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CURRENT ALGEBRA AND WESS-ZUMINO MODEL IN TWO DIMENSIONS

V G KNIZHNIK and A B ZAMOLODCHIKOV

Landau Institute of Theoretical Physics, Moscow, USSR

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We investigate quantum field theory in two dimensions invariant with respect to conformal (Virasoro) and non-abelian current (Kac-Moody) algebras The Wess-Zumino model is related to the special case of the representations of these algebras, the conformal generators being quadratically expressed in terms of currents The anomalous dimensions of the Wess-Zumino fields are found exactly, and the multipoint correlation functions are shown to satisfy linear differential equations In particular, Witten's non-abelean bosonisation rules are proven

1. Introduction

In recent papers [1-3] some novel important properties of the two-dimensional σ-model with Wess-Zumino action

$$S_{\lambda,k}(g)=\frac{1}{4\lambda^2}\int \mathrm{tr}\left(\partial_\mu g^{-1}\partial_\mu g\right)\mathrm{d}^2\xi+k\Gamma(g) \tag{1.1}$$

have been discovered. The matrix field $g(\xi)$ in (1.1) is taken to be an element of some semisimple group G, $\xi^\mu=(\xi^1,\xi^2)$ are the coordinates of two-dimensional space, λ^2 and k are dimensionless coupling constants, k being necessarily integer [1,2]. The Wess-Zumino term $\Gamma(g)$ is defined by the integral

$$\Gamma(g)=\frac{1}{24\pi}\int \mathrm{d}^3X\mathrm{e}^{\alpha\beta\gamma}\mathrm{tr}\left(g^{-1}\partial_\alpha g g^{-1}\partial_\beta g g^{-1}\partial_\gamma g\right) \tag{1.2}$$

over the three-dimensional ball with coordinates X^α; the boundary being identified with two-dimensional space [2]. The boundary values $g(\xi)$ determine (1.2) modulo 2π [1].

If $k=0$, the action (1.1) reduces to the usual σ-model which is well known to be asymptotically free and effectively massive. This model has been exactly solved in [4,5]. Under the choice $k=1,2,\ldots$ the character of the theory changes drastically as has been shown by Witten [2], Polyakov and Wiegmann [3]. The renormalization group possesses the infrared-stable fixed point

$$\lambda^2 = \frac{4\pi}{k}, \tag{1.3}$$

and therefore the effective theory is massless and its large-distance behavior is governed by the action

$$S_k(g) = kW(g), \tag{1.4}$$

where

$$W(g) = \left\{ \frac{1}{16\pi} \int \mathrm{tr}\left(\partial_\mu g^{-1} \partial_\mu g \right) \mathrm{d}^2\xi + \Gamma(g) \right\}. \tag{1.5}$$

In further discussions this theory (1.4) will be referred to as the Wess-Zumino model*.

The most important property of the action (1.4) is its invariance with respect to infinite-dimensional current (Kac-Moody) algebra [2, 3, 8]. The action (1.4) remains unchanged under the transformations

$$g(\xi) \to \Omega(z) g(\xi) \bar{\Omega}^{-1}(\bar{z}), \tag{1.6}$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are arbitrary G-valued matrices analytically depending on the complex coordinates

$$z = \xi^1 + i\xi^2,$$
$$\bar{z} = \xi^1 - i\xi^2, \tag{1.7}$$

respectively. (Here we imply the euclidean version of the theory; in Minkowski space-time the variables (1.7) are the light-cone coordinates.) One can easily ensure symmetry (1.6) using the following remarkable relation [3, 8]:

$$W(gh^{-1}) = W(g) + W(h) + \frac{\mathrm{tr}}{16\pi} \int \left(g^{-1} \partial_{\bar{z}} g h^{-1} \partial_z h \right) \mathrm{d}^2\xi, \tag{1.8}$$

satisfied by the functional (1.5). Note that the group (1.6) generalizes the usual $G \times G$ symmetry of the chiral field, and it can also be represented as the direct product of the "left" and "right" gauge groups; we shall denote it by $G(z) \times G(\bar{z})$.

* The fixed-point theory (1.4) deserves special interest because of its analogies with the quantum Liouville theory related to the Polyakov string [7].

The symmetry (1.6) gives rise to an infinite number of conserved currents which can be derived from the equations

$$\partial_{\bar{z}} J = 0, \qquad \partial_z \bar{J} = 0, \tag{1.9}$$

where the basic currents J and $\bar{J}$:

$$J = J^a t^a = -\tfrac{1}{2} k \partial_z g g^{-1},$$

$$\bar{J} = \bar{J}^a t^a = -\tfrac{1}{2} k g^{-1} \partial_{\bar{z}} g, \tag{1.10}$$

correspond to the generators of the groups $G(z)$ and $G(\bar{z})$ respectively [2, 3]. Here t^a are the antihermitian matrices representing (for the field $g(\xi)$) the Lie algebra

$$[t^a, t^b] = f^{abc} t^c \tag{1.11}$$

of the group G; f^{abc} are the structure constants. Due to (1.9) we can write

$$J^a = J^a(z), \qquad \bar{J}^a = \bar{J}^a(\bar{z}). \tag{1.12}$$

The variations of the fields (1.10) under the infinitesimal transformations (1.6) with

$$\Omega(z) = I + \omega(z) = I + \omega^a(z) t^a, \tag{1.13a}$$

$$\bar{\Omega}(\bar{z}) = I + \bar{\omega}(\bar{z}) = I + \bar{\omega}^a(\bar{z}) t^a, \tag{1.13b}$$

are described by the formulae

$$\delta_\omega J(z) = [\omega(z), J(z)] + \tfrac{1}{2} k \omega'(z),$$

$$\delta_{\bar{\omega}} \bar{J}(\bar{z}) = [\bar{\omega}(\bar{z}), \bar{J}(\bar{z})] + \tfrac{1}{2} k \bar{\omega}'(\bar{z}), \tag{1.14}$$

which shows that the generators $J(\bar{J})$ of the group $G(z)$ $(G(\bar{z}))$ represent the Kac-Moody algebra [2] with the central charge k*. Since $\delta_\omega \bar{J} = \delta_{\bar{\omega}} J = 0$, the generators J and $\bar{J}$ are commutative.

Under the choice $G = O(N)$ the group $G(z) \times G(\bar{z})$ describes the symmetry of the free massless N-component Majorana fermion theory with the action

$$S_f(\psi, \bar{\psi}) = \tfrac{1}{2} \int \sum_{\alpha=1}^{N} [\psi_\alpha \partial_{\bar{z}} \psi_\alpha + \bar{\psi}_\alpha \partial_z \bar{\psi}_\alpha] \, d^2\xi, \tag{1.15}$$

* Obviously, there are no divergent renormalizations of the integer-valued "coupling constant" k in the theory (1.1). However, finite renormalization of the type $k_0 \to k = k_0 + \Delta k$, where k_0 stands in front of the "bare" action (1.4) and Δk is some integer, cannot be excluded a priori. The one-loop computation shows that $\Delta k = 0$

where ψ and $\bar{\psi}$ are the "left" and "right" components of the Fermi fields. Actually, the theories (1.4) and (1.15) are related. Witten [2] has shown that the Wess-Zumino model with $G = O(N)$ and $k = 1$ is equivalent to the free fermion theory (1.15), the fields (1.10) being equal to the corresponding currents of (1.15):

$$J_{\alpha\alpha'}(z) = :\psi_\alpha(z)\psi_{\alpha'}(z):, \qquad \bar{J}_{\beta\beta'}(\bar{z}) = :\bar{\psi}_\beta(\bar{z})\bar{\psi}_{\beta'}(\bar{z}):. \tag{1.16}$$

This result follows directly from the observation that the current of the models (1.14) and (1.15) satisfies the same algebra. Moreover, Witten suggested the local expression for the field $g(\xi) = g_{\alpha\beta}(\xi)$ in terms of the Fermi fields $\psi, \bar{\psi}$:

$$Mg_{\alpha\beta}(z, \bar{z}) = :\psi_\alpha(z)\bar{\psi}_\beta(\bar{z}):, \tag{1.17}$$

where M is the mass parameter dependent on the regularization scheme. The same formulae (with slight specifications, see [8] and sect. 4) relate the model (1.4) with $G = U(N)$, $k = 1$ to the theory of N-component charged Fermi fields. Formulae (1.16) and (1.17) are Witten's non-abelian bosonization rules.

Since the conformal anomaly vanishes at the fixed point (1.3), the Wess-Zumino theory (1.4) is invariant also with respect to the infinite-dimensional group of coordinate transformations:

$$z \to \zeta(z), \qquad \bar{z} \to \bar{\zeta}(\bar{z}), \tag{1.18}$$

with arbitrary analytic functions ζ and $\bar{\zeta}$; these transformations constitute the conformal group of two-dimensional space. In conformal quantum theory the local fields, like $g(z, \bar{z})$ in (1.4), can acquire anomalous dimensions, i.e. they are transformed as

$$g(z, \bar{z}) \to \left(\frac{d\zeta}{dz}\right)^{\Delta} \left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\bar{\Delta}} g(\zeta, \bar{\zeta}) \tag{1.19}$$

(with real positive Δ and $\bar{\Delta}$) under the substitutions (1.18). Obviously, for the spinless field $g(z, \bar{z})$ of (1.4) Δ and $\bar{\Delta}$ must be equal.

In this paper we investigate the Wess-Zumino model with an arbitrary integer k. Using the infinite-dimensional symmetry (1.6), we compute exactly the anomalous dimensions and develop the method for computing the multipoint correlation functions (Green's functions)

$$\langle g(z_1, \bar{z}_1) \ldots g(z_N, \bar{z}_N) \rangle; \tag{1.20}$$

some of them will also be constructed explicitly. We apply the technique similar to that proposed in [9] for the conformal field theories in two dimensions. In particular, we will find the field $g(z, \bar{z})$ (as well as some other "composite" fields of the theory) to be associated with the degenerate representation of symmetry algebra (semidirect product of current algebra and Virasoro algebra) of the model (1.4); therefore the correlation functions (1.20) satisfy special linear differential equations. Together with the general requirements of crossing symmetry, these equations determine the functions (1.20) completely. In the particular case $G = U(N)$ or $O(N)$ and $k = 1$, the relations (1.17) follow from our result. At $k > 1$ the correlation functions (1.20) turn out to be more complicated and the theory (1.4) can hardly be connected with free fields in any local way. In fact, the field $g(\xi)$ and other local fields possess nontrivial anomalous dimensions in the theory with $k > 1$.

Polyakov and Wiegmann [3] have managed to solve the model (1.1) (with arbitrary λ^2) exactly by means of the Bethe ansatz technique. Our approach is based completely on the symmetry (1.6) and conformal symmetry and therefore is restricted to the fixed-point theory (1.4). However, our approach provides much more detailed information about the theory (1.4); in particular, the computation of correlation functions like (1.20) remain beyond the powers of the Bethe ansatz method. It is also worth noting that the correlation functions of the model (1.4) studied in this paper describe exact infrared asymptotics of the general model (1.1).

2. General properties of conformal quantum field theory invariant with respect to current algebra

The stress-energy tensor $T_{\mu\nu}(\xi)$ of a conformal quantum field theory satisfies, besides the usual equation $\partial_\mu T^{\mu\nu}(\xi) = 0$, the zero trace condition $T^\mu_\mu(\xi) = 0$. In two dimensions these two equations can be reduced to

$$\partial_{\bar z} T = 0, \qquad \partial_z \bar T = 0, \tag{2.1}$$

where

$$T = T_{11} - T_{22} + 2iT_{12},$$
$$\bar T = T_{11} - T_{22} - 2iT_{12}. \tag{2.2}$$

In view of (2.1) we shall write

$$T = T(z), \qquad \bar T = \bar T(\bar z). \tag{2.3}$$

The fields (2.2) represent the generators of the infinitesimal conformal transformations

$$z \to z + \varepsilon(z), \tag{2.4a}$$

$$\bar z \to \bar z + \bar\varepsilon(\bar z), \tag{2.4b}$$

in the field theory, the field T being associated with the infinitesimal substitutions

(2.4a) of the variable z and $\bar{T}$ plays the same role for $\bar{z}$ [9]. If, besides the conformal symmetry, the field theory is invariant with respect to the transformations (1.6), there are also the local fields

$$J(z)=J^a(z)t^a, \qquad \bar{J}(z)=\bar{J}^a(\bar{z})t^a, \tag{2.5}$$

satisfying eqs. (1.9) and representing the generators of $G(z)$ and $G(\bar{z})$ respectively. These statements have the following precise meaning. Consider correlation functions of the form

$$\langle T(z)A_{j_1}(z_1,\bar{z}_1)\dots A_{j_N}(z_N,\bar{z}_N)\rangle, \tag{2.6a}$$

$$\langle J^a(z)A_{j_1}(z_1,\bar{z}_1)\dots A_{j_N}(z_N,\bar{z}_N)\rangle, \tag{2.6b}$$

where $A_{j_k}(z_k,\bar{z}_k)$ are the arbitrary local fields. These correlators are single-valued analytic functions of z, possessing poles at $z=z_1,z_2,\dots,z_N$. The order and residue of each of these poles, say z_k, are determined by the transformation properties of the corresponding field $A_{j_k}(z_k,\bar{z}_k)$ with respect to the conformal (2.4a) and gauge (1.13a) transformations*. In fact, the following relations are valid:

$$\delta_\varepsilon A_j(z,\bar{z})=\oint_{C_z}T(\zeta)\varepsilon(\zeta)A_j(z,\bar{z})\,d\zeta,$$

$$\delta_\omega A_j(z,\bar{z})=\oint_{C_z}J^a(\zeta)\omega^a(\zeta)A_j(z,\bar{z})\,d\zeta, \tag{2.7}$$

where $\delta_\varepsilon A_j$ and $\delta_\omega A_j$ are the variations of the field A_j under the infinitesimal transformations (2.4a) and (1.13a); the integration contour surrounds the point $\zeta=z$. Formulae (2.7) are understood as the relations between correlation functions. The same equations, with the substitutions $T\to\bar{T}$, $J\to\bar{J}$, are valid for the variations $\delta_{\bar{\varepsilon}}$ and $\delta_{\bar{\omega}}$.

Generally, the variations of the fields $T(z)$ and $J^a(z)$ themselves are given by the formulae

$$\delta_\varepsilon T(z)=\varepsilon(z)T'(z)+2\varepsilon'(z)T(z)+\tfrac{1}{12}c\varepsilon'''(z), \tag{2.8a}$$

$$\delta_\varepsilon J^a(z)=\varepsilon(z)J^{a\prime}(z)+\varepsilon'(z)J^a(z), \tag{2.8b}$$

$$\delta_\omega J^a(z)=f^{abc}\omega^b(z)J^c(z)+\tfrac{1}{2}k\omega^{a\prime}(z), \tag{2.8c}$$

* Here and below we constantly consider the correlators functions of the complex coordinates $\xi\in\mathbf{C}^2$. In the space $\mathbf{C}^2$ the coordinates z and $\bar{z}$ (1 7) are independent complex variables and the conformal group (like the group (1 6)) can be considered as the direct product $\Gamma(z)\times\Gamma(\bar{z})$ of two (identical) groups Γ of analytic substitutions of one variable.

where the prime denotes the derivative. The variations $\delta_{\bar{\varepsilon}}$ and $\delta_{\bar{\omega}}$ of the fields $\bar{T}$ and $\bar{J}$ are given by the same equations, whereas the variations $\delta_{\bar{\varepsilon}}$ and $\delta_{\bar{\omega}}$ of T and J vanish. The values of the real parameters c and k in (2.8) are not completely fixed by general principles. There are, however, the strong limitations of the values of these parameters: c must be positive and k must be an integer. It can be shown that otherwise the positivity condition of quantum field theory cannot be satisfied*. The equations (2.8) determine the algebra of the generators of the symmetry group in field theory.

According to (2.7), eqs. (2.8) can be rewritten in the form of operator product expansions:

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{1}{z-z'}T'(z') + \cdots , \tag{2.9a}$$

$$T(z)J^a(z') = \frac{1}{(z-z')^2}J^a(z') + \frac{1}{z-z'}J^{a\prime}(z') + \cdots , \tag{2.9b}$$

$$J^a(z)J^b(z') = \frac{k\delta^{ab}}{(z-z')^2} + \frac{f^{abc}}{z-z'}J^c(z') + \cdots , \tag{2.9c}$$

where the terms regular at $z \to z'$ are omitted from r.h.s.'s. The definition of the fields $T(z)$ and $J^a(z)$ (as well as $\bar{T}$ and $\bar{J}$) should be supplemented with the requirement of regularity at $z = \infty$, which is equivalent to the asymptotic conditions

$$T(z) \sim z^{-4}, \qquad J^a(z) \sim z^{-2} \qquad \text{as } z \to \infty . \tag{2.10}$$

Any local field $A_j(z, \bar{z})$ of the theory is an "isotopic" tensor corresponding to some finite-dimensional representation of the "left" and "right" (global) groups G. Besides that, it is characterized by the anomalous dimensions $(\Delta_j, \bar{\Delta}_j)$ describing its transformation

$$A_j \to \lambda^{\Delta_j}\bar{\lambda}^{\bar{\Delta}_j}A_j \tag{2.11}$$

under the (complex) dilatations $z \to \lambda z$, $\bar{z} \to \bar{\lambda}\bar{z}$. In fact, the difference $s_j = \Delta_j - \bar{\Delta}_j$ is the spin of the field A_j (the spin s_j of the local fields can take integer or half-integer values only) whereas the sum $d_j = \Delta_j + \bar{\Delta}_j$ coincides with the conventional anomalous dimension. Applying arguments similar to those presented in [9] for the conformal theory, one can prove that there are the fields (like in [9] we shall call them "primary fields") which transform according to (1.19) and (1.6) under

* At $0 < c < 1$ the positivity condition selects also an infinite discrete set of allowed values of c [11] This limitation is not significant here because in the Wess-Zumino model $c > 1$, see sect. 3

arbitrary conformal and gauge transformations. Let us note that the matrices $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ may correspond, in the general case, to different representations of G. Introducing the notation $\phi_l(z, \bar{z})$ for the primary fields and $(\Delta_l, \bar{\Delta}_l)$ for the corresponding dimensions, one can write down the singular terms of the operator product expansions:

$$T(\zeta)\phi_l(z,\bar{z}) = \frac{\Delta_l}{(\zeta - z)^2}\phi_l(z,\bar{z}) + \frac{1}{\zeta - z}\frac{\partial}{\partial z}\phi_l(z,\bar{z}) + \cdots, \tag{2.12a}$$

$$J^a(\zeta)\phi_l(z,\bar{z}) = \frac{t_l^a}{\zeta - z}\phi_l(z,\bar{z}) + \cdots, \tag{2.12b}$$

which are determined by the transformation properties of the field ϕ_l with respect to the infinitesimal transformations (2.4a) and (1.3a). Here t_l^a is the "left" representation of the generators of G for the field ϕ_l. Similar formulae (with the substitutions $\Delta_l \to \bar{\Delta}_l$ and $t_l^a \to \bar{t}_l^a$, where $\bar{t}_l^a$ corresponds to the "right" representation) are valid for the expansions of the products $\bar{T}\phi_l$ and $\bar{J}\phi_l$. Eqs. (2.12) allow one to determine explicitly the z-dependence of the correlation functions (2.6) provided all the fields A_j involved are primary ones:

$$\langle T(z)\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N)\rangle = \sum_{j=1}^{N}\left\{\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right\} \times\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N)\rangle, \tag{2.13a}$$

$$\langle J^a(z)\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N)\rangle = \sum_{j=1}^{N}\frac{t_j^a}{z-z_j}\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N)\rangle, \tag{2.13b}$$

where the matrices t_j^a are applied to the "left" isotopic indices of the field $\phi_j(z, \bar{z})$. The relations (2.13) are the Ward identities corresponding to the conformal and gauge symmetries. Combining (2.13) with asymptotic conditions (2.10) one can easily derive the well-known Ward identities:

$$\sum_{j=1}^{N}\left\{z_j^{n+1}\frac{\partial}{\partial z_j} + (n+1)\Delta_j z_j^n\right\}\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N)\rangle = 0, \tag{2.14a}$$

where $n = -1, 0, +1$ and

$$\sum_{j=1}^{N} t_j^a\langle\phi_1(z_1,\bar{z}_1)\ldots\phi_N(z_N,\bar{z}_N)\rangle = 0, \tag{2.14b}$$

which are manifestations of the invariance with respect to the regular subgroups

$SL_2 \subset \Gamma(z)$ of projective conformal transformations and $G \subset G(z)$ of global gauge transformations.

The variations δ_ε and δ_ω can be understood as some operators, defined by the r.h.s.'s of eqs. (2.7), acting on the fields A_j. It is convenient to expand the functions $\varepsilon(\zeta)$ and $\omega(\zeta)$ as a power series in $(\zeta - z)$ and to introduce the corresponding operators:

$$L_n A_j(z,\bar{z}) = \oint_{C_z} T(\zeta)(\zeta - z)^{n+1} A_j(z,\bar{z})\, d\zeta,$$

$$J_n^a A_j(z,\bar{z}) = \oint_{C_z} J^a(\zeta)(\zeta - z)^n t_j^a A_j(z,\bar{z})\, d\zeta. \tag{2.15}$$

The primary fields ϕ_l satisfy the equations

$$L_n\phi_l = J_n^a\phi_l = 0 \qquad \text{for } n > 0,$$

$$L_0\phi_l = \Delta_l\phi_l, \qquad J_0^a\phi_l = t_l^a\phi_l, \tag{2.16}$$

which are the direct consequence of (2.12). Eqs. (2.15) define the operators L_n and J_n^a with negative values of n as well as with positive ones. In general, the local fields $L_{-n}A_j$ and $J_{-n}^a A_j$ with $n > 0$ do not vanish. In particular, the operators L_{-1} and $\bar{L}_{-1}$ reduce to simple differential ones:

$$L_{-1}\phi_l(z,\bar{z}) = \frac{\partial}{\partial z}\phi_l(z,\bar{z}), \qquad \bar{L}_{-1}\phi_l(z,\bar{z}) = \frac{\partial}{\partial \bar{z}}\phi_l(z,\bar{z}). \tag{2.17}$$

Evidently, the regular terms omitted in the operator product expansions (2.12) can be expressed in terms of the fields $L_{-n}\phi_l$, $J_{-n}^a\phi_l$ with $n = 2, 3, \ldots$.

Due to the singular terms in (2.9), the operators L_n and J_n^a are not commutative but satisfy the relations

$$[L_n, L_m] = (n - m)L_{n+m} + \tfrac{1}{12}c(n^3 - n)\delta_{n+m}, \tag{2.18a}$$

$$[L_n, J_m^a] = -mJ_{n+m}^a, \tag{2.18b}$$

$$[J_n^a, J_m^b] = f^{abc}J_{n+m}^c + \tfrac{1}{2}kn\delta^{ab}\delta_{n+m,0}. \tag{2.18c}$$

The commutation relations (2.18a) and (2.18c) are known as Virasoro algebra and Kac-Moody algebra respectively. The complete algebra (2.18), which is the semidirect product of these algebras, will be denoted here as $\mathscr{Q}$. Obviously, the operators $\bar{L}_n$ and $\bar{J}_n^a$ defined by the same formulae as (2.15) except with the fields $\bar{T}$ and $\bar{J}$ constitute the same algebra which will be denoted by $\bar{\mathscr{Q}}$.

The complete system of local fields $\{A_j\}$ involved in the theory includes, besides the primary fields ϕ_l, all the fields of the form

$$L_{-n_1}\dots L_{-n_N}\bar{L}_{-\bar{n}_1}\dots\bar{L}_{-\bar{n}_{N'}}J^{a_1}_{-m_1}\dots J^{a_M}_{-m_M}\bar{J}^{b_1}_{-\bar{m}_1}\dots\bar{J}^{b_{M'}}_{-\bar{m}_{M'}}\phi_l, \tag{2.19}$$

with arbitrary positives $n, \bar{n}, m, \bar{m}$. Like in [9], we shall denote by $[\phi_l]_{\mathscr{Q}}$ the totality of the fields (2.19) associated with some primary field ϕ_l. This infinite set of fields corresponds, obviously, to the highest weight representation (Verma modulus) of the algebra $\mathscr{Q}$ (more precisely, $[\phi_l]_{\mathscr{Q}}$ is the direct product of the highest weight representations of $\mathscr{Q}$ and $\bar{\mathscr{Q}}$), the primary field ϕ_l being associated with the highest weight vector. The dimensions of the fields (2.19) are

$$\Delta_l^{\{n,m\}} = \Delta_l + \sum_{i=1}^{N} n_i + \sum_{i=1}^{M} m_i, \qquad \bar{\Delta}_l^{\{\bar{n},\bar{m}\}} = \bar{\Delta}_l + \sum_{i=1}^{N'} \bar{n}_i + \sum_{i=1}^{M'} \bar{m}_i, \tag{2.20}$$

and their isotropic properties are self-evident from (2.19). The complete set of fields

$$\{A_j\} = \bigoplus_l [\phi_l]_{\mathscr{Q}} \tag{2.21}$$

form the closed operator algebra [9].

It is worth noting that the fields $T(z), J(z), \bar{T}(\bar{z}), \bar{J}(\bar{z})$ are not primary ones; they belong to $[I]_{\mathscr{Q}}$, where I is the identity operator. Namely,

$$T(z) = L_{-2}I, \qquad J^a(z) = J^a_{-1}I,$$
$$\bar{T}(\bar{z}) = \bar{L}_{-2}I, \qquad \bar{J}^a(\bar{z}) = \bar{J}^a_{-1}I. \tag{2.22}$$

3. Wess-Zumino model

The relations presented in the previous section concern any quantum field theory with the symmetry algebra $\mathscr{Q} \times \bar{\mathscr{Q}}$. The expression (1.10) of currents in terms of the g-field can be considered as the peculiarity of the Wess-Zumino model. In the quantum theory these expressions have the following meaning. We assume that the complete set of fields $\{A_j\}$ contains the spinless primary field $g(z, \bar{z})$ (which corresponds to the representation t^a of the left and right global groups G and have the dimensions $\Delta_g = \bar{\Delta}_g = \Delta$) satisfying the equations

$$\kappa \frac{\partial}{\partial z} g(z, \bar{z}) = :J^a(z) t^a g(z, \bar{z}):, \tag{3.1a}$$

$$\kappa \frac{\partial}{\partial \bar{z}} g(z, \bar{z}) = :\bar{J}^a(\bar{z}) g(z, \bar{z}) t^a:, \tag{3.1b}$$

where κ is the numerical factor (which will be computed later) and the local products of fields on the r.h.s. of (3.1) are regularized in a particular way. Eqs. (3.1) can be understood as the special property of the operator product expansions of $J^a(\zeta)t^a g(z,\bar{z})$ and $\bar{J}^a(\bar{z})g(z,\bar{z})t^a$. Consider, for example, the first of these products which has the general form

$$J^a(\zeta)t^a g(z,\bar{z}) = \frac{c_g}{\zeta - z} g(z,\bar{z}) + \sum_{n=1}^{\infty} (\zeta - z)^{n-1} t^a J^a_{-n} g(z,\bar{z}), \tag{3.2}$$

where the constant c_g is defined as

$$t^a t^a = c_g I. \tag{3.3}$$

The operator coefficient accompanying the zero power of $(\zeta - z)$ in (3.2) has to coincide (up to a numerical factor) with the derivative $\partial_z g$, i.e.

$$J^a(\zeta)t^a g(z,\bar{z}) = \frac{c_g}{\zeta - z} g(z,\bar{z}) + \kappa \frac{\partial}{\partial z} g(z,\bar{z}) + \mathrm{O}(\zeta - z). \tag{3.4}$$

So the product in (3.1a) is defined as

$$:J^a(z)t^a g(z,\bar{z}): \stackrel{\text{def}}{=} \lim_{\zeta \to z} \left[J^a(\zeta) - \frac{t^a}{\zeta - z} \right] t^a g(z,\bar{z}). \tag{3.5}$$

A similar definition applies to the r.h.s of (3.1b).

The peculiarity of the operator product expansion (3.4) allows one to determine immediately the anomalous dimension Δ of the field g. Comparing (3.2) and (3.4) one gets

$$\chi \equiv (J^a_{-1}t^a - \kappa L_{-1})g = 0, \tag{3.6}$$

where (2.17) is taken into account. From the mathematical point of view, this relation means that the representation $[g]_{\mathfrak{g}}$ of the algebra (2.18) is degenerate, the field χ (defined by (3.6)) being associated with the "null vector". The field χ should satisfy the equations

$$L_0\chi = (\Delta + 1)\chi, \qquad J_0^a\chi = t^a\chi, \tag{3.7a}$$

$$L_n\chi = J_n^a\chi = 0 \qquad \text{for } n > 0, \tag{3.7b}$$

since otherwise eq. (3.6) does not make sense. Note that eqs. (3.7a) are satisfied identically whereas (3.7b) has to be solved for $n = 1$ only because the other eqs. (3.7b) can be derived from these by means of (2.18). Using (2.16) and (2.18) one can ascertain that eqs. (3.7b) are satisfied provided

$$c_g + 2\Delta\kappa = 0,$$

$$c_V + k + 2\kappa = 0, \tag{3.8}$$

where c_V is defined as

$$f^{acd}f^{bcd} = c_V\delta^{ab}. \tag{3.9}$$

Eqs. (3.8) provide the values of the anomalous dimension of the field g:

$$\Delta = \frac{c_g}{c_V + k}, \tag{3.10}$$

and of the parameter κ in (3.1):

$$\kappa = -\tfrac{1}{2}(c_V + k). \tag{3.11}$$

There is another way of deriving eq. (3.6). In the classical Wess-Zumino theory the stress-energy tensor is expressed quadratically in terms of the currents (1.10). Let us assume that a similar relation holds in the quantum theory, i.e.

$$2\kappa T(z) = :J^a(z)J^a(z):,$$

$$2\kappa\bar{T}(\bar{z}) = :\bar{J}^a(\bar{z})\bar{J}^a(\bar{z}):, \tag{3.12}$$

where the numerical parameter κ coincides, as we shall see below, with (3.11). This assumption can be considered as another definition of the Wess-Zumino model, equivalent to (3.1). Like (3.1), the relations (3.12) have to be understood in terms of the operator product expansions. For instance, the expansion of the product $J^a(z)J^a(z')$ has the following nonvanishing terms at $z \to z'$:

$$J^a(z)J^a(z') = \frac{kD}{(z-z')^2} + 2\kappa T(z') + O(z-z'), \tag{3.13}$$

where $D = \delta^{aa}$ is the dimension of the group G. Evaluating the multipoint correlation functions

$$\langle J^{a_1}(z_1)\dots J^{a_N}(z_N)\rangle, \tag{3.14}$$

and performing the expansion (3.13) one can find that the definition (3.13) of the stress-energy tensor is consistent with (2.9) only if

$$c = \frac{kD}{c_V + k}, \tag{3.15}$$

and the parameter κ is given by (3.11). Formula (3.13) is equivalent to the following relation between the generators (2.15):

$$2\kappa L_n = \sum_{m=-\infty}^{\infty} :J_m^a J_{n-m}^a:, \tag{3.16}$$

where the symbol : : denotes the conventional normal ordering: the operators J_n with negative n are always placed to the left of the operators with $n > 0$*. Applying the eq. (3.16) with $n = -1$ to the field g one gets exactly (3.6). Note that in this way of reasoning no particular properties of the field g are specified. Therefore, the equation

$$(J_{-1}^a t_l^a - \kappa L_{-1})\phi_l = 0 \tag{3.17}$$

is satisfied with any primary field ϕ_l as well. Hence any primary field ϕ_l in the Wess-Zumino theory is degenerate and its dimensions are given by

$$\Delta_l = \frac{c_l}{c_V + k}, \qquad \bar{\Delta}_l = \frac{\bar{c}_l}{c_V + k}, \tag{3.18}$$

where $c_l = t_l^a t_l^a$, $\bar{c}_l = \bar{t}_l^a \bar{t}_l^a$. In particular, any primary field which is scalar with respect to the left and right gauge group both have vanishing dimensions and therefore it is proportional to the identity operator I. It is also worth noting that all the fields (2.19) belonging to any representation $[\phi_l]_{\mathcal{G}}$ are expressed (by means of (3.16)) in terms of the fields

$$J_{-m_1}^{a_1} \dots J_{-m_N}^{a_N} \bar{J}_{-\bar{m}_1}^{b_1} \dots \bar{J}_{-\bar{m}_M}^{b_M} \phi_l. \tag{3.19}$$

Therefore in the Wess-Zumino theory the representations $[\phi_l]_{\mathcal{G}}$ of the algebra $\mathcal{G} \times \bar{\mathcal{G}}$ are in fact the highest weight representations of the current algebra (2.18c) only; they can be denoted by $[\phi_l]_J$.

As in general conformal theory [9], the correlation functions (1.20) of the degenerate field g satisfy some linear differential equations. Consider the relation

$$t_i^a \langle J^a(z) g(z_1, \bar{z}_1) \dots g(z_N, \bar{z}_N) \rangle = \left\{ \frac{c_g}{z - z_i} + \sum_{j \neq i}^{N} \frac{t_i^a t_j^a}{z - z_j} \right\} \langle g(z_1, \bar{z}_1) \dots g(z_N, \bar{z}_N) \rangle, \tag{3.20}$$

which is a direct consequence of the Ward identity (2.13b); here the matrices t_i^a act

* The fact that the Virasoro algebra belongs to the enveloping algebra of a Kac-Moody algebra is well known by mathematicians [12]. Earlier, it had been discovered in the study of two-dimensional field theory models [10].

on the left indices of the field $g(z_i, \bar{z}_i)$. Substituting the operator product expansion (3.4) on the l.h.s. of (3.20) and taking the limit $z \to z_i$ one gets the equation

$$\left\{ \kappa \frac{\partial}{\partial z_i} - \sum_{j \neq i}^{N} \frac{t_i^a t_j^a}{z_i - z_j} \right\} \langle g(z_1, \bar{z}_1) \dots g(z_N, \bar{z}_N) \rangle = 0 . \qquad (3.21)$$

Since i can take any of N values, $i = 1, 2, \dots, N$, we have in fact the system of linear differential equations. The same equations with substitutions $z \to \bar{z}$, $t^a \to \bar{t}^a$ ($\bar{t}^a$ acting on g from the right) are also valid. Clearly, the equations (3.21) with obvious modifications are satisfied by any correlation function

$$\langle \phi_{l_1}(z_1, \bar{z}_1) \dots \phi_{l_N}(z_N, \bar{z}_N) \rangle \qquad (3.22)$$

of primary fields ϕ_l in the Wess-Zumino theory. The correlation functions (1.20) can be computed as the solutions of these differential equations with appropriate analytical characteristics; an example of this computation is given in the next section.

Let us consider the operator expansion of the product $g(z, \bar{z}) g(0,0)$. One could expect, as in [9], the following form of this expansion:

$$g(z, \bar{z}) g(0,0) = \sum_l C_{gg}^{(l)} z^{\Delta_l - 2\Delta} \bar{z}^{\bar{\Delta}_l - 2\Delta} [\phi_l(0,0) + \cdots] , \qquad (3.23)$$

where $C_{gg}^{(l)}$ are numerical "structure constants" and ϕ_l denotes some primary fields of the theory having the dimensions $(\Delta_l, \bar{\Delta}_l)$. Here we omitted, in each term of the sum (3.23), the infinite power series in z and $\bar{z}$ with the fields belonging to $[\phi_l]_J$ as the coefficients. What kinds of primary fields could appear in (3.23)? The product $g(z, \bar{z}) g(0,0)$ transforms as the tensor product of two irreducible representations of (for instance) the left global group G. It can be decomposed in the set of some irreducible representations:

$$g(z, \bar{z}) g(0,0) = \sum_l P_l \{ g(z, \bar{z}) g(0,0) \} , \qquad (3.24)$$

where P_l are the projectors (acting on the left indices of the product $g \otimes g$) each selecting the subspace of l's representation. Let us substitute the expansion (3.23) for some pair of fields, say $g(z_1, \bar{z}_1) g(z_2, \bar{z}_2)$, in (3.21) and take into account the most singular (at $z_1 \to z_2$) contribution of each term of (3.23). Assuming $P_l \phi_l = \phi_l$ and using the identity

$$\{ t^a g(z, \bar{z}) \} \{ t^a g(0,0) \} = \sum_l \tfrac{1}{2} (c_l - 2c_g) P_l \{ g(z, \bar{z}) g(0,0) \} , \qquad (3.25)$$

where $c_l = t_l^a t_l^a$, one gets the characteristic equation

$$-\kappa(\Delta_l - 2\Delta) = \tfrac{1}{2}(c_l - 2c_g) \tag{3.26}$$

for (3.21) and recovers (3.18). So the primary fields ϕ_l corresponding to the representations l entering the decomposition (3.24) could only appear in the operator product expansion (3.23). Surely this result was obvious before, and we presented the above computation as the self-consistency check.

Note that some of the coefficients $C^{(l)}$ in the expansions like (3.23) could vanish. In fact, there are special selection rules which forbid most of the a priori conceivable primary fields to appear in the operator algebra, generated by the fields g*. Apparently, in the general case a finite number of primary fields ϕ_l form the closed operator algebra of the Wess-Zumino model (a similar phenomenon takes place in the "minimal" conformal theories introduced in [9]). In the next section we shall meet an example ($k = 1$) of this situation.

It is worth saying some words about the composite fields of the model (1.4). Let us consider the primary field $\phi_1^{ab}(z, \bar{z})$, which transforms as adjoint representations of left and right groups G. According to (3.18) its dimensions are

$$\Delta_1 = \bar{\Delta}_1 = \frac{c_V}{c_V + k}\,. \tag{3.27}$$

This field is naturally identified with the composite field

$$\phi_1^{ab} = \mathrm{tr}(g^{-1}t^a g t^b)\,. \tag{3.28}$$

The fields

$$K^a = J^b_{-1}\phi_1^{ba}, \qquad \bar{K}^a = \bar{J}^b_{-1}\phi_1^{ab}, \tag{3.29}$$

have the dimensions $(\Delta_1 + 1, \Delta_1)$ and $(\Delta_1, \Delta_1 + 1)$ respectively. They transform according to the formulae

$$\delta_{\bar{\omega}} K^a(z, \bar{z}) = f^{abc}\bar{\omega}^b(\bar{z})K^c(z, \bar{z})\,,$$

$$\delta_\omega K^a(z, \bar{z}) = \tfrac{1}{2}k\omega^b(z)\phi_1^{ba}(z, \bar{z})\,. \tag{3.30}$$

Therefore they apparently coincide with the "wrong currents" of the model (1.4)

$$K^a \sim \mathrm{tr}(t^a g^{-1}\partial_z g)\,, \qquad \bar{K}^a \sim \mathrm{tr}(t^a \partial_{\bar{z}} g g^{-1})\,, \tag{3.31}$$

* These selection rules have the following origin In a general integer case of k, the highest weight representation $[g]_J$ contains null vectors Like (3 6), these null-vectors give rise to some extra matrix equations for the correlation functions (1 20) which, contrary to (3 21), contain no derivatives The selection rules come from the consistency condition of these matrix equations and (3 21)

Note that the "currents" (3.31) possess anomalous dimensions and therefore in conformal theory they cannot be conserved. Finally, the field

$$S(z,\bar{z}) = J^a_{-1}\bar{J}^b_{-1}\phi_1^{ab}, \tag{3.32}$$

which has the dimensions (Δ_1+1, Δ_1+1), corresponds to the lagrangian density

$$S \sim \mathrm{tr}\left(\partial_\mu g^{-1}\partial_\mu g\right). \tag{3.33}$$

This enables one to predict the slope of the β-function of the model (1.1) at the fixed point (1.3):

$$\left.\frac{\mathrm{d}\beta(\lambda^2,k)}{\mathrm{d}\lambda^2}\right|_{\lambda^2=4\pi/k} = \frac{2c_V}{c_V+k}. \tag{3.34}$$

At $k\to\infty$ this equation is in agreement with the one-loop result [2].

4. Correlation functions

The projective Ward identities (2.14a) determine the two- and three-point correlation functions up to a numerical factor. Here we shall compute the four-point functions

$$G(z_i,\bar{z}_i) = \langle g(z_1,\bar{z}_1)g^{-1}(z_2,\bar{z}_2)g^{-1}(z_3,\bar{z}_3)g(z_4,\bar{z}_4)\rangle \tag{4.1}$$

for the Wess-Zumino model, combining the differential equations (3.21) with the general requirement of crossing symmetry. In fact, the crossing symmetry of four-point functions ensures the associativity of the complete operator algebra* and therefore the self-consistency of the field theory in general.

Firstly, let us note that the correlation function (4.1) depends essentially on two variables; due to the Ward identities (2.14a) it can be represented in the form

$$G(z_i,\bar{z}_i) = \left[(z_1-z_4)(z_2-z_3)(\bar{z}_1-\bar{z}_4)(\bar{z}_2-\bar{z}_3)\right]^{-2\Delta}G(x,\bar{x}), \tag{4.2}$$

where x and $\bar{x}$ are the anharmonic quotients

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}, \qquad \bar{x} = \frac{(\bar{z}_1-\bar{z}_4)(\bar{z}_3-\bar{z}_4)}{(\bar{z}_1-\bar{z}_4)(\bar{z}_3-\bar{z}_2)}, \tag{4.3}$$

and Δ is the dimension (3.10) of the field g. Further computation depends on the choice of the group G. Here we elaborate the case of unitary groups.

* See ref. [9] for the details. The bootstrap approach based on the operator algebra was originally proposed by Polyakov [14].

Let $G = SU(N)$, $g(z, \bar{z})$ being the $N \times N$ unitary matrix, $\det g = 1$, which transforms as the fundamental representation of $SU(N) \times SU(N)$. Let us denote by $\alpha_i(\beta_i)$ the tensor indices of the fields $g(z_i, \bar{z}_i)$ corresponding to the left (right) $SU(N)$ group; this way, in (4.1) we imply $g(z_i, \bar{z}_i) = g_{\alpha_i}^{\beta_i}(z_i, \bar{z}_i)$ and $g^{-1}(z_i, \bar{z}_i) = g_{\beta_i}^{-1\alpha_i}(z_i, \bar{z}_i)$. The correlation function (4.2) enjoys the $SU(N) \times SU(N)$ invariant decomposition

$$G(x, \bar{x}) = \sum_{A, B = 1,2} (I_A)(\bar{I}_B) G_{AB}(x, \bar{x}), \tag{4.4}$$

with the scalar coefficients $G_{AB}(x, \bar{x})$. The matrices I and $\bar{I}$ are defined as

$$I_1 = \delta_{\alpha_1}^{\alpha_2} \delta_{\alpha_3}^{\alpha_4}, \qquad \bar{I}_1 = \delta_{\beta_2}^{\beta_1} \delta_{\beta_3}^{\beta_4},$$

$$I_2 = \delta_{\alpha_1}^{\alpha_4} \delta_{\alpha_3}^{\alpha_2}, \qquad \bar{I}_2 = \delta_{\beta_2}^{\beta_4} \delta_{\beta_3}^{\beta_1}. \tag{4.5}$$

The correlation function (4.1) satisfies the differential equations (3.21) and the same equations with respect to $\bar{z}$. By means of direct computation these equations can be converted to the form*

$$\frac{\partial G}{\partial x} = \left[\frac{1}{x} P + \frac{1}{x-1} Q\right] G,$$

$$\frac{\partial G}{\partial \bar{x}} = G\left[\frac{1}{\bar{x}} P^t + \frac{1}{\bar{x}-1} Q^t\right], \tag{4.6}$$

where G denotes 2×2 matrix G_{AB}, the matrices P and Q are given by

$$P = \frac{1}{2N\kappa}\begin{pmatrix} N^2-1 & N \\ 0 & -1 \end{pmatrix}, \qquad Q = \frac{1}{2N\kappa}\begin{pmatrix} -1 & 0 \\ N & N^2-1 \end{pmatrix}, \tag{4.7}$$

and the mark t means the matrix transposition. The parameter κ, the same as in (3.11), in the case under consideration is equal to

$$\kappa = -\tfrac{1}{2}(N+k). \tag{4.8}$$

The general solution of eqs. (4.6) can be given in terms of hypergeometric functions; it is conveniently represented in the form

$$G_{AB}(x, \bar{x}) = \sum_{p, q = 0,1} U_{pq} \mathcal{F}_A^{(p)}(x) \mathcal{F}_B^{(q)}(\bar{x}), \tag{4.9}$$

* Note that precisely eqs (4.6) appeared earlier in the paper by Dashen and Frishman [10], in their study of the conformally invariant solution of the SU(N) Thirring model

with arbitrary constants U_{pq} and the functions $\mathcal{F}$ given by

$$\mathcal{F}_1^{(0)}(x) = x^{-2\Delta}(1-x)^{\Delta_1-2\Delta}F\left(-\frac{1}{2\kappa},\frac{1}{2\kappa},1+\frac{N}{2\kappa},x\right),$$

$$\mathcal{F}_2^{(0)}(x) = -(2\kappa+N)^{-1}x^{1-2\Delta}(1-x)^{\Delta_1-2\Delta}F\left(1-\frac{1}{2\kappa},1+\frac{1}{2\kappa},2+\frac{N}{2\kappa},x\right), \tag{4.10a}$$

$$\mathcal{F}_1^{(1)}(x) = x^{\Delta_1-2\Delta}(1-x)^{\Delta_1-2\Delta}F\left(-\frac{N-1}{2\kappa},-\frac{N+1}{2\kappa},1-\frac{N}{2\kappa},x\right),$$

$$\mathcal{F}_2^{(1)}(x) = -Nx^{\Delta_1-2\Delta}(1-x)^{\Delta_1-2\Delta}F\left(-\frac{N-1}{2\kappa},-\frac{N+1}{2\kappa},-\frac{N}{2\kappa},x\right). \tag{4.10b}$$

Here Δ is the dimension (3.10) of the field g:

$$\Delta = \frac{N^2-1}{2N(N+k)}, \tag{4.11}$$

whereas

$$\Delta_1 = \frac{N}{N+k} \tag{4.12}$$

is the dimension of the composite field (3.28). Note that the functions $\mathcal{F}_A^{(p)}(x)$ play here a role similar to the "conformal blocks" in conformal theory [9]; the functions $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ in (4.9) describe the "s-channel" contribution of all the fields belonging to the representations $[I]_J$ and $[\phi_1^{ab}]_J$, respectively. It is not out of place to name them the current blocks.

Now we have to take into account the local properties of the field g and impose the crossing symmetry (4.2); this requirement proves to determine the constants U_{pq} in (4.9). According to the local properties, the correlation functions must be single-valued while considered in the euclidian domain $\bar{x} = x^*$, where the star denotes the complex conjugation. Concentrating attention in the vicinity of the point $x = \bar{x} = 0$ one immediately recognizes that (4.9) is compatible with this requirement only if

$$U_{10} = U_{01} = 0. \tag{4.13}$$

The crossing symmetry of the four-point function (4.1) requires

$$G_{AB}(x,\bar{x}) = \sum_{A',B'=1,2} E_{AA'}G_{A'B'}(1-x,1-\bar{x})E_{B'B}, \tag{4.14}$$

where $E_{12} = E_{21} = 1$, $E_{11} = E_{22} = 0$. Let us substitute (4.9) into (4.14) and use the relation

$$\mathcal{F}_A^{(p)}(x) = \sum_{q,A'} C_q^p E_{AA'} \mathcal{F}_{A'}^{(q)}(1-x), \tag{4.15}$$

which can be verified directly. The elements of the "crossing matrix" C_q^p in (4.15) are given by

$$C_0^0 = -C_1^1 = N\frac{\Gamma(-N/2\kappa)\Gamma(N/2\kappa)}{\Gamma(-1/2\kappa)\Gamma(1/2\kappa)},$$

$$C_0^1 = -N\frac{\Gamma^2(-N/2\kappa)}{\Gamma(-(N-1)/2\kappa)\Gamma(-(N+1)/2\kappa)}, \qquad C_0^1 C_1^0 + C_0^0 C_1^1 = 1. \tag{4.16}$$

Eqs. (4.14) are satisfied provided $U_{11} = hU_{00}$:

$$h = \frac{1}{N^2}\frac{\Gamma((N-1)/(N+k))\Gamma((N+1)/(N+k))}{\Gamma((k+1)/(N+k))\Gamma((k-1)/(N+k))}\frac{\Gamma^2(k/(N+k))}{\Gamma^2(N/(N+k))}. \tag{4.17}$$

Finally we obtain

$$G_{AB}(x,\bar{x}) = M^{-8\Delta}\{\mathcal{F}_A^{(0)}(x)\mathcal{F}_B^{(0)}(\bar{x}) + h\mathcal{F}_A^{(1)}(x)\mathcal{F}_B^{(1)}(\bar{x})\}, \tag{4.18}$$

where the overall factor is the matter of the g-field normalization; the normalization in (4.18) corresponds to the two-point function

$$\langle g_{\alpha_1}^{\beta_1}(z,\bar{z})g_{\beta_2}^{-1\alpha_1}(0,0)\rangle = M^{-4\Delta}\delta_{\alpha_1}^{\alpha_2}\delta_{\beta_1}^{\beta_2}(z\bar{z})^{-2\Delta}. \tag{4.19}$$

In the general case the function (4.18) possesses the power-like singularities at $x = \infty$, $\bar{x} = \infty$, which correspond to the contributions of the composite fields

$$\phi_{\mathrm{A}} = g_{\{\alpha_1}^{\{\beta_1}g_{\alpha_2\}}^{\beta_2\}}(z,\bar{z}), \tag{4.20a}$$

$$\phi_{\mathrm{S}} = g_{[\alpha_1}^{[\beta_1}g_{\alpha_2]}^{\beta_2]}(z,\bar{z}), \tag{4.20b}$$

in the operator product expansion of $g(z_1,\bar{z}_1)g(z_2,\bar{z}_2)$ in (1.1). Here the braces (square brackets) denote the antisymmetrization (symmetrization). The dimensions of these fields are

$$\Delta_{\mathrm{A}} = \frac{(N-2)(N+1)}{N(N+k)}, \qquad \Delta_{\mathrm{S}} = \frac{(N+2)(N-1)}{N(N+k)}, \tag{4.21}$$

in agreement with (3.18).

Note that at $k=1$ the second term in (4.18) vanishes and the function (4.4) becomes*

$$G(x,\bar{x}) = [x\bar{x}(1-x)(1-\bar{x})]^{1/N} \times\left[(I_1)\frac{1}{x}+(I_2)\frac{1}{1-x}\right]\left[(\bar{I}_1)\frac{1}{\bar{x}}+(\bar{I}_2)\frac{1}{1-\bar{x}}\right]. \tag{4.22}$$

Obviously, this result has the following meaning. Let us consider the field

$$\tilde{g}(z,\bar{z}) = e^{i\sqrt{4\pi/N}\gamma\varphi(z,\bar{z})}g(z,\bar{z}), \tag{4.23}$$

where g is the above SU(N) Wess-Zumino field and φ is the free massless boson field so that (4.23) corresponds to the U(N) = SU(N) × U(1) group. In fact, the parameter γ can be chosen arbitrarily and the dimension of the field $\tilde{g}$ is

$$\tilde{\Delta} = \Delta + \Delta(\gamma), \tag{4.24}$$

where

$$\Delta(\gamma) = \gamma^2/2N. \tag{4.25}$$

The four-point function $\tilde{G}(z_i, \bar{z}_i)$ of the fields (4.23) is given by

$$\tilde{G}(x,\bar{x}) = [x\bar{x}(1-x)(1-\bar{x})]^{-2\Delta(\gamma)}G(x,\bar{x})M^{-4\Delta(\gamma)}, \tag{4.26}$$

and under the choice

$$\gamma = 1 \tag{4.27}$$

coincides with that of the bilinears of the free massless charged Fermi fields:

$$M\tilde{g}_\alpha^\beta(z,\bar{z}) = :\psi_\alpha(z)\bar{\psi}^{+\beta}(\bar{z}):,$$
$$M\tilde{g}_\beta^{-1\alpha}(z,\bar{z}) = :\bar{\psi}_\beta(\bar{z})\psi^{+\alpha}(z):, \tag{4.28}$$

governed by the action

$$S_f(\psi,\bar{\psi}) = \int\left[\psi^{+\alpha}\partial_{\bar{z}}\psi_\alpha + \bar{\psi}^{+\beta}\partial_z\bar{\psi}_\beta\right]d^2\xi. \tag{4.29}$$

* In this case the operators (3.28) and (4.20b) decouple from the operator algebra generated by the field g, this is the simplest example of the selection rules mentioned in the previous section

Clearly, this result holds for any multipoint correlation functions (1.20). In fact, there is no need in computing the multipoint functions to prove this. It is sufficient to verify that the stress-energy tensor

$$T_{\mathrm{f}}(z)=\sum_{\alpha=1}^{N}:\psi^{+\alpha}\partial_z\psi_\alpha: \tag{4.30}$$

is related to the fermion currents $J^a=:\psi^+t^a\psi:$ and $J=:\psi^+\psi:$ as [10]

$$-T_{\mathrm{f}}(z)=\frac{1}{N+1}:J^a(z)J^a(z):+\frac{1}{2N}:J(z)J(z):, \tag{4.31}$$

and to note that the singlet current J is expressed in terms of the free massless boson field

$$J(z)=i\sqrt{N}\,\partial_z\varphi(z,\bar{z}). \tag{4.32}$$

Note also that (4.28) remains valid at arbitrary values of γ in (4.23) provided the ψ's are understood as the fermions of the N-component Thirring model with the isoscalar current coupling

$$S_{\mathrm{f}}^{(\gamma)}(\psi,\bar{\psi})=S_{\mathrm{f}}(\psi,\bar{\psi})+\frac{\gamma^2-1}{2\gamma^2}\int J(z)\bar{J}(\bar{z})\,\mathrm{d}^2\xi. \tag{4.33}$$

Clearly, the relation (1.17) corresponding to the model (1.4) with $G=\mathrm{O}(N)$ and $k=1$ can be proved in the same way.

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