QUANTUM GEOMETRY OF FERMIONIC STRINGS

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The formalism of the previous paper is extended to the case of supersymmetric strings. The effective theory which sums up fermionic surfaces is described by the supersymmetric Liouville equation. At $D = 10$ effective decoupling of the Liouville dilaton takes place and our theory coincides with the old ones. At $D = 3$ our theory is equivalent to the three-dimensional Ising model, which is thus reduced to the two-dimensional supersymmetric Liouville theory.

In the previous paper [1] I have developed the procedure for quantizing Bose strings, or, which is the same, for summing up random surfaces. It is very urgent to extend these results to fermionic strings because, as was shown in refs. [2,3], the three-dimensional Ising model can be reduced to the free Fermi string theory. In this paper we shall generalize the construction of ref. [1] to the fermionic case and show how to compute critical exponents. Let us begin from the supersymmetric extension of the Bose string lagrangian (see ref. [4] and references therein).

$$S = \frac{1}{2} \int [\sqrt{g}\, g^{\alpha\beta} \partial_\alpha x\, \partial_\beta x + \bar{\psi} \mathrm{i}\gamma^\alpha \partial_\alpha \psi + \bar{\chi}_\alpha \gamma^\beta \gamma^\alpha (\partial_\beta x + \tfrac{1}{2}\chi_\beta \psi)\psi]\ \mathrm{d}^2\xi\ , \tag{1}$$

here the surface is parametrized by $x_A = x_A(\xi)$, $A = 1, ..., D$; ψ is a two-component Majorana spinor, $g_{\alpha\beta}$ is a metric tensor and χ_α is a spin 3/2 "gravitino" field. one has to treat $g_{\alpha\beta}$ and χ_α as independent variables in order to ensure local supersymmetry. Our strategy as in the previous paper will be to integrate out the x and ψ fields first and then to examine the resulting theory of "induced supergravity". Again, the integration can be explicitly performed because of the following reason. We have gauge freedom in the definition of $g_{\alpha\beta}$ and χ_α which consists of general coordinate transformations, involving two arbitrary functions and a supergauge transformation, involving an arbitrary Majorana spinor. Because of that we can choose the "superconformal" gauge

$$g_{\alpha\beta}(\xi) = e^2(\xi)\delta_{\alpha\beta}\ , \quad \chi_\alpha(\xi) = \gamma_\alpha \chi\ . \tag{2}$$

If we try to substitute (2) into (1) we find that the e and χ dependence disappear, due to the relation $\gamma_\beta \gamma_\alpha \gamma^\beta = 0$. That means that the functional integral over x, ψ is defined by the trace anomalies of the energy–momentum tensor $T_{\alpha\beta}$ and the supercurrent S_α given by $g^{\alpha\beta} T_{\alpha\beta}$ and $\gamma^\alpha S_\alpha$, respectively. The most simple way to compute the integral

$$\mathrm{e}^{-W} = \int D\psi\, Dx\, \mathrm{e}^{-S}\ , \tag{3}$$

is to observe that $W[e, \chi]$ must possess two symmetries, which are remnants of the local supersymmetry that are not destroyed by the superconformal gauge (2). Namely, in the language of superspace $(\xi_1 \xi_2 \theta_1 \theta_2)$, we have the transformations

$$\delta z = u(z), \quad \delta\theta = 0\ ; \quad \delta z = \epsilon(z)\theta\ , \quad \delta\theta = \epsilon(z)\ , \tag{4}$$

where $z = \xi_1 + \mathrm{i}\xi_2$, $\theta = \theta_1 + \mathrm{i}\theta_2$; $u(z)$ is an analytic function and $\epsilon(z)$ is an analytic Grassman parameter. Another condition on $W[e, \chi]$ is that being determined by the trace anomaly it must be local in e and χ and contain only coupling constants of zero and positive dimension. There exists only one expression which satisfies all these claims. It is the direct supersymmetric extension of the Liouville lagrangian, given in ref.

[1]. Let us introduce the superfields:

$$E = e + \theta\bar{\chi} + \chi\bar{\theta} + (\bar{\theta}\theta)\lambda\,, \quad \phi = \log E\,, \tag{5}$$

here λ is a subsidary field which will be excluded later, the complex field χ is connected with the previous Majorana spinor $\binom{\chi_1}{\chi_2}$ by the relation $\chi = \chi_1 + i\chi_2$. In terms of the superfield ϕ, the effective lagrangian we are looking for is given by

$$W[e,\chi] = A \int [(d\phi)(\overline{d\phi}) + \mu e^{\phi}]\, d^2z\, d^2\theta\,, \tag{6}$$

here A is yet undetermined constant, μ is some arbitrary mass scale, $d = \partial/\partial\theta + \theta\partial/\partial z$.

The superanalytic symmetry group (4) is realized in the following way:

$$\delta E = (\partial/\partial\theta - \theta\partial/\partial z)\,[\epsilon(z)E]\,,$$

$$\delta\phi = \epsilon(z)\,(\partial/\partial\theta - \theta\partial/\partial z)\phi - \theta\partial\epsilon/\partial z\,. \tag{7}$$

In terms of original fields $\varphi = \log e$ and χ, the action (6) can be rewritten as:

$$W = A \int [\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}i\chi^T(\gamma\cdot\partial)\chi + \tfrac{1}{2}\mu(\chi^T\gamma_5\chi)e^{\varphi} + \tfrac{1}{2}\mu^2 e^{2\varphi}]\,. \tag{8}$$

The constant A is most easily determined by the comparison of (3) and (8) with perturbation theory and is found to be

$$A = -D/8\pi\,, \tag{9}$$

where D is the number of space–time dimensions.

Our next problem is to compute the contribution of ghosts, associated with the gauge conditions (2). The fermionic part of these conditions can be rewritten as

$$\chi_\alpha + \gamma_5\epsilon_{\alpha\beta}\chi_\beta = 0\,, \tag{10}$$

where $\epsilon_{\alpha\beta}$ is a standard antisymmetric tensor.

Under supersymmetry we have $\delta\chi_\alpha = \nabla_\alpha\omega$ where ω is a Majorana spinor and ∇_α is a spinorial derivative. It is possible to show, following the reasoning of ref. [1], that the integration measure is given by

$$d\mu(g_{ab},\chi) \propto D\varphi(\xi)\,D\chi(\xi)\,\det{}^{1/2}\mathcal{L}_B\,\det{}^{-1/2}\mathcal{L}_F\,, \tag{11}$$

where $\mathcal{L}_B$ is a supersymmetric extension of the ghost operator introduced in ref. [1], and $\mathcal{L}_F$ is given by

$$\mathcal{L}_F\omega = \nabla^\alpha(\nabla_\alpha\omega + \gamma_5\epsilon_{\alpha\beta}\nabla_\beta\omega)\,. \tag{12}$$

For the case $\chi = 0$ and $g_{\alpha\beta} = e^2(\xi)\delta_{\alpha\beta} \equiv \rho\delta_{\alpha\beta}$ the operator (12) is reduced to

$$\mathcal{L}_F = e^{-3}\partial_{\bar{z}}(e\partial_z)\,. \tag{13}$$

The determinant of (13) is easily computed, giving

$$\tfrac{1}{2}\log\det\mathcal{L}_F = \frac{11}{24\pi}\int [\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}\mu^2 e^{2\varphi}]\, d^2\xi \quad \text{at } \chi = 0 \tag{14}$$

(we use here the general formula):

$$\tfrac{1}{2}\log\det[\rho^{-j-1}\partial_{\bar{z}}(\rho^j\partial_z)] = \frac{1 + 6j(j+1)}{12\pi}\int [\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}\mu^2 e^{2\varphi}]\, d^2\xi\,, \tag{15}$$

the Dirac operator corresponds to $j = -\frac{1}{2}$, bosonic ghosts to $j = 1$, fermionic ghosts to $j = +\frac{1}{2}$ and scalar fields to $j = 0$. Collecting together (14), (11), (8) and (9) we obtain the final answer:

$$Z = \int D\varphi(\xi)\,D\chi(\xi) \times \exp\left(-\frac{10-D}{8\pi}\int \mathcal{L}\, d^2\xi\right), \tag{16}$$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}i\bar{\chi}(\gamma\cdot\partial)\chi + \tfrac{1}{2}\mu\,(\bar{\chi}\gamma_5\chi)e^{\varphi} + \tfrac{1}{2}\mu^2 e^{2\varphi}\,.$$

This is a two-dimensional, renormalizable, completely integrable field theory, which as we just proved describes the sum over random surfaces with fermionic structure. It will be a subject of another paper to investigate this theory in full detail. Here we shall give only a one-loop estimate for the critical exponents of the fermionic string, alias the three-dimensional Ising model. Let us examine the renormalization of the mass in (16). In order to do this we introduce some background field φ_c, which minimizes the classical action:

$$S_{cl} = \frac{10-D}{8\pi}\int [\tfrac{1}{2}(\partial_\mu\varphi_c)^2 + \tfrac{1}{2}\mu^2 e^{2\varphi_c}]\, d^2\xi \tag{17}$$

and compute the one-loop correction to (17). The corresponding quadratic action has the form

$$S_{II} = \frac{10-D}{8\pi}\int [\tfrac{1}{2}(d\psi)(\overline{d\psi}) + \tfrac{1}{2}\mu\, e^{\phi_c}\psi^2]\, d^2\xi\, d^2\theta\,, \tag{18}$$

where $\phi = \phi_c + \psi$, ψ is a real superfield. The cut-off procedure for the gaussian integral of (18) needs some care. Namely, we should not destroy under the renormalization the analytic supersymmetry (4). This can be achieved by definition of the determinant through the following eigenvalue problem:

$$-d(d\psi) = (\lambda_n + \mu)\, e^{\phi} c \psi \,. \tag{19}$$

The resulting effective action in terms of λ_n is given by

$$S_{\mathrm{eff}} = S_{\mathrm{cl}} + \frac{1}{2} \int_{\varepsilon}^{\infty} \frac{d\tau}{\tau} \sum_n [\exp(i\tau\lambda_n^{B}) - \exp(i\tau\lambda_n^{F})] \,, \tag{20}$$

where ε is a proper time cut-off, $\lambda_n^{B,F}$ are Bose and Fermi branches of the spectrum. The divergent part of expression (20) can be evaluated by the use of the standard WKB short-time expansion. As a result, the mass μ acquires a logarithmically divergent renormalization. Detailed calculations will be given in a separate paper. It is a very important effect, because, as is well known, in phase transition theory the dependence of a physical mass μ on a bare mass μ_0 defines the critical behaviour of the specific heat [5]. So, by solving the Liouville problem and finding the $\mu(\mu_0)$ dependence we can check the scaling laws and find the critical exponents.

This solution is certainly possible to find due to the complete integrability of the theory. There still will be some further problems, since up to now we dealt with topologically trivial surfaces. It might happen that summation over topologies will introduce some extra renormalization of the physical mass. Obviously, there is still a lot of work ahead.

References

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