\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu \tag{24}$$

The first point that we must check is now that the physics depends on θ if and only if $m \neq 0$.

In fact, if $m = 0$, θ can be eliminated explicitly by the substitution $\sigma \to \sigma - \theta/2k\sqrt{\pi}$. Thus, for $m = 0$, the physics is θ independent.

But for $m \neq 0$, θ cannot be eliminated. In this case, as discussed by Coleman and reveiwed in sect. 3, the Feynman diagrams will be explicitly θ dependent, so the physics will depend on θ.

Next, we must see that for $m = 0$, the U(1) problem is solved; there is no observable U(1) symmetry and of course, this being two space-time dimensions, no Goldstone boson.

The Lagrangian (22) has for $m = 0$ a naive chiral symmetry $\psi \to e^{i\beta\gamma_5}\psi$. But it is known that at the quantum level, the chiral current has an anomalous divergence,

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = \frac{ke}{\pi}\,\epsilon^{\mu\nu}\partial_\mu A_\nu\,.$$

In terms of σ, chiral symmetry is $\sigma \to \sigma + \beta/\sqrt{\pi}$, and the chiral current is $(1/\sqrt{\pi})$ $\partial_\mu\sigma$,

The term $(ke/\sqrt{\pi})\,\sigma\,\epsilon^{\mu\nu}\partial_\mu A_\nu$ is present in (24) because of the axial anomaly. Without this term the free field equation $\partial_\mu(\partial^\mu\sigma) = 0$ would tell us that the chiral current $(1/\sqrt{\pi})\,\partial_\mu\sigma$ was conserved. With this term present, the equation of motion for σ is

$$\partial_\mu\left(\frac{1}{\sqrt{\pi}}\,\partial_\mu\sigma\right) = \frac{ke}{\pi}\,\epsilon^{\mu\nu}\partial_\mu A_\nu\,.$$

The right-hand side of this equation is the anomalous divergence of the axial current. Note that the anomaly becomes a canonical equation when (22) is rewritten in terms of (24).

Without the term $(ke/\sqrt{\pi})\,\sigma\epsilon^{\mu\nu}\partial_\mu A_\nu$ in (24) the σ particle would be massless. It is easy to see, however, that this term gives σ a mass. In fact, by diagonalizing the σ^2, F^2, and σ-A_μ mixing terms in (24) to find the σ and A_μ propagators, one finds that σ acquires a mass $ke/\sqrt{\pi}$. Thus, the mixing, which is a manifestation of the axial anomaly, gives a mass to the would-be massless σ particle. This is an analogue to the acquisition of a mass by the would-be massless η in four dimensions.

At the same time, there is no physical chiral symmetry in (24). Indeed, the fact that the σ particle has a mass means that there is no physical chiral symmetry. For we have noted that chiral symmetry is $\sigma \to \sigma + c$, and the Green functions of a massive field are not invariant under any such symmetry.

These two facts, the absence of a physical U(1) symmetry and the absence of a massless particle (which, in two dimensions, could not in any case have been a Goldstone boson), represent the resolution of the U(1) problem.

Some additional comments about the resolution of this problem are appropriate. In models of this kind, the mass of the η can be regarded as coming from the mixing of an isoscalar $\bar{\psi}\psi$ state with the A_μ field. In fact, we have noted that the mass comes from σ mixing with A_μ, and σ, in turn, as we see from its definition $\partial_\mu \sigma = \sqrt{\pi}\epsilon_{\mu\nu}\bar{\psi}\gamma_\nu\psi$, is a $\bar{\psi}\psi$ composite field.

It is also useful to think about the η directly in the ψ representation, before bosonization. In this language the η mass comes from the one-loop vacuum polarization of the ψ field. In fact, the photon propagator with the one-loop vacuum polarization included contains in this model, as in the Schwinger model, a massive particle pole at a mass $ke/\sqrt{\pi}$.

This massive particle, the "η," can also be considered as a $\psi\bar{\psi}$ state with the mass coming from the lowest-order annihilation diagram. The η appears, *via* diagrams of the sort shown in fig. 2, as a massive pole in the two-point function of $\bar{\psi}\gamma^\mu\psi$. But the diagrams of fig. 2 describe the propagation of a quark-antiquark pair with repeated pair annihilation and recreation *via* the Coulomb force.

In the absence of annihilation diagrams, only the first diagram of fig. 2 would contribute. This diagram is known from the theory of the Schwinger model to have a pole at $p^2 = 0$. The pole is shifted by the annihilation diagrams to a non-zero mass, $ke/\sqrt{\pi}$.

In particular, suppose that one includes in this model several flavors of fermions, so that there is a non-trivial flavor symmetry. Then in the flavor non-singlet channel (the two-point function of non-singlet currents) only the first diagram of fig. 2 contributes, and there is a massless pole, but the singlet channel has an extra contribution from annihilation diagrams and has no massless pole.

Thus in models of this type, the simple quark model picture of ref. [8], according to which the η mass comes from the annihilation diagrams which are possible in and only in the singlet channel, is perfectly valid. I have belabored here this point, which

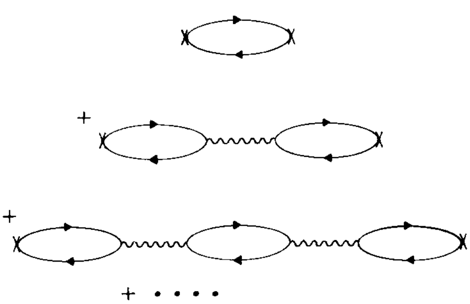

Fig. 2. The quark-antiquark annihilation diagrams which give a mass to the η in two-dimensional models which do not have instantons. In the diagrams, × represents an insertion of a current operator.

comes essentially from ref. [13], because we will see below that, in contradiction to instanton lore, the SU(N) sigma model behaves in the same way.

To conclude this discussion, we must see that there is, despite the anomaly, a discrete k-fold chiral symmetry in (22) or (24), and that it is spontaneously broken.

In general, it is not easy to decide by studying the particle structure and operator matrix elements in a theory whether a discrete symmetry exists and is spontaneously broken, or whether it does not exist. When a continuous symmetry is spontaneously broken, there is a Goldstone boson, but there is, in general, no such universal signal for a spontaneously broken discrete symmetry.

However, in one space dimension, a spontaneously broken discrete symmetry always leaves a trace: the existence of solitons. In other words, when a discrete symmetry is spontaneously broken, there are always solitons, finite-energy states interpolating between two different vacua.

We wish to show that (22) or (24) has a k-fold discrete chiral symmetry, $\psi \to e^{2\pi i\gamma_5/k}\psi$ or $\sigma \to \sigma + (1/k)\sqrt{\pi}$. We will do this by looking for soliton states, that is, states in which the vacuum expectation value of the σ field (or of $\bar{\psi}\psi$) is different at $x = -\infty$ from its value at $x = +\infty$.

First, we must discuss what are the physical states in (22). At this point, we encounter a rather surprising, and dramatic, effect of the massless fermions. Without massless fermions, (22) is a confining theory, and has no physical ϕ or ϕ^* particles, but only neutral bound states. With massless fermions, however, the confinement is lost, and there are free ϕ and ϕ^* particles.

The reason for this is that the σ-A_μ mixing which gives the η a mass also, as we have said, modifies the A_μ propagator from having a pole at $p^2 = 0$ to having a pole at $p^2 = k^2e^2/\pi$. Thus, the Coulomb force, which corresponds to a pole at $p^2 = 0$, has disappeared, being replaced by a Yukawa force. With only Yukawa forces between them, the ϕ and ϕ^* particles are unconfined.

Perhaps surprisingly, it turns out that the unconfined ϕ and ϕ^* particles are the solitons corresponding to a spontaneously broken discrete chiral symmetry.

To see this, we write down Gauss' law, the equation of motion of the gauge field. From (24), we see that Gauss' law is

$$\frac{\mathrm{d}}{\mathrm{d}x}E + \frac{ke}{\sqrt{\pi}}\frac{\mathrm{d}\sigma}{\mathrm{d}x} = J_0 , \tag{25}$$

where $E = F_{01} = \partial_0 A_1 - \partial_1 A_0$ is the electric field, and $J_0 = i\phi^* \overleftrightarrow{D}_0 \phi$ is the charge density of the ϕ field.

Integrating this equation from $x = -\infty$ to $x = +\infty$, we find

$$E(\infty) - E(-\infty) + \frac{ke}{\sqrt{\pi}}(\sigma(\infty) - \sigma(-\infty)) = eQ , \tag{26}$$

where $Q = \int \mathrm{d}x\, J_0(x)$ is the particle number of the ϕ field.

However, $E(\infty) = E(-\infty) = 0$ for all states. For, as just mentioned, the $A_\mu - \sigma$

mixing has screened the Coulomb potential and given the photon a mass; with a massive photon and screened Coulomb potential, the electric field must vanish at infinity.

Setting $E(\infty) = E(-\infty) = 0$ in (26), we get

$$\sigma(\infty) - \sigma(-\infty) = \frac{\sqrt{\pi}}{k} Q \,. \tag{27}$$

This equation tells us that all states of non-zero Q are solitons in the sense of interpolating between two values of the σ field.

The ϕ and ϕ^* particles have $Q = \pm 1$, and these are the minimum non-zero values of Q. Thus, the minimum change in σ in passing from $x = -\infty$ to $x = +\infty$ is $\sqrt{\pi}/k$. This tells us that we are dealing with a spontaneously broken discrete symmetry, $\sigma \to \sigma + \sqrt{\pi}/k$.

The last point that must be established is that in theories of this kind, unlike theories with instantons, the discrete chiral symmetry is spontaneously broken even when k is large. In fact, as k increases, the photon mass becomes larger and larger; the Coulomb force is more and more strongly screened. As a result, the interactions become weaker and weaker, and perturbation theory becomes better and better. But perturbation theory, as we have seen, indicates that the discrete symmetry is spontaneously broken.

For most purposes, it is not necessary or useful to regard the ϕ and ϕ^* particles in this theory as solitons. However, if one wants to know whether models like this have a discrete chiral symmetry of the sort discussed in ref. [6], the answer is that they do; the symmetry is spontaneously broken, and ϕ and ϕ^* are solitons.

4.2. A simple coupling to fermions

We will consider two ways of coupling the SU(N) sigma model to massless fermions. In this section we consider a simple model that illustrates the main points, and in the next section we consider a supersymmetric model.

The simplest way to include massless fermions is to introduce an SU(N) singlet fermion ψ of charge k. The Lagrangian, including all auxiliary fields, is

$$\begin{aligned} \mathcal{L} &= (\partial_\mu - iA_\mu)\, n_i^* (\partial_\mu + iA_\mu)\, n^i - \lambda(n_i^* n^i - 1) \\ &\quad + \bar{\psi}(i\partial\!\!\!/ - kA\!\!\!/)\,\psi + \frac{\theta}{2\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu \,. \end{aligned} \tag{28}$$

The model has a naive chiral symmetry, $\psi \to \mathrm{e}^{i\beta\gamma_5}\psi$, but the corresponding chiral current has, as in the model in subsect. 4.1. an anomalous divergence,

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = \frac{k}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu = \frac{k}{\pi}\partial_\mu(n_i^*\epsilon_{\mu\nu}\partial_\nu n^i) \,.$$

Because of this anomaly and the fact that $\int \mathrm{d}^2x\,\epsilon^{\mu\nu}\partial_\mu A_\nu$ is not necessarily zero, the

U(1) problem will be resolved; there will be no physical U(1) symmetry and no massless particle.

However, we want to see whether instanton reasoning describes these effects correctly.

According to instanton lore, the resolution of the U(1) problem is an instanton effect, because $\int d^2x \epsilon^{\mu\nu}\partial_\mu A_\nu$ is non-zero only for instantons. In particular, the resolution of this problem should be invisible in Feynman diagrams, and in the $1/N$ expansion.

We will, on the contrary, see that the resolution of the U(1) problem, together with all its characteristic features (the η mass, the absence of a continuous chiral symmetry, the existence of a spontaneously broken discrete chiral symmetry) can be seen by summing the bubble diagrams of the $1/N$ expansion.

Also, we will see that it is not the instanton picture, but the quark-model picture of the η that is valid in this model: its mass comes from one-photon annihilation diagrams. (There is a kinematical difference between one and three dimensions. In one dimension, one-photon annihilation is possible for a pseudoscalar, while in three dimensions, two photons are required. Thus, a simple quark model in one dimension would predict that the η mass comes from one-photon annihilation. As is known, there are no true photon degrees of freedom in one space dimension; by "one-photon annihilation" I mean annihilation diagrams involving one gauge field line that are available only for flavor singlet states.)

Actually, it is very easy to verify these statements. We can write down by inspection an effective Lagrangian for (28) that is analogous to (17). The steps that led to (17), integration over n and n^* and approximate evaluation of the resulting functional determinant, can be carried out in the same way for (28) and are unaffected by the presence of the term in the action involving the fermions. Therefore, the effective Lagrangian for (28), analogous to (17), is the sum of (28) plus the fermion action $\bar{\psi} i \not{D} \psi$:

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} = {} & \left(\partial_\mu - iA_\mu M\sqrt{\frac{12\pi}{N}}\right) n_i^* \left(\partial_\mu + iA_\mu M\sqrt{\frac{12\pi}{N}}\right) n^i \\
& - M^2 n_i^* n^i - \tfrac{1}{4} F_{\mu\nu}^2 \\
& + \bar{\psi}\left(i\not{\partial} - kM\sqrt{\frac{12\pi}{N}}\not{A}\right)\psi \\
& + \frac{\theta M}{2\pi}\sqrt{\frac{12\pi}{N}}\,\epsilon^{\mu\nu}\partial_\mu A_\nu\,.
\end{aligned} \tag{29}
$$

(The reason for the factor $M\sqrt{12\pi/N}$ in the $\bar{\psi}\not{A}\psi$ term is that A_μ was rescaled by this factor in arriving at (17).)

We already know the main properties of (29), because (29) coincides with the model (22) discussed above, apart from the inessential fact of having N scalar fields instead of one. Moreover, perturbation theory in $1/N$ in (29) just corresponds to perturbation theory in e^2 in (22).

Thus we immediately know the following about (29): the physics is θ independent; there is no continuous U(1) symmetry, and only a k-fold discrete symmetry which is spontaneously broken for any k; and the simple quark-model picture of the η, based on annihilation diagrams, is correct.

Furthermore, these effects are visible in perturbation theory in $1/N$, and in particular $m_\eta^2 = 12\, k^2M^2/N$, since m_η^2 was e^2/π in (22), and here e is $\sqrt{12\pi k^2 M^2/N}$.

These conclusions depend in no way on the use of an effective Lagrangian (29), which is here introduced only to minimize labor. One reaches the same conclusions, with only slightly more work, by calculating the exact Green functions in an expansion in $1/N$, starting with exact formulas like (12).

4.3. Supersymmetric coupling to fermions

The simple model of subsect. 4.2 shows all of the essential properties of the SU(N) sigma model when coupled to massless fermions. However, there is another coupling to massless fermions that is interesting to study for its elegance and also because of various special checks that are possible on the calculation. This is the supersymmetric form of the SU(N) sigma model, which was introduced by Cremmer and Scherk [19],

This model contains, in addition to n^i, an N-component Dirac Fermi field ψ^i, $i = 1 \dots N$, which is constrained to satisfy $\Sigma_i n_i^* \psi^i = 0$. One way to write the supersymmetric Lagrangian is the following:

$$\begin{aligned} \mathcal{L} = \frac{N}{g^2} \, [&(\partial_\mu - iA_\mu)\, n_i^* (\partial_\mu + iA_\mu)\, n^i + \overline{\psi}_i (i\partial\!\!\!/ - A\!\!\!/)\, \psi^i \\ &- \tfrac{1}{2}(\sigma^2 + \pi^2) - \sqrt{\tfrac{1}{2}}\overline{\psi}(\sigma + i\pi\gamma_5)\psi \\ &- \lambda(n_i^* n^i - 1) \\ &+ \overline{\chi} n_i^* \psi^i + \overline{\psi}_i n^i \chi] \; . \end{aligned} \tag{30}$$

Here χ and $\overline{\chi}$ are Lagrange multiplier fields that enforce the constraint $n_i^* \psi^i = 0$, λ is a Lagrange multiplier that enforces $n_i^* n^i = 1$, σ and π are auxiliary fields which can, if one wishes, be eliminated, leading to four Fermi interactions, and A_μ is an auxiliary gauge field.

The conserved supersymmetry current is

$$J_\mu = D_\alpha n_i^* \gamma^\alpha \gamma_\mu \psi^i \; . \tag{31}$$

The corresponding supercharge is a *complex* Dirac spinor Q_α which satisfies, in perturbation theory, the algebra

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0 \,,$$

$$\{Q_\alpha, \overline{Q}_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu \,. \tag{32}$$

We will see, though, that because of certain non-perturbative effects, to be discussed, this algebra is modified.

This model has a naive chiral symmetry

$$\psi \to e^{i\beta\gamma_5}\psi \,, \qquad (\sigma + i\pi) \to e^{-2i\beta}(\sigma + i\pi) \,.$$

The chiral current, however, has an anomaly proportional to the instanton density $\partial_\mu(n_i^* \epsilon_{\mu\nu}\partial_\nu n^i)$, or $\epsilon^{\mu\nu}\partial_\mu A_\nu$. In contrast to what one might expect from instanton lore, we will see that the U(1) problem is resolved within the $1/N$ expansion.

Specifically, we will find in the $1/N$ expansion that σ acquires a non-zero vacuum expectation value. This explicitly breaks the U(1) symmetry; there is, nonetheless, no massless particle. The π, which is the would-be Goldstone boson, turns out, far from being massless, to be degenerate with the σ.

After rescaling n and ψ and integrating over them, one obtains from (30) the following effective action for the integrations over the other fields:

$$\begin{aligned} I_{\text{eff}} = &-N \operatorname{Tr} \ln(-(\partial_\mu + iA_\mu)^2 - \lambda) \\ &+ N \operatorname{Tr} \ln(i\partial\!\!\!/ - A\!\!\!/ - \sqrt{\tfrac{1}{2}}\,(\sigma + i\pi\gamma_5)) \\ &- \frac{iN}{2g^2}\int d^2x\,(\sigma^2 + \pi^2) + \frac{iN}{g^2}\int d^2x\,\lambda \\ &+ \text{terms involving } \chi \text{ and } \chi^* \,. \end{aligned} \tag{33}$$

The terms involving χ and χ^* have been omitted because the arguments to follow do not depend on them.

We now must find a stationary point around which to expand (33), in order to do the integrations over the auxiliary fields. As in sect. 3, the stationary point value of λ turns out to be non-zero, and in fact it is the same value found in sect. 3 (except that λ is normalized differently here). Likewise, exactly as discussed by Gross and Neveu for the SU(N) Thirring model, (33) must be expanded around a non-zero value of σ (or π). Making an arbitrary choice that π has zero vacuum expectation value, one finds the stationary point

$$\lambda = \tfrac{1}{2}\sigma^2 = \Lambda^2 \exp(-4\pi/g^2) \,,$$

$$\pi = 0 \,. \tag{34}$$

From the vacuum expectation values of λ and σ, the bosons n^i get a mass squared λ, and the fermions ψ^i get a mass squared $\frac{1}{2}\sigma^2$. The equality indicated in (34) is not an accident but is a consequence of supersymmetry. We will designate as M^2 the physical Fermi-Bose mass $\Lambda^2 \exp(-4\pi/g^2)$.

One might fear that because of the vacuum expectation value of σ, the π will be massless. To investigate this and other questions, it is necessary to expand (33) in powers of σ, π, and A_μ around the stationary point. As usual in the $1/N$ expansion, only the quadratic terms in this expansion are important, the others being suppressed by powers of $1/\sqrt{N}$. The quadratic terms in (33) turn out to be the σ^2, π^2, and A_μ^2 terms and a π-A_μ mixing term.

The term quadratic in π turns out to be, in momentum space,

$$\frac{N}{8\pi} k^2 F(k^2) , \tag{35}$$

where

$$F(k^2) = \int_0^1 \mathrm{d}\alpha \frac{1}{M^2 - (\alpha - \alpha^2) k^2} .$$

Because (35), which would ordinarily be the inverse of the π propagator, vanishes at $k^2 = 0$, it seems at first sight that we have a massless π, and no resolution of the U(1) problem.

Before jumping to this conclusion, we must consider the $A_\mu \pi$ mixing term, which is a manifestation of the anomaly. It turns out to be

$$N\sqrt{2} \frac{M}{4\pi} F(k^2) \,\epsilon_{\mu\nu} k_\nu , \tag{36}$$

while the term quadratic in A_μ is

$$\frac{N}{4\pi} F(k^2) \left(-g_{\mu\nu} k^2 + k_\mu k_\nu\right) . \tag{37}$$

Inverting the combined system (35)–(37), we find for the π propagator

$$\langle \pi(k)\, \pi(-k) \rangle = \frac{8\pi}{N} \frac{1}{k^2 - 4M^2} \frac{1}{F(k^2)} . \tag{38}$$

Thus, the pole has moved from $k^2 = 0$ to $k^2 = 4M^2$. The π has acquired a mass $2M$, twice the n-ψ mass.

If one considers the σ field, one finds with much less effort (since no consideration of mixing diagrams is required) the same formula

$$\langle \sigma(k)\, \sigma(-k) \rangle = \frac{8\pi}{N} \frac{1}{k^2 - 4M^2} \frac{1}{F(k^2)} . \tag{39}$$

This identity of the σ and π propagators is, of course, not an accident, but a consequence of supersymmetry.

Indeed, if one calculates the λ propagator one finds the same formula as (38) and (39), except for a Dirac numerator. In particular, there is a singularity at $k^2 = 4M^2$. (The relevant calculations have already been done by Alvarez [24], in connection with a related model.) What is happening is that the two bosons, σ and π, and the two fermions, χ and χ^*, are combining into a supersymmetry multiplet, with a common mass $2M$.

Thus, instanton lore cannot possibly be right in claiming that the π mass should be exponentially small as $N \to \infty$, because supersymmetry requires that the π is degenerate with the σ, χ, and χ^*, and all analyses would agree that they have masses of order one for $N \to \infty$.

(The reason that the "η" or π mass squared is of order one in this model, but of order $1/N$ in (22), is that this model has N Fermi fields, and therefore the coefficient of the anomaly is N times as large.)

In short, we have established that the U(1) problem is resolved in leading order of the $1/N$ expansion. For σ has acquired a vacuum expectation value without there being a massless particle.

A further very interesting consequence of the π-A_μ mixing is that it screens the gauge field A_μ, just as in (22). When one diagonalizes the joint π-A_μ kinetic energy, one finds that the A_μ propagator, like the π propagator, has no singularity at $k^2 = 0$, which would correspond to a Coulomb potential, but, like the π, it couples to the singularity at $k^2 = 4M^2$. The mixing screens the long-range Coulomb potential, leaving only weak Yukawa-like forces.

In consequence, this is *not* a model with confinement. In contrast to the SU(N) sigma model without fermions, the supersymmetric model has free, unconfined n and ψ particles and antiparticles. We will discuss later a check on this conclusion that comes from known exact results at $N = 2$.

We still must establish that this model has, in leading order in $1/N$, a discrete chiral symmetry. This is a subgroup of the naive U(1) chiral symmetry. It is not explicitly broken by the anomaly, but we will see that it is spontaneously broken. Because there are N flavors, the discrete symmetry is $\psi \to e^{\pi i \gamma_5/N}\psi$.

As in the discussion of (22), the spontaneously broken symmetry manifests itself in the appearance of solitons, that is, states in which $\pi(x = +\infty)$ does not equal $\pi(x = -\infty)$. The states with this soliton property are the ordinary (unconfined!) n and ψ particles. (However, the change in π from $x = -\infty$ to $x = +\infty$ in one of these states will turn out to be small, of order $1/N$, which is why it is not necessary or useful, for most purposes, to think of these states as solitons.)

To see this, we will repeat the argument that we gave in connection with (22). Let us collect from (30), (36), and (37), the terms in the effective Lagrangian that involve the gauge field. We will take the non-relativistic limit, and so set $F(k^2)$ equal to $1/M^2$, which is its value at $k^2 = 0$. The non-relativistic limit of the effective Lagrangian turns out to be

$$L_{\mathrm{eff}} = -\frac{N}{16\pi M^2}F_{\mu\nu}^2 + \frac{A_\mu \epsilon_{\mu\nu}\partial_\nu \pi}{4\pi M} N\sqrt{2} + A_\mu J^\mu , \tag{40}$$

where

$$J^\mu = i n_i^* \overleftrightarrow{D}^\mu n^i + \overline{\psi}_i \gamma^\mu \psi^i$$

is the electric current of the elementary fields.

From (40) one derives a kind of Gauss' law:

$$\frac{N}{4\pi M^2}\frac{\mathrm{d}E}{\mathrm{d}x}+\frac{N\sqrt{2}}{4\pi M}\frac{\mathrm{d}\pi}{\mathrm{d}x}=J^0\,. \tag{41}$$

Integrating from $x = -\infty$ to $x = +\infty$ and realizing that because of the screening of the gauge field, the electric field E vanishes as $x \to \pm\infty$, we find

$$\pi(\infty)-\pi(-\infty)=\frac{4\pi M}{N\sqrt{2}}Q\,. \tag{42}$$

where $Q = \int \mathrm{d}x J^0$ is the total charge. Thus, the n and ψ particles, and any other particles that may have non-zero Q, are solitons, in the sense of interpolating between different values of the π field.

Recalling that the vacuum expectation value of σ is related to the physical mass M by $M^2 = \frac{1}{2}\sigma^2$, we see that (42) can be rewritten

$$\frac{\pi(\infty)-\pi(-\infty)}{\sigma}=\frac{2\pi}{N}Q\,. \tag{43}$$

On the other hand, $\sigma(\infty) = \sigma(-\infty)$ up to and including terms of order $1/N$, because σ does not appear in (41). And finally, for the particular states we are considering, because of the arbitrary choice of stationary point in (34), $\pi(\infty)$ and $\pi(-\infty)$ are of order $1/N$.

For states with $Q = 1$, which is the minimum possible value of Q, we can summarize our results:

$$\frac{\pi(\infty)-\pi(-\infty)}{\sigma}=\frac{2\pi}{N}\,,$$

$$\sigma(\infty)=\sigma(-\infty)=\sigma\,,$$

$$\pi(\infty)\,,\ \pi(-\infty)\sim 1/N\,. \tag{44}$$

These results can be interpreted as the first non-trivial term in an expansion in powers of $1/N$ of an exact formula

$$(\sigma(\infty)+i\pi(\infty))=\mathrm{e}^{2\pi i/N}(\sigma(-\infty)+i\pi(-\infty))\,.$$

(Because of our choice (34), we are dealing not with all solutions of this equation, but only with the ones near $\pi = 0$.) This equation indicates that we are dealing with an N-fold discrete symmetry, which has been spontaneously broken.

Thus, the $1/N$ expansion exhibits in leading order the absence of a physical continuous chiral symmetry and the presence of a spontaneously broken discrete one.

Before leaving this model, it is interesting to note that a check can be made on the results stated above, and on the qualitative validity even for small N of the

$1/N$ expansion, by comparing to known results for $N = 2$.

What happens for $N = 2$ is known because the SU(2) model coincides with the O(3) version of the O(N) supersymmetric sigma model [25].

The O(3) supersymmetric sigma model is known [26] to have two somewhat unusual properties: the physical states are all solitons (they are states that interpolate between the vacua of a spontaneously broken discrete chiral symmetry), and they are doublets or spinors of the O(3) symmetry. (That the soliton states are isospinors was explained in ref. [26] for the $(\bar{\psi}\psi)^2$ model only, but the discussion carries over in the same way for the O(N) supersymmetric σ model.)

These known results about the SU(2) or O(3) model are precisely what one obtains by extrapolating to $N = 2$ the results given above for large N. In fact, we found that the n and ψ particles in this model are *not* confined; they transform as the N of SU(N), and for $N = 2$, they are the doublets of ref. [26]. Also, we found in eq. (43) that these states, which all have non-zero Q, are solitons; this again agrees with the conclusions of ref. [26].

In sect. 3 we found that the n particles are confined in the absence of massless fermions. Extrapolating to $N = 2$, we predicted from this that there are no doublets in the spectrum, since the doublets are charged and confined. As noted in sect. 3, this agrees with the known result at $N = 2$, that the spectrum consists of a triplet. But in the supersymmetric model, which has no confinement, a similar extrapolation predicts that there will be doublets in the spectrum, and this, again, agrees with the known result at $N = 2$. This is an indication that the large N expansion is qualitatively correct even at the smallest physical value of N, $N = 2$.

There is another non-perturbative check on the claim surrounding equation (43) that the n and ψ particles of this model are solitons. This involves a purely algebraic argument.

The supersymmetry algebra (32) that is valid in ordinary perturbation theory leads to a paradox when compared with the results of the $1/N$ expansion.

For massless particles (32) has irreducible representations of dimension two. However, for massive particles the irreducible representations of (32) have dimension four.

In perturbation theory, the n and ψ particles are massless. For given SU(N) quantum numbers, there is one n and one ψ particle, and these two states are just right to make a representation of (32).

However, in the $1/N$ expansion, the n and ψ particles acquire masses. These particles, once they acquire masses, cannot be arranged in multiplets of (32), since four states would be required for given SU(N) quantum numbers. What is wrong?

A similar problem arises, and has been resolved, in connection with the Higgs phenomenon in four-dimensional supersymmetric gauge theories [27]. The resolution of this problem, in our model, is as follows. We have claimed that this model has a spontaneously broken discrete symmetry. Therefore, we can define topological charges

$$S = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\mathrm{d}\sigma}{\mathrm{d}x}, \qquad P = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\mathrm{d}\pi}{\mathrm{d}x}, \tag{45}$$

which are zero in ordinary perturbation theory, but non-zero in the $1/N$ expansion, because there is symmetry breaking in this expansion.

As in ref. [27], the naive supersymmetry algebra (32) is not correct, but must be modified to include the central charges,

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0\ ,$$
$$\{Q_\alpha, \bar{Q}_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu + c(\delta_{\alpha\beta} S + (\gamma_5)_{\alpha\beta} P)\ , \tag{46}$$

where c is a constant.

Now, (46) has representations of dimension four, but it also has representations of dimension two *for states which have S or P non-zero.* Thus, if the n and ψ particles have non-zero values of the topological charges, they can be arranged in two-dimensional representations of (46). Since we have argued that these particles are in fact solitons, with non-zero values of the topological quantum numbers, the paradox is resolved.

However, if there are states with $Q = 0$, which by (43) cannot be solitons, they must come in four-dimensional representations of (46). Indeed this occurs. The σ, π, χ, and χ^* particles have $Q = 0$, and form a four-dimensional representation of (46).

Finally, in the second paper of ref. [25], it was pointed out that the O(3) (or SU(2)) supersymmetric non-linear sigma model has a bosonized form. This result can be rederived much more easily in the SU(2) language. The constraint $n_i^* \psi^i = 0$ on the Fermi field ψ^i of the SU(N) model can, for SU(2), be conveniently solved by writing $\psi^i = \epsilon^{ij} n^*{}_j \chi$, where χ is an SU(2) singlet Fermi field. In terms of χ, the Lagrangian (30) takes a form which can be routinely bosonized, by use of the standard formulas, reproducing the result of ref. [25].

In the bosonized form of the theory, the fermions are replaced by a logarithmic interaction among instantons. This is the form of the theory which makes the instanton degrees of freedom most conspicuous, but it does not seem to be a form very close to the actual physical properties of the theory.

5. Implications for quantum chromodynamics

We have now seen, in a fairly detailed way, that predictions based on instanton lore are not reliable in the two-dimensional SU(N) sigma model. Quantitative predictions based on instantons are wrong, and qualitatively the main phenomena in these theories cannot be understood in terms of instantons; the effective Lagrangians that describe these effects were constructed without, and do not contain, instantons.

Now we must ask why this has occurred, and whether it is likely to occur in four-dimensional quantum chromodynamics. I will give here a heuristic answer to this question. (The analogy that follows was suggested, in part, by work of Migdal [28].)

As argued in sect. 2, the starting point of all discussions of instantons is a claim that the topological charge is quantized; if it is not quantized, one should not think about instantons at all, but about a smooth distribution of topological charge.

Our argument in sect. 3 that the topological charge is quantized in the two-dimensional SU(N) sigma model started with the observation that to have finite action, the field n^i must be constant at spatial infinity up to a phase factor,

$$n^i \to n_0^i \, \mathrm{e}^{i\sigma(x)} , \qquad \text{as } |x| \to \infty , \tag{47}$$

where n_0^i is a constant. This boundary condition involves spontaneously broken SU(N) symmetry because any choice of n_0^i breaks the symmetry.

In perturbation theory, the boundary condition (47) which leads us to think in terms of instantons is reasonable, because from the standpoint of perturbation theory the SU(N) symmetry really is spontaneously broken. However, in the $1/N$ expansion we see that the SU(N) symmetry is actually restored. When it is restored, the boundary condition (47), and all conclusions that one might draw from it, are invalidated. Instanton physics is an example of a consequence of (47) that is invalidated when the SU(N) symmetry is restored.

What is the analogous situation in QCD?

The argument for quantization of the topological charge in QCD starts by saying that for finite action, the gauge field must approach a pure gauge at infinity,

$$A_\mu \to g^{-1} \partial_\mu g , \qquad \text{as } |x| \to \infty . \tag{48}$$

Is this boundary condition reasonable? We must remember that in quantum field theory, the correct boundary condition is always that the fields approach the vacuum at infinity, where by "the vacuum" I mean field configurations that are typical of the vacuum.

From the point of view of perturbation theory, the vacuum is mostly pure gauge; in fact, in perturbation theory we construct the vacuum as an expansion around pure gauge. Therefore, from the point of view of perturbation theory, the boundary condition (48) is reasonable.

However, taking as given a supposed experimental fact of quark confinement, quark confinement tells us that in reality the vacuum cannot be regarded as being mostly pure gauge. In fact, the expectation value of Wilson's loop in any state that is mostly pure gauge will (just as in perturbation theory) not show a confining potential. Thus, in a quark-confining theory the boundary condition (48) is not reasonable and conclusions based on it are likely to be wrong.

In particular, instantons and instanton physics are likely to be just as misleading in four-dimensional quantum chromodynamics as in the two-dimensional SU(N) sigma model.

Predictions that depend only on the existence of the topological charge and of the axial anomaly will, I believe, be valid in QCD. This includes the existence of the vacuum angle, the resolution of the U(1) problem, and others that have been discussed at length in this paper.

However, predictions that depend on thinking about instantons and an instanton gas (even a dense gas) will not be correct.

For instance, if QCD remains a confining theory as $N \to \infty$, as 't Hooft argued [9], then I expect that the θ dependence and the resolution of the U(1) problem will be visible in the $1/N$ expansion, in contradiction to expectations based on instantons. For any approximation that yields confinement will have to include, under one guise or another, fields that fluctuate at infinity, corresponding to a vacuum that is not mainly pure gauge. Fields that fluctuate at infinity and do not approach pure gauge generically have non-zero $\int \mathrm{d}x\, F\tilde{F}$, and once we include such fields we will find the characteristic effects of non-zero $\int \mathrm{d}x\, F\tilde{F}$: the θ dependence and the resolution of the U(1) problem. In such a picture, it is most probable that the η mass would appear, in the $1/N$ expansion, in the first diagrams that split the singlet and non-singlet channels. These are the first non-planar diagrams, the first diagrams that permit $q\bar{q}$ annihilation into glue states.

In summary, the proposed correspondence between two and four dimensions is as follows.

The boundary condition $n^i \to n_0^i \mathrm{e}^{i\sigma}$ in two dimensions corresponds to the boundary condition $A_\mu \to g^{-1}\partial_\mu g$ in four dimensions.

Restoration of the SU(N) symmetry in two dimensions, which signals that the boundary condition $n^i \to n_0^i \mathrm{e}^{i\sigma}$ is misleading, corresponds to quark confinement in four dimensions, which signals that the condition $A_\mu \to g^{-1}\partial_\mu g$ is misleading.

Finally, Coleman's theorem on the absence of continuous symmetry breaking in two dimensions [29], which tells us in advance of any calculation that in a correct calculation the symmetry would be restored, corresponds to the experimental fact of quark confinement, which tells us, in advance of our ability to do calculations, that a correct calculation would yield confinement.

I believe that this heuristic line of reasoning is convincing, but is certainly not conclusive by itself. It should be noted that several other considerations make this point of view attractive.

(i) It removes the conflict between instantons and the quark model concerning the η.

(ii) It removes the conflict between instantons and the large N expansion.

(iii) The instanton calculations have internal difficulties, infrared divergences that get worse when higher-order processes are considered, and this suggests that we should seek a different physical picture.

I will comment on these points in turn.

About the η, much has been said above, and I will only stress here that because the simple quark model is usually so successful, it is attractive to try to reconcile the discrepancy between the quark model and field theoretic reasoning.

As for the second point, the $1/N$ expansion, it should by now be clear that the $1/N$ expansion and instantons, as ways of thinking about the strong interactions, are in conflict.

Assuming that the quark masses are negligibly small, $1/N$ is the only parameter in

QCD. To lose the possibility of expanding in this parameter would be a serious set-back to our chances of some day understanding the strong interactions. Therefore, it is important to understand whether an expansion in $1/N$ can be expected to be qualitatively right, or whether it misses qualitative effects.

It cannot be emphasized too strongly that in attempting to understand the strong interactions with instantons, there is no expansion parameter. We cannot say, for instance, that we are expanding in "h" since h is not dimensionless. The only dimensionless parameter in QCD, apart from $1/N$, is g^2, which, as we know, is absorbed into defining the scale of lengths.

It is very attractive to believe that QCD behaves like the two-dimensional model discussed in this paper because this removes the few discrepancies with what is otherwise the very attractive idea of the $1/N$ expansion.

A technical point should be added here.

In sect. 1 it was argued that instanton effects are of order e^{-N}. Here I would like to clarify one of the arguments that was given. (The argument that follows shows only that instanton effects are smaller than any power of $1/N$.)

The $1/N$ expansion is a systematic expansion of integrals such as (12) in powers of a parameter, $1/N$. Although such an expansion might or might not reproduce correctly this or that physical phenomenon, we should expect that it will be valid at least as an asymptotic expansion of the Green functions.

Therefore, we can calculate the Green functions correctly to any finite order in $1/N$ simply by following the rules of the $1/N$ expansion, that is, by summing certain classes of Feynman diagrams. As a result, any "instanton effects," that is, any effects that must be included apart from summing the Feynman diagrams, must be smaller than any power of $1/N$, as was to be shown.

Finally, the last reason that one might find the point of view in this paper attractive is that instanton calculations have certain internal difficulties, in the form of infrared divergences, which may suggest that one should seek an alternative physical picture. The divergence in integration over the instanton size is well known. It appears in the one instanton contribution to the vacuum energy and in the two point function of the electromagnetic current [30]

However, it has been shown by Zakrzewski [31] that there are additional divergences that appear in instanton calculations, which become worse when one considers higher-order processes. For instance, when one considers two-point functions of operators other than the electromagnetic current, one finds, in general, that in addition to the scale integration, the integration over the instanton position also diverges. And in examining two-instanton contributions to the vacuum energy, one finds, because of a similar effect, that it does not have the correct behavior proportional to the volume V of space, but rather has a term $V \ln V$.

Of course, it is possible that a resolution of these problems can be found within instanton reasoning. But another interpretation, in line with the discussion in this paper, is that one should be seeking a different physical picture.

6. Conclusions

It may be useful to conclude this paper with some remarks about the connection of the analysis here with previous work.

't Hooft showed that the U(1) problem is resolved in QCD because, although $F\tilde{F}$ is a total divergence, nevertheless its space-time integral is not necessarily zero.

The existence of instantons is the most general way to see that $\int d^4xF\tilde{F}$ is not necessarily zero. Because instantons exist, even if one assumes at infinity the boundary condition $A_\mu \to g^{-1}\partial_\mu g$ that is most unfavorable for getting a non-zero value of this integral, still one encounters fields for which the integral is non-zero.

't Hooft's arguments show that regardless of what else happens in QCD, the U(1) problem will be resolved, this following from the fact that $\int d^4xF\tilde{F}$ is not necessarily zero.

While instantons show that even if quarks were not confined in QCD, the U(1) problem would still be resolved, a different physical picture of the resolution of this problem may be more appropriate if quarks are confined.

If quarks are confined, the true vacuum state is far from being mainly pure gauge. Any approximation to the (infinite space-time volume) path integral includes, under one guise or another, fields whose asymptotic behavior is the behavior characteristic of the fields that dominate the vacuum state found in that approximation. An approximation that yields quark confinement will inevitably include fields that in no sense approach a pure gauge at infinity. For such fields, there is no reason for the crucial integral $\int d^4xF\tilde{F}$ to vanish. (Remember that to show it vanishes, two ingredients are required: the gauge field must be a pure gauge at infinity; and this pure gauge must be topologically trivial. My claim is that any quark-confining approximation will include fields that violate the first condition.) Therefore, the effects that arise when one includes fields for which this integral does not vanish, the θ dependence and the resolution of the U(1) problem, will appear in any approximation that yields quark confinement.

In particular, the $1/N$ expansion may well be such an expansion [9]. If so, I believe that in QCD, as in the Schwinger model, it will be inappropriate to consider "instantons." that is, classical fields whose contributions must be added on to the sum of the Feynman diagrams. The sum of the planar diagrams, if it yields confinement, will have invalidated the boundary condition, $A_\mu \to g^{-1}\partial_\mu g$ at infinity, which motivates a consideration of instantons, as well as yielding the effects (the θ dependence and η mass) for which instantons are puportedly responsible.

The two-dimensional model considered in this paper shows that such behavior is possible. Perhaps the arguments in this paper will persuade the reader that it is attractive to believe that such behavior occurs also in QCD.

I would like to thank A. Patrasciou for discussions.

Appendix A

The "Higgs model" in one space dimension

We wish to comment here on the confusing question of whether two-dimensional scalar electrodynamics has a phase transition. These comments will not resolve this question, but may help clarify the issues. Thus, consider the Lagrangian:

$$\mathcal{L} = (\partial_\mu - ie\,A_\mu)\,\phi^*(\partial_\mu + ie\,A_\mu)\,\phi - \lambda\,(\phi^*\phi)^2$$
$$- M^2\phi^*\phi - \tfrac{1}{4}F_{\mu\nu}^2\,. \tag{49}$$

We wish to argue that this theory has a phase transition as a function of M^2.

One expects a phase transition for a very simple reason. For positive M^2 this theory has unbroken U(1) symmetry, so that the ϕ and ϕ^* particles interact *via* Coulomb potentials. This corresponds, in one space dimension, to a confining phase. On the other hand, for negative M^2 we expect broken U(1) symmetry and a Higgs phase. However, several subtleties arise when one tries to show clearly that this phase transition really exists. Some of the subtleties are related to instantons.

The most straightforward way to prove that a phase transition exists is to find a qualitative criterion (an "order parameter") that distinguishes the positive M^2 theory from the negative M^2, so that there must be a phase transition between them.

At first sight a qualitative criterion seems easy to find. We have confinement for positive M^2 but not for negative M^2. However, in attempting to make this precise one finds trouble.

What is confinement? One might try to define confinement by introducing the electric charge $Q = \int \mathrm{d}x\, J_0$, J_0 being the charge density, and saying that Q annihilates all the physical states.

Here, however, we encounter an embarrassing fact. Q also annihilates all states in the Higgs theory, $M^2 < 0$. This follows from Gauss' law, $\mathrm{d}E/\mathrm{d}x = eJ_0$, from which we learn that

$$Q = \int \mathrm{d}x\, J_0 = \frac{1}{e}\int \mathrm{d}x\, \frac{\mathrm{d}E}{\mathrm{d}x} = \frac{1}{e}(E(\infty) - E(-\infty))\,.$$

But in a Higgs theory, the electric field is screened, so $E(\infty) = E(-\infty) = 0$ for all states. Therefore $Q = 0$ for all states.

Of course, we feel intuitively that Q is zero for completely different reasons in the two cases: because of confinement for $M^2 > 0$, and because of screening for $M^2 < 0$. To make this intuition into an argument that a phase transition exists, we must somehow make precise the idea that for $M^2 > 0$, the physical states, although neutral, are bound states of charged constituents, while for $M^2 < 0$ they do not have such an interpretation. I will suggest below a way to make this precise, but first let us consider other possible approaches.

One might try to ask how the theory reacts to an external test charge. For $M^2 > 0$ an external test charge of strength e can form neutral, and hence finite-energy bound states with one of the ϕ or ϕ^* particles. However, a fractional test charge cannot combine with the ϕ and ϕ^* particles into a neutral bound state. Therefore, there do not exist, for $M^2 > 0$, finite-energy states with a fractional test charge.

Ordinarily we would expect the Higgs theory, $M^2 < 0$, to behave differently and to have finite-energy states with external charges of any strength. This is true in any number of space dimensions except one, but in one dimension it is not true, because of instantons [14,15], and therefore we have failed again to distinguish $M^2 > 0$ from $M^2 < 0$.

Another unusual property of this theory is the vacuum angle θ. But θ exists both for $M^2 > 0$ (where it corresponds to a background electric field) and for $M^2 < 0$ (where it is related to instantons). Thus, the existence of θ is not a qualitative distinction between the two regimes.

The qualitative property which I think makes precise the idea that $M^2 > 0$ is a confining theory and $M^2 < 0$ a Higgs theory appears when we consider the form of the θ dependence.

For $M^2 < 0$, the amplitudes depend on θ only through factors of $\cos\theta$ associated with instanton amplitudes. These factors are, of course, completely analytic as a function of θ.

However, it is easy to see from ref. [12] that for $M^2 > 0$ the physics is not analytic as a function of θ. For in this theory, θ corresponds to a background electric field. As one increases θ, the electric field increases, and eventually it becomes energetically favorable to create a $\phi\phi^*$ pair, which then separate to plus and minus infinity, so as to reduce the magnitude of the electric field. At the values of θ at which a pair is created, the vacuum expectation value of the electric field changes discontinuously, and all other physical quantities are non-analytic, or, in some cases, discontinuous. It was found in ref. [12] that the points of non-analyticity are $\theta = (2n + 1)\pi$.

Thus, for $M^2 < 0$ the physics is analytic as a function of θ, while for $M^2 > 0$ it is not. This qualitative difference shows that a phase transition exists. It is not just a mathematical abstraction, but rather is closely related to our intuition that for $M^2 > 0$, but not for $M^2 < 0$, the physical states are bound states of charged constituents. For the non-analyticity corresponds precisely to the liberation, and separation, of a pair of these charged constituents. This gives a precise basis for saying that the positive M^2 theory is a theory of confinement while the negative M^2 theory is a theory of screening or a Higgs theory (but one which because of instantons would confine fractional test charges).

Appendix B

θ, dependence in the Kogut-Susskind model

One conclusion that follows from the viewpoint of this paper is that the θ dependence and the U(1) problem are closely related to confinement. In the case of the

U(1) problem, Kogut and Susskind made a similar suggestion several years ago [32] for somewhat different reasons.

Kogut and Susskind considered (in four dimensions) a phenomenological model of confinement, in which $1/k^4$ photon propagators were introduced in perturbation theory, and found that the resolution of the U(1) problem could be seen in perturbation theory in such models.

Can also the θ dependence be seen in perturbation theory in such models? Here I will only state a partial result coming from a limited study of this question, which deserves further study. To answer this question, one may add to the Kogut-Suskind Lagrangian a term $\theta \int d^4x F_{\mu\nu}\widetilde{F}_{\mu\nu} f(x)$, where $f(x)$ is a test function that is taken to be one only at the end of the calculation. Furthermore, it is useful to expand in powers of θ near $\theta = 0$. The term linear in θ turns out to have a non-zero limit as f approaches one. It is a correction to the energy of a system of electric charges of the form $\theta P \cdot \mu$, P and μ being the electric and magnetic dipole moments. This is analogous to Coleman's result that in one space dimension a non-zero θ corresponds to an extra term θP in the energy. However, the terms of higher order in θ seem to diverge as f approaches one, and the nature of the true answer is not clear.

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