Nuclear Physics B154 (1979) 1–20

QUANTUM FLUCTUATIONS OF INSTANTONS IN THE NON-LINEAR σ MODEL

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Received 24 January 1979

Quantum fluctuations of instantons in the two-dimensional non-linear O(3) σ model are computed completely. The calculation of the instanton contribution is reduced to the study of the Coulomb gas. It is shown that infrared divergences disappear if we take into account the contribution of instantons having arbitrary topological number.

1. Introduction

Some features of the two-dimensional non-linear σ model (or the continuum classical Heisenberg ferromagnet in two space dimensions) bear many similarities to a four-dimensional non-Abelian gauge theory. Both are scale invariant and asymptotically free. Both possess exact multi-instanton solutions.

Since the non-linear σ model in two dimensions is much simpler than the Yang–Mills theory in four dimensions, this model seems to be an ideal testing ground for any speculations about the effects of instantons in four-dimensional gauge theories.

The purpose of this paper is to compute the instanton contribution to the Green functions of the non-linear σ model. The model under consideration can be described by the Lagrangian

$$\mathscr{L}=\frac{1}{2f}\sum_{a=1}^{3}\partial_\mu\sigma^a\partial_\mu\sigma^a\,,$$

where $\sigma^a(x_0, x_1)$ is a three component unit vector: $\sum_{a=1}^{3}\sigma^a\sigma^a=1$; $\mu=0,1$.

The Euclidean Green functions of the model can be represented in the form

$$\int\phi(\sigma)\exp(-S)\prod \mathrm{d}\sigma(x)\Big/\int\exp(-S)\prod\mathrm{d}\sigma(x)\,, \tag{1.1}$$

where $\phi(\sigma)$ is an arbitrary functional and S denotes the Euclidean action:

$$S=\frac{1}{2f}\int\sum_{a=1}^{3}(\partial_\mu\sigma^a)^2\,\mathrm{d}^2x\,. \tag{1.2}$$

The functional integrals in (1.1) can be calculated by means of the steepest descent method in the weak coupling case. To apply this method we must know the extremals of the Euclidean action (1.2), instantons. These instantons were found in [1].

It is convenient to introduce the new field and space variables by the description of instantons. Namely, we use the complex variable $z = x_0 + ix_1$ instead of the space and time coordinates x_1, x_0 and the field

$$w = \frac{\sigma^1 + i\sigma^2}{1+\sigma^3} \tag{1.3}$$

obtained from the field $(\sigma^1, \sigma^2, \sigma^3)$, taking values on the unit sphere, by stereographic projection.

In terms of these variables

$$S = \frac{4}{f}\int (1+|w|^2)^{-2}(\partial_z w \partial_{\bar z}\bar w + \partial_{\bar z} w \partial_z \bar w)\, \mathrm{d}^2 x\,, \tag{1.4}$$

where

$$\partial_z = \tfrac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar z} = \tfrac{1}{2}(\partial_x + i\partial_y), \qquad \mathrm{d}^2 x = \tfrac{1}{2} i\, \mathrm{d}z\, \mathrm{d}\bar z\,.$$

The topological charge,

$$q = \frac{1}{\pi}\int (1+|w|^2)^{-2}(\partial_z w \partial_{\bar z}\bar w - \partial_{\bar z} w \partial_z \bar w)\, \mathrm{d}^2 x\,, \tag{1.5}$$

is an integer for every field having finite Euclidean action. (The integrand in (1.4) can be considered as the Jacobian of the map σ of the z-plane into the two-dimensional sphere S^2 so that q can be interpreted as the degree of this map.)

It follows from (1.4), (1.5) that

$$S = \frac{4\pi}{f} q + \frac{8}{f}\int (1+|w|^2)^{-2} |\partial_{\bar z} w|^2\, \mathrm{d}^2 x\,. \tag{1.6}$$

We see therefore that the minimal value of S on the fields having topological charge $q \geqslant 0$ is equal to $(4\pi/f)q$; this value is achieved on the fields satisfying $\partial_{\bar z} w = 0$. Hence the field

$$v(z) = P_0(z)/P_1(z)\,, \tag{1.7}$$

where $P_0(z)$, $P_1(z)$ are polynomials, is an instanton; the topological charge of the instanton (1.7) is equal to the maximal degree of $P_0(z)$, $P_1(z)$. It is convenient to write the general q-instanton solution (i.e., instanton having topological charge q) in the form

$$v(z) = c\frac{(z-a_1)\dots(z-a_q)}{(z-b_1)\dots(z-b_q)}\,. \tag{1.8}$$

(The instantons which cannot be represented in the form (1.8) are not essential because the set of these instantons has zero measure.) The paper is organized as follows.

In sect. 2 we review the renormalization procedure for the computation of the quantum fluctuations of instantons.

In sect. 3 we derive the instanton contribution up to a constant multiplier. (The constant multiplier will be calculated in appendix C.) We obtain the result that the contribution of instantons in Euclidean Green functions can be written in the form

$$I_q(\phi)=K^q(q!)^{-2}\int \phi(a,b,c)\,\mathrm{e}^{-\epsilon_q(a,b)}\frac{\mathrm{d}^2c}{\pi(1+|c|^2)^2}\prod_j^q \mathrm{d}^2a_j\,\mathrm{d}^2b_j\,, \tag{1.9}$$

where

$$\epsilon_q(a,b)=-\sum_{i<j}^q \log|a_i-a_j|^2-\sum_{i<j}^q \log|b_i-b_j|^2+\sum_{i,j}^q \log|a_i-b_j|^2\,.$$

The proof of (1.9) was sketched in [13].

The analysis of this answer will be performed in sect. 4. The expression (1.9) can be interpreted as the energy of a system consisting of q Coulomb particles at the points $a_1, ..., a_q$ and of q Coulomb particles of opposite charge at points $b_1, ..., b_q$. In other words the one-instanton solution can be considered as a pair of Coulomb particles of opposite charges and the q-instanton solution can be considered as q such pairs. (These Coulomb particles can be called instanton quarks.) We see that the study of the instanton contribution can be reduced to the study of the classical Coulomb system (CCS). This system was considered in many papers [5–9]. Using the results of these papers we find that the infrared troubles arising in the dilute instanton gas approximation disappear by correct calculation of the instanton contribution. There exists a phase transition in the CCS: this system is in the plasma phase for high temperatures and in the molecular phase for low temperatures. One can check that the temperature of the Coulomb system of instanton quarks is high in this sense, i.e., the instantons break up into unbounded instanton quarks. Therefore, the dilute instanton gas approximation cannot be reasonable in our case (this approximation corresponds to the molecular phase).

We use the Coleman–Fröhlich correspondence between the CCS, the massive Thirring model (MTM) and the sine-Gordon model (SGM) to calculate completely the instanton contribution in some Green functions of our theory. We find that these functions decrease exponentially. To explain this result we must remember that there exists Debye screening in the Coulomb plasma.

The instantons in a four-dimensional gauge theory in the group SU(2) are similar in many ways to the instantons in the model under consideration. In the gauge theory one can also define instanton quarks. Every instanton having topological number 1 can be considered as two instanton quarks and every q-instanton solution can be considered as q positive and q negative quarks. In the case when the gauge group is SU(n) there exist n types of instanton quark and every q-instanton solution consists of qn quarks (q quarks of each type). The SU(n) gauge theory is very similar to the two-dimensional theory with the non-linear fields taking values in $(n-1)$ dimensional complex projective space, $\mathrm{CP}(n-1)$, having the same structure of instanton quarks. The instanton contribution in the $\mathrm{CP}(n-1)$ theory will be described in a forthcoming paper [14]. One can think that general properties of the system of instanton quarks coincide with the properties of the corresponding system in the case of σ model.

In the appendices we prove some general results and use the ones in appendix C to complete the calculation of instanton determinants. Namely, in appendix A we analyse the connection between Pauli–Villars and proper time regularizations of determinants and in appendix B we give a method for calculation of the determinant if the eigenvalues and their degeneracies depend polynomially on the number of eigenvalues.

2. Regularization of the instanton contribution

Let us use the steepest descent method for calculation of the functional integral entering in (1.1). The contribution of the q-instanton solution in the numerator of (1.1) will be denoted by $J_q(\phi)$ so that the instanton contribution in (1.1) can be represented as

$$I(\phi)=\sum_q J_q(\phi)/\sum_q J_q(1)\,. \tag{2.1}$$

To avoid the infrared divergences we will suppose that the field $\sigma^a(x_0, x_1)$ is defined on a sphere with the usual metric

$$g_{\mu\nu}(x)=\delta_{\mu\nu}(1+(x_1^2+x_0^2)/4R^2)^{-2}\,.$$

To return to the Euclidean metric we take $R\to\infty$ at the final step of the calculation. We will always replace the field σ^a by the complex field (1.3). Using (1.6) we represent the action of the field $v(z)+\nu(z)$, where $v(z)$ is an instanton (1.7), $\nu(z)$ is a small variation of the instanton, in the form

$$S=\frac{4\pi}{f}q+\frac{8}{f}\int \rho_0^{-2}|\partial_{\bar z}\nu|^2\,\mathrm{d}^2x \tag{2.2}$$

(here $\rho_0=1+|v|^2$, the higher-order terms are omitted). To work with non-singular expressions we introduce the functions

$$\rho=\rho_0\prod_i |z-b_i|^2\,,$$

$$\tilde\nu=2\nu\rho^{-1}\prod_i (z-b_i)^2\,.$$

Then

$$S=\frac{4\pi}{f}q+\frac{2}{f}\int\bar{\tilde{\nu}}\Delta_v\tilde{\nu}\sqrt{g}\,\mathrm{d}^2x\,, \tag{2.3}$$

where

$$\Delta_v=-g^{-1/2}\rho\partial_z\rho^{-2}\partial_{\bar{z}}\rho\,. \tag{2.4}$$

The scalar product of variations ν_1, ν_2 can be written in the form

$$2\int\rho_0^{-2}(\bar{\nu}_1\nu_2+\nu_1\bar{\nu}_2)\sqrt{g}\,\mathrm{d}^2x=\tfrac{1}{2}\int(\bar{\tilde{\nu}}_1\tilde{\nu}_2+\tilde{\nu}_1\bar{\tilde{\nu}}_2)\sqrt{g}\,\mathrm{d}^2x$$

$$=\int\sum_{a=1}^{3}\delta_1\sigma^a\delta_2\sigma^a\sqrt{g}\,\mathrm{d}^2x\,.$$

This scalar product is real so that Δ_v acts in real Hilbert space. The q-instanton contribution is given by

The q-instanton contribution is given by

$$J_q(\phi)=f^{-2q}\exp\left(-\frac{4\pi q}{f}\right)\int\phi(v)\det{}^{-1/2}\left(\frac{2}{\pi}\Delta_v\right)\mathrm{d}\mu_0\,, \tag{2.5}$$

where $\mathrm{d}\mu_0$ denotes the measure on the manifold of instantons induced by the metric on this manifold. (We do not take into account the zero modes by definition of the determinants.) Let us introduce the complex parameters $\varphi_0=c$, $\varphi_1=a_1,\ldots,\varphi_q=a_q$, $\varphi_{q+1}=b_1,\ldots,\varphi_{2q}=b_q$ on the manifold of instantons of the form (1.8). Then the metric on this manifold can be written in the form

$$\sum_{k,j}\left(\int 4\rho_0^{-2}\left(\overline{\frac{\partial v}{\partial\varphi_k}}\right)\left(\frac{\partial v}{\partial\varphi_j}\right)\sqrt{g}\,\mathrm{d}^2x\right)\delta\bar{\varphi}_k\delta\varphi_j=\sum_{k,j}N_{kj}\delta\bar{\varphi}_k\delta\varphi_j\,, \tag{2.6}$$

where $\delta\varphi_j$ denotes the variation of parameter φ_j. The matrix N_{kj} can be interpreted as the metric tensor on the manifold of instantons. Therefore we can write

$$\mathrm{d}\mu_0=(q!)^{-2}\det N\prod_k\mathrm{d}^2\varphi_k\,, \tag{2.7}$$

where $\mathrm{d}^2\varphi=\frac{1}{2}i\,\mathrm{d}\varphi\,\mathrm{d}\bar{\varphi}$. The combinatorial factor $(q!)^{-2}$ is necessary to avoid double counting. (The permutations of zeros $a_1,\ldots,a_q$ and the permutations of poles $b_1,\ldots,b_q$ do not change the instanton (1.8).) We will prove a useful identity:

$$\det N=2^{4q+2}|c|^{4q}\prod_{k>j}|a_k-a_j|^2|b_k-b_j|^2\prod_{l,m}|a_l-b_m|^2\det M\,, \tag{2.8}$$

where

$$M_{kj}=\int\rho^{-2}\bar{z}^kz^j\sqrt{g}\,\mathrm{d}^2x\,. \tag{2.9}$$

To verify (2.8) we note that the expression $(\partial v/\partial\varphi_k)\prod_j(z-b_j)^2$ can be represented in the form $\sum_{k=0}^{2q} U_{jk} z^k$. Using this representation we obtain $N = U^+ M U$ and hence $\det N = \det M |\det U|^2$. It is easy to check that $\det U = 0$ if $\varphi_i = \varphi_j$; $1 \leq i, j \leq 2q$ (in this case two lines in U coincide up to sign). Noting that $\det U$ is a polynomial of degree $q(2q-1)$ we obtain

$$\det U = 2^{2q+1} c^{2q} \prod_{k>j} (a_k - a_j)(b_k - b_j) \prod_{l,m} (a_l - b_m) .$$

This proves (2.8).

Expression (2.5) is formal because it contains an infinite-dimensional determinant. To regularize this determinant we introduce a proper time cutoff. In other words we define $\log \det_\epsilon \Delta_v$ as

$$-\sum_i \int_\epsilon^\infty e^{-\lambda_i t} \frac{dt}{t}$$

where λ_i run over non-zero eigenvalues of Δ_v. In other words

$$\log \det_\epsilon \Delta_v = -\int_\epsilon^\infty (\mathrm{Sp}\, e^{-t\Delta_v} - p) \frac{dt}{t} , \tag{2.10}$$

where p is the number of zero modes of Δ_v, i.e., $p = 4q + 2$. The asymptotic of $\mathrm{Sp} \exp(-t\, \Delta_v)$ for $t \to 0$ can be calculated by the semiclassical method; we obtain

$$\mathrm{Sp}\, e^{-t\Delta_v} \simeq \alpha_1 t^{-1} + \alpha_0 , \tag{2.11}$$

where $\alpha_1 = R^2$, $\alpha_0 = 2q$. Using (2.11) we get that the divergent part of $\log \det_\epsilon \Delta_v$ for $\epsilon \to 0$ is equal to

$$\alpha_1 \epsilon^{-1} + (\alpha_0 - p) \log \epsilon = R^2 \epsilon^{-1} - (2q + 2) \log \epsilon .$$

It is convenient to replace $J_q(\phi)$ by

$$I_q(\phi) = \frac{J_q(\phi)}{(4\pi V \det_\epsilon^{-1/2}((2/\pi)\Delta_0))} ,$$

where $V = 4\pi R^2$ is the area of the sphere, Δ_0 denotes the operator Δ_v for the trivial instanton $v = 0$. (One can show that the denominator of this expression is equal to $J_0(1)$). By calculation of $I_q(\phi)$ the linear divergent terms in $\log \det_\epsilon \Delta_v$ and $\log \det_\epsilon \Delta_0$ cancel. Using this usual one-loop renormalization of coupling constant we find that logarithmic divergences in $I_q(\phi)$ cancel too. (The analogous renormalization procedure in the Yang–Mills case is described in detail in [3]).

The regularized determinant $\det' \Delta_v$ of Δ_v will be defined by the formula

$$\log \det' \Delta_v = \lim_{\epsilon \to 0} (\log \det_\epsilon \Delta_v - \alpha_1 \epsilon^{-1} - \alpha_0) . \tag{2.12}$$

In other words $\log \det' \Delta_v$ is the finite part of $\log \det_\epsilon \Delta_v$. We see that after removing of the cutoff $I_q(\phi)$ can be represented through the regularized determinants:

$$I_q(\phi) = W^q \int \phi(v)\, \mathrm{d}\mu\,, \tag{2.13}$$

where

$$\begin{aligned} \mathrm{d}\mu &= (\det' \Delta_0/\det' \Delta_v)^{1/2}\, \mathrm{d}\mu_0\,, \\ W &= k_0 f_{\text{phys}}^{-2} \exp\left(-4\pi/f_{\text{phys}}\right)\nu^2\,. \end{aligned} \tag{2.14}$$

ν is the subtraction point, f_{phys} is the physical coupling constant and k_0 is a constant, depending of the cutoff method.

3. Calculation of determinants

In this section we study the dependence of $\det' \Delta_v$ on instanton parameters. It follows from (2.10) that

$$\delta \log \det_\epsilon \Delta_v = \int_\epsilon^\infty \mathrm{Sp}\,(\delta\, \Delta_v\, \mathrm{e}^{-t\Delta_v})\, \mathrm{d}t\,, \tag{3.1}$$

where

$$\delta\, \Delta_v = \rho^{-1}\delta\rho\, \Delta_v + \Delta_v \rho^{-1}\delta\rho - 2\rho\partial_z\rho^{-3}\delta\rho\partial_{\bar z}\rho\,.$$

Noting that α_1 and α_0 do not depend on instanton parameters, we obtain from (2.12) and (3.1)

$$\delta \log \det' \Delta_v = \int_0^\infty \mathrm{Sp}\,(\delta\, \Delta_v\, \mathrm{e}^{-t\Delta_v})\, \mathrm{d}t.$$

Let us define the operator $\tilde\Delta_v$ by the formula

$$\tilde\Delta_v = -\rho^{-1}\partial_z\rho^2 g^{-1/2}\partial_{\bar z}\rho^{-1}\,.$$

It is evident that

$$\Delta_v\rho^{-1}\partial_z\rho = \rho^{-1}\partial_{\bar z}\rho\, \tilde\Delta_v$$

and therefore

$$\begin{aligned} \delta \log \det' \Delta_v &= 2\int \mathrm{Sp}\, \rho^{-1}\delta\rho(\Delta_v\, \mathrm{e}^{-t\Delta_v} - \tilde\Delta_v\, \mathrm{e}^{-t\tilde\Delta_v})\, \mathrm{d}t \\ &= -2\int \frac{\partial}{\partial t}\mathrm{Sp}\, \rho^{-1}(\mathrm{e}^{-t\Delta_v} - \mathrm{e}^{-t\tilde\Delta_v})\, \mathrm{d}t\,. \end{aligned} \tag{3.2}$$

To calculate the integral in (3.2) we must study the behaviour of the integrand for $t \to 0$ and for $t \to \infty$. For $t \to 0$

$$\langle z|\mathrm{e}^{-t\Delta_v}|z\rangle = \frac{\sqrt{g}}{4\pi}t^{-1} + \frac{1}{8\pi}\frac{|\partial_{\bar{z}}v|^2}{(1+|v|^2)^2} + \ldots = \frac{\sqrt{g}}{4\pi}t^{-1} + \frac{1}{2\pi}\partial_\mu\partial_\mu \log\rho + \ldots$$

$$\langle z|\mathrm{e}^{-t\tilde{\Delta}_v}|z\rangle = \frac{\sqrt{g}}{4\pi}t^{-1} - \frac{1}{2\pi}\partial_\mu\partial_\mu\left(\log\rho - \frac{1}{4\pi}\log g\right),$$

where the omitted terms tend to zero. (These formulae can be obtained by semi-classical methods or by means of the results of [2]). The asymptotics for $t \to \infty$ are governed by zero modes $\psi_0, \ldots, \psi_{4q+1}$ of Δ_v. (The operator $\tilde{\Delta}_v$ has no zero modes). We get

$$\delta \log \det{}' \Delta_v = A_1 + A_2, \tag{3.3}$$

where

$$A_1 = \frac{1}{\pi}\delta\int \log\rho\,\partial_\mu\partial_\mu \log\rho\, \mathrm{d}^2x - \frac{1}{4\pi}\int \delta(\log\rho)\partial_\mu\partial_\mu \log g\, \mathrm{d}^2x$$

$$+\frac{1}{\pi}\oint ((\delta\log\rho)\partial_\mu \log\rho - \log\rho\,\partial_\mu\delta\log\rho)\,\mathrm{d}\sigma_\mu \equiv B_1 + B_2 + B_3, \tag{3.4}$$

$$A_2 = -2\int \rho^{-1}\delta\rho\,\pi(x)\,\mathrm{d}x,$$

$$\pi(x) = \sum_i \bar{\psi}_i(x)\psi_i(x), \tag{3.5}$$

(we assume that $\langle\psi_i, \psi_j\rangle = \delta^i_j$).

The last integral in formula (3.4) is taken over an infinitely large circle. It can be easily calculated. We obtain

$$B_3 = 4q(1+|c|^2)^{-1}\delta|c|^2.$$

The second term in formula (3.4) can be calculated in the limit $R \to \infty$:

$$B_2 = 4(1+|c|^2)^{-1}\delta|c|^2.$$

The first term in (3.4) can be represented in the following form.

$$B_1 = \frac{1}{\pi}\delta\int \log\rho_0\partial_\mu\partial_\mu \log\rho\,\mathrm{d}^2x + \frac{1}{\pi}\delta\sum_k\int \log|z-b_k|^2\partial_\mu\partial_\mu \log\rho\,\mathrm{d}^2x$$

$$= \frac{1}{\pi}\delta\int \log(1+|v|^2)\frac{|\partial_z v|^2}{(1+|v|^2)^2}\tfrac{1}{2}i\,\mathrm{d}z\,\mathrm{d}\bar{z} + \frac{\delta}{\pi}\int\sum_k(\partial_\mu\partial_\mu \log|z-b_k|^2)\log\rho\,\mathrm{d}^2x$$

$$+\frac{1}{\pi}\delta\sum_k\oint((\partial_\mu \log\rho)\log|z-b_k|^2 - \log\rho\,\partial_\mu \log|z-b_k|^2)\,\mathrm{d}\sigma_\mu. \tag{3.6}$$

The variation of the first integral vanishes because this integral is equal to the topological number:

$$\frac{1}{\pi}\int \log(1+|v|^2)\frac{|\partial_z v|^2}{(1+|v|^2)^2}\tfrac{1}{2}i\, dz\, d\bar{z} = \frac{q}{\pi}\int \log(1+|v|^2)\frac{d^2 v}{(1+|v|^2)^2}.$$

The factor q arises because the point in z plane has in general q inverse images in V plane. To find the second term in (3.6) we have to take into account the fact that $\partial_\mu\partial_\mu \log(x-a)^2 = 4\pi\delta(x-a)$. The last term in (3.6) can be easily calculated. Thus we obtain

$$B_1 = 4\delta\Big(\sum_k \log|c|^2 \prod_j |a_j - b_k|^2\Big) - 4q(1+|c|^2)^{-1}\delta|c|^2,$$

and hence

$$A_1 = 4\delta \log\Big(\prod_{jk} |a_j - b_k|^2\Big) + 4(1+|c|^2)^{-1}\delta|c|^2 + 4q|c|^{-2}\delta|c|^2. \tag{3.7}$$

Let us calculate A_2. It is convenient to choose the standard basis of zero modes of Δv in the form $\chi_k = \rho^{-1}z^{k/2}$ if k is even, $\chi_k = i\rho^{-1}z^{(k-1)/2}$ if k is odd. This system of zero modes is not orthonormal and therefore the expression of A_2 through χ_k takes the form

$$A_2 = -2\int \rho^{-3}\delta\rho\bar{\chi}_i\chi_k\sqrt{g}\, d^2x\, r_{kj}^{-1} = \mathrm{Sp}\,\delta r\, r^{-1} = \delta \log\det r, \tag{3.8}$$

where

$$r_{kj} = \int \rho^{-2}\bar{\chi}_k\chi_j\sqrt{g}\, d^2x, \tag{3.9}$$

the matrix $(r^{-1})_{kj}$ is inverse to (3.9). It is easy to check that

$$\det r = (\det M)^2,$$

where the matrix M is defined by (2.9). Combining (3.3) and (3.8) we see that the formula (3.7) represents the variation of the expression $\log\det'\Delta_v - 2\log\det M$. It is easy now to obtain from (2.7), (2.8), (2.14) and (3.7) the following expression for $d\mu$:

$$d\mu = K_q(q!)^{-2}\prod_{k>j}|a_k - a_j|^2|b_k - b_j|^2 \prod_{l,m}|a_l - b_m|^{-2}\frac{d^2c}{\pi(1+|c|^2)^2}\prod_j d^2a_j\, d^2b_j. \tag{3.10}$$

The constant factor K_q is undetermined. In the case when, for each j, $|a_j - b_j| \ll |a_j|, |b_j|$, the instanton solution can be considered as a superposition of distant instantons with topological number 1. In this case expression (3.10) can be represented as a product of factors corresponding to one-instanton solutions:

$$d\mu \simeq K_q(q!)^{-2}\Big(\prod_j |a_j - b_j|^{-2}\, d^2a_j\, d^2b_j\Big)\frac{d^2c}{\pi(1+|c|^2)^2}.$$

It is natural to think that the constant factor breaks up into product of one-instanton factors:

$$K_q = L^q , \tag{3.11}$$

where $L = K_1$. Eq. (3.11) will be proved in appendix C. The value of K_1 can be obtained from the study of one-instanton contribution [15].

4. Analysis of the instanton contribution

In this section we will study the sum of instanton contributions in Green functions.

Taking into account the instantons having arbitrary topological number we obtain from the results of sect. 3 the following answer for the Green functions in the steepest descent approximation:

$$I(\phi) = \frac{\sum_q \int \Psi(a, b, c) \frac{K^q}{(q!)^2} \exp\left(-\epsilon_q(a, b)\right) \frac{d^2 c}{\pi(1+|c|^2)^2} \prod_j d^2 a_j \, d^2 b_j}{\sum_q \int \frac{K^q}{(q!)^2} \exp\left(-\epsilon_q(a, b)\right) \prod_j d^2 a_j \, d^2 b_j} , \tag{4.1}$$

where $(a, b, c) = (a_1, \ldots, a_n; b_1, \ldots, b_n; c)$, a_i, b_i, c are two-dimensional vectors (or complex numbers), $K = LW$

$$\epsilon_q(a, b) = -\sum_{i<j}^{q} \log|a_i - a_j|^2 - \sum_{i<j}^{q} \log|b_i - b_j|^2 + \sum_{i,j}^{q} \log|a_i - b_j|^2 ,$$

$$\Psi(a, b, c) = \phi(w), \qquad \text{where } w \text{ is given by (1.8) and } W \text{ is given by (2.14)} .$$

It is easy to see that the denominator Z_{inst} in (4.1) can be expressed through the partition function Ξ of the two-dimensional neutral classical Coulomb system (CCS) in the grand canonical ensemble:

$$\Xi = \sum_q \frac{K^q}{(q!)^2} \int \exp\left(-\frac{1}{T}\epsilon_q(a, b)\right) \prod_j d^2 a_j \, d^2 b_j . \tag{4.2}$$

Namely, if $T = 1$, then

$$Z_{\text{inst}} = \Xi . \tag{4.3}$$

The constant K plays the role of fugacity of the Coulomb system. The expression (4.1) coincides with the correlation function of the CCS

$$\frac{\sum_q (K^q/(q!)^2) \int \varphi(a, b) \exp\left(-(1/T)\epsilon_q(a, b)\right) \prod_j d^2 a_j d^2 b_j}{\sum_q (K^q/(q!)^2) \int \exp\left(-(1/T)\epsilon_q(a, b)\right) \prod_j d^2 a_j \, d^2 b_j} . \tag{4.4}$$

Let us consider, for example, the instanton contribution $G^{\text{inst}}(x, y)$ in the Green function

$$G(x, y) = \langle \Delta \log |w(x)|\,, \Delta \log |w(y)| \rangle\,,$$

corresponding to the functional $\phi(w) = \Delta \log |w(x)| \Delta \log |w(y)|$. It is easy to see that

$$\varphi(a, b) = \psi(a, b, c) = \rho(x)\rho(y)\,,$$

where $\rho(x) = 2\pi(\sum_i \delta(x - a_i) - \delta(x - b_i))$. One can interpret $\rho(x)$ as the charge density, we see that

$$G^{\text{inst}}(x, y) = \langle \Delta \log |w(x)|, \Delta \log |w(y)| \rangle_{\text{inst}} = \langle \rho(x), \rho(y) \rangle_{\text{CCS}}\,. \tag{4.5}$$

In a similar way one can assert that the instanton contribution in the Green function corresponding to the functional

$$\phi(w) = \Delta \log |w(x_1)| \dots \Delta \log |w(x_n)|$$

is given by the formula

$$G_n^{\text{inst}}(x_1, \dots, x_n) = \langle \rho(x_1), \dots, \rho(x_n) \rangle_{\text{CCS}}\,.$$

We see that the instantons are closely related with the classical Coulomb system for $T = 1$; this Coulomb system will be called a system of instanton quarks. It is well-known that ultraviolet divergences occur in the CCS, if the temperature T satisfies $T \leqslant 1$. We see that there exist ultraviolet divergences in our case and therefore we will make the ultraviolet cutoff. (One can consider, for example, the lattice Coulomb system or Coulomb particles with cores). In more complicated models such as the CP$(n-1)$ theory and gauge theories, ultraviolet troubles do not occur.

Of course, in the calculation of (4.2) and (4.4) we must first consider the Coulomb system in a box (i.e., make the spatial cutoff). If the size L of the box tends to infinity, then there exists a limit of the pressure $p = (1/L^2)T \log \Xi$. An analogous statement is correct for the correlation functions of the CCS. These assertions cannot be deduced from general results of statistical mechanics because the Coulomb potential is extremely long range. However, one can give an independent proof of these assertions in the Coulomb case (see [5] for instance). The existence of the limits for $L \to \infty$ denotes that infrared divergences do not occur in our problem. It is well-known that there exists a phase transition in the CCS [6–9]. This system is in the plasma phase for $T > T_{\text{cr}}$ and in the molecular phase for $T < T_{\text{cr}}$. In the molecular phase the Coulomb particles form dipoles; in other words only the configurations $a_1, \dots, a_n;\ b_1, \dots, b_n$ satisfying

$$|a_i - b_i| \ll |a_i - a_j| \tag{4.6}$$

are essential. The temperature of the Coulomb gas of instanton quarks is higher than T_{cr} and therefore this gas is in the plasma phase. (The calculations of [7–9] show that $T_{\text{cr}} \approx \frac{1}{2}$, hence $T_{\text{cr}} < 1$). As we mentioned above, the configurations satisfying (4.6) (dipole configurations) correspond to distant instantons. (One can interpret $\frac{1}{2}(a_i+b_i)$ as the position of ith instanton and $\frac{1}{2}|a_i-b_i|$ as the size of this instanton). The dilute instanton gas approximation can be obtained if we take into account in (4.1) only configurations satisfying (4.6). This approximation is not reasonable in our problem because the gas of instanton quarks is in the plasma phase. Notice that in the dilute instanton gas approximation infrared divergences occur, however, as we have mentioned above, these divergences disappear in the exact solution.

Let us study the instanton contribution in certain Green functions. We assume in our calculations that the ultraviolet cutoff is removed and the standard renormalization procedure is performed. It is convenient to use the Coleman–Fröhlich correspondence between the CCS, massive Thirring model (MTM) and sine-Gordon model (SGM) [4, 5].

This correspondence takes place when the inverse temperature $1/T$ of the CCS, the constant β in the SGM:

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\varphi)^2+M^2\cos\beta\varphi\,, \tag{4.7}$$

and the constant g in the MTM:

$$\mathscr{L}=\bar\psi i\partial_\mu\gamma_\mu\psi+m\bar\psi\psi+g(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\mu\psi)\,, \tag{4.8}$$

are connected by

$$\frac{1}{T}=\frac{\beta^2}{4\pi}=\frac{1}{1+g/\pi}\,.$$

In our case $T=1$; $\beta=2\sqrt{\pi}$; $g=0$ and we see that the Coulomb system of instanton quarks is equivalent to free fermions.

The complete instanton contribution in the partition function can be represented as the vacuum energy functional of the free Dirac field (FDF):

$$Z_{\rm inst}=\int\prod_x \mathrm{d}\bar\psi(x)\,\mathrm{d}\psi(x)\exp\left(\int(\bar\psi i\gamma_\mu\partial_\mu\psi+m\bar\psi\psi)\,\mathrm{d}^2x\right)\,. \tag{4.9}$$

It is interesting to note that the mass of the Dirac field which will play the role of the inverse correlation radius in our system, $m=c(\mu/f_{\rm phys})\,\mathrm{e}^{-2\pi/f_{\rm phys}}$ (here c is an inessential constant), is connected with $f_{\rm phys}$ by the usual formula of the renormalization group.

Now we can compute the instanton contribution (4.1) to the Green function (4.5)

$$G^{\rm inst}(x,y)=\langle\Delta\log|w(x)|,\Delta\log|w(y)|\rangle_{\rm inst}=\langle\rho(x),\rho(y)\rangle_{\rm CCS}\,.$$

To compute this function it is convenient to have an expression for the generating functional of the charge density of the Coulomb plasma. This expression can be obtained in the standard way using (CCS)$\rightleftharpoons$(SGM)

$$
\begin{aligned}
&\langle \mathrm{e}^{(i/\sqrt{\pi})\int \eta(x)\rho(x)\,\mathrm{d}x}\rangle_{\mathrm{ccs}} \\
&= \frac{\sum_q (K^q/(q!)^2)\int \mathrm{e}^{-\epsilon_q(a,b)}\,\mathrm{e}^{i\sum_i 2\sqrt{\pi}(\eta(a_i)-\eta(b_i))}\prod_j \mathrm{d}^2a_j\,\mathrm{d}^2b_j}{\sum_q (K^q/(q!)^2)\int \mathrm{e}^{-\epsilon_q(a,b)}\prod_j \mathrm{d}^2a_j\,\mathrm{d}^2b_j} \\
&= \int \prod_x \mathrm{d}\varphi(x)\exp\left[-\int\left[\tfrac{1}{2}(\partial_\mu\varphi)^2 + M^2\cos 2\sqrt{\pi}(\varphi+\eta)\right]\mathrm{d}^2x\right] \\
&= \int \prod_x \mathrm{d}\varphi(x)\exp\left[-\int\left[\tfrac{1}{2}(\partial_\mu(\varphi-\eta))^2 + M^2\cos 2\sqrt{\pi}\varphi\right]\mathrm{d}^2x\right].
\end{aligned}
\tag{4.10}
$$

From (4.10) we get

$$
\left\langle \frac{i}{\sqrt{\pi}}\Delta\log|w(x)|, \frac{i}{\sqrt{\pi}}\Delta\log|w(y)|\right\rangle_{\mathrm{inst}} = \langle\Delta\varphi(x),\Delta\varphi(y)\rangle_{\mathrm{SG}} + \Delta_x\delta(x-y)\,, \tag{4.11}
$$

and using the relation

$$
\left\langle \frac{1}{\sqrt{\pi}}\partial_\mu\varphi(x), \frac{1}{\sqrt{\pi}}\partial_\nu\varphi(y)\right\rangle_{\mathrm{SG}} = \epsilon_{\mu\alpha}\epsilon_{\nu\beta}\langle\bar\psi\gamma^\alpha\psi(x),\bar\psi\gamma^\beta\psi(y)\rangle_{\mathrm{FDF}}\,, \tag{4.12}
$$

we can compute this function exactly:

$$
G^{\mathrm{inst}}(x,y) = \int \frac{\mathrm{d}^2k}{4\pi}\mathrm{e}^{ik(x-y)}m^2\sqrt{\frac{k^2}{k^2+m^2}}\log\left\{\frac{\sqrt{k^2+m^2}+\sqrt{k^2}}{\sqrt{k^2+m^2}-\sqrt{k^2}}\right\}. \tag{4.13}
$$

It may be interesting to express the sum of instanton contributions in some correlation functions of the σ model through the Green functions of the free Dirac field. To do it we can use the Euclidean version of Mandelstam's representation of the Fermi field in the MTM through the SG field [10]. These formulae were obtained in [11], and have a form

$$
\begin{aligned}
&\psi_1(x) = C\,\mathrm{e}^{\gamma\int^x \mathrm{d}x_\mu\epsilon_{\mu\nu}\partial_\nu\varphi + i\beta\varphi(x)}\,, \qquad \psi_2(x) = (-i)C\,\mathrm{e}^{\gamma\int^x \mathrm{d}x_\mu\epsilon_{\mu\nu}\partial_\nu\varphi - i\beta\varphi(x)}\,, \\
&\psi_1^+(x) = C\,\mathrm{e}^{-\gamma\int^x \mathrm{d}x_\mu\epsilon_{\mu\nu}\partial_\nu\varphi - i\beta\varphi(x)}\,, \qquad \psi_2^+(x) = (i)C\,\mathrm{e}^{-\gamma\int^x \mathrm{d}x_\mu\epsilon_{\mu\nu}\partial_\nu\varphi + i\beta\varphi(x)}
\end{aligned}
\tag{4.14}
$$

Here C is an inessential renormalization constant,

$$
\gamma^2 = \pi\frac{1+g/2\pi}{1-g/2\pi}\,, \qquad \beta^2 = \pi\frac{1-g/2\pi}{1+g/2\pi}\,.
$$

In our case $\gamma=\beta=\sqrt{\pi}$. Using (4.11) and the Cauchy-Riemann equation,

$$
\epsilon_{\mu\nu}\partial_\nu\log|w(z)| = \partial_\mu\arg w\,,
$$

we can obtain:

$$\langle w(x), w^{-1}_{(y)}\rangle_{\text{inst}} = \langle \exp[\log|w(x)| + i \arg w(x)] \exp[-\log|w(y)| - i \arg w(y)]\rangle_{\text{inst}}$$
$$= \frac{\langle \psi_1^+(x), \psi_1(y)\rangle_m}{\langle \psi_1^+(x), \psi_1(y)\rangle_0}.$$

Here $\langle \psi_1^+(x), \psi_1(y)\rangle_m$ denotes the Green function of the free Dirac field with mass m and $\langle \psi_1^+(x), \psi_1(y)\rangle_0$ the Green function of the massless Dirac field.

The Green functions (4.13), (4.15) decrease exponentially if $m|x-y| \gg 1$ and we see that m plays the role of the correlation length in our σ model.

We do not study the instanton–anti-instanton contributions and θ-vacuum problem here. We plan to return to these problems in our next paper.

We are grateful to A.G. Aronov, A.A. Belavin, F.A. Berezin, E.B. Bogomolny, V.N. Gribov, A.A. Migdal, A.M. Polyakov and Yu.S. Tyupkin for many fruitful discussions.

Appendix A

In this appendix we will establish the connection between Pauli–Villars regularization and proper time regularization of determinants.

Let A be a non-negative elliptic operator on a compact manifold. Then for $t \to 0$,

$$\operatorname{Sp} \exp(-tA) = \sum_{k \geq 0} \alpha_k t^{-k} + O(t^\delta),$$

and for $t \to \infty$

$$\operatorname{Sp} \exp(-tA) = p + O(t^{-\delta_1})$$

(here k run over a finite set of non-negative numbers, p denotes the number of zero modes of A). Therefore we can represent $\operatorname{Sp} \exp(-tA)$ in the form

$$\operatorname{Sp} e^{-tA} = p + \sum_{k>0} \alpha_k t^{-k} + (\alpha_0 - p)\theta(1-t) + \sigma(t), \tag{A.1}$$

where $\sigma(t) = O(t^\delta)$ for $t \to 0$, $\sigma(t) = O(t^{-\delta_1})$, for $t \to \infty$, $\delta_1 > 0$, $\delta > 0$. If we use the proper time regularization of A (see sect. 2) then

$$\log \det{}' A = -\int_0^\infty t^{-1}\sigma(t)\,dt. \tag{A.2}$$

Let us take coefficients c_j and masses M_j, $j = 0, 1, \ldots, l$ satisfying the conditions $M_0 = 0$, $M_j \to \infty$ for $j > 0$; $c_0 = 1$; $\sum_j c_j = 0$; $\sum_j c_j M_j^{2p} = 0$ for $p = 1, 2, \ldots, l$. (Here l

denotes the largest integer, such that there exists k in (A.1) satisfying $k \leq l$, i.e., $l = [k_{\max}]$.) The Pauli–Villars cutoff of the determinant is defined by the formula

$$\log \det_{\mathrm{PV}} A = -\int_0^\infty \sum_j c_j \,\mathrm{e}^{-M_j^2 t} (\mathrm{Sp}\,\mathrm{e}^{-tA} - p) \frac{\mathrm{d}t}{t}. \tag{A.3}$$

In terms of eigenvalues λ_i of operator A this expression can be represented in the form

$$\log \det_{\mathrm{PV}} A = -\int_0^\infty \sum_j \sum_{\lambda_i \neq 0} c_j \,\mathrm{e}^{-(\lambda_i + M_j^2)t} \frac{\mathrm{d}t}{t} = \sum_j \sum_{\lambda_i \neq 0} c_j \log(\lambda_i + M_j^2). \tag{A.4}$$

These expressions are convergent due to the conditions imposed on c_j and M_j.

The connection between the Pauli–Villars cutoff and proper time regularization of the determinant is given by the formula

$$\begin{aligned} \log \det' A = {} & \log \det_{\mathrm{PV}} A - \sum_{k=[k]>0} (-1)^k \frac{\alpha_k}{k!} \sum_j c_j M_j^{2k} \log M_j^2 \\ & - \sum_{k \neq [k]} \Gamma(-k) \alpha_k \sum_j c_j M_j^{2k} - (\alpha_0 - p) \sum_{j \neq 0} c_j \log M_j^2 + \gamma(\alpha_0 - p) + \mathrm{O}(M_j^{-2\delta}), \end{aligned} \tag{A.5}$$

where $\gamma = -\Gamma'(1) = 0.5772$ is Euler's constant. The proof of (A.5) is based on the formulae

$$\int_0^\infty t^{-k-1} \sum_j c_j \,\mathrm{e}^{-tM_j^2} \mathrm{d}t = \begin{cases} \dfrac{(-1)^{k+1}}{k!} \sum_j c_j M_j^{2k} \log M_j^2, & \text{if } k = [k] > 0, \\[2ex] -\Gamma(-k) \sum_j c_j M_j^{2k}, & \text{if } k \neq [k], \end{cases}$$

$$\begin{aligned} \int_0^1 \sum_j c_j \,\mathrm{e}^{-tM_j^2} \frac{\mathrm{d}t}{t} &= \int_0^1 (1 - \mathrm{e}^{-t}) \frac{\mathrm{d}t}{t} - \int_1^\infty \mathrm{e}^{-t} \frac{\mathrm{d}t}{t} + \int_0^\infty \left(\mathrm{e}^{-t} + \sum_{j \neq 0} c_j \,\mathrm{e}^{-tM_j^2} \right) \frac{\mathrm{d}t}{t} + \mathrm{O}(M_j^{-2}) \\ &= \gamma - \sum_{j \neq 0} c_j \log M_j^2, \end{aligned}$$

$$\int_0^\infty \mathrm{e}^{-tM_j^2} \sigma(t) \frac{\mathrm{d}t}{t} = \mathrm{O}(M_j^{-2\delta}).$$

We can also compare the regularizations considered above with the zeta-function regularization defined as

$$\log \det_\zeta A = -\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s)|_{s=0} ,$$

where

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\mathrm{Sp}\,\mathrm{e}^{-tA}-p)\,\mathrm{d}t .$$

There exists the following equality [3]:

$$\log \det' A = \log \det_\zeta A + \gamma(\alpha_0 - p) . \tag{A.6}$$

Appendix B

We will now calculate the regularized determinant of operator A having eigenvalues $\lambda_\nu = (\nu+a)(\nu+b)$ with multiplicities $d_\nu = c\nu + f$. It is convenient to use the notations $a = a_0 + \alpha$; $b = b_0 + \beta$ where $0<\alpha, \beta \leqslant 1$; a_0 and b_0 are integer. Then we will show that

$$\begin{aligned}\log \det' A \equiv Z(a,b,c,f) &= -\gamma\tfrac{1}{2}f(a+b-2a_0+1)c\zeta_\mathrm{R}'(-1,\alpha) - c\zeta_\mathrm{R}'(-1,\beta)\\ &\quad -(f-ca)\zeta_\mathrm{R}'(0,\alpha)-(f-cb)\zeta'(0,\beta)-\sum_{\nu=0}^{b_0-a_0-1}(c\nu+f-cb_0)\log(\nu+\beta)\\ &\quad +\tfrac{1}{4}c(b-a)^2 .\end{aligned} \tag{B.1}$$

It will be convenient to use the Pauli–Villars cutoff and the formula (A.5) connecting $\log \det_{\mathrm{PV}} A$ and $\log \det' A$.

Taking formula (A.4) we will sum over all λ_ν with $\nu \leqslant \Lambda - a_0$ where Λ is a big integer number. It can be shown that the tail is $\mathrm{O}(\Lambda^{-1})$. Thus we can write

$$\begin{aligned}\log \det_{\mathrm{PV}} A &= \sum_i c_i \sum_{\nu=-a_0}^{\Lambda-a_0}(c\nu+f)\log((\nu+a)(\nu+b)+M_i^2)+\mathrm{O}(\Lambda^{-1})\\ &\equiv \sum_i c_i D_i + \mathrm{O}(\Lambda^{-1}) .\end{aligned} \tag{B.2}$$

First we will calculate D_i. It is convenient to use the variables $t = b - a_0 \equiv t + \beta$; $g = f - a_0 c$. Then

$$\begin{aligned}D_0 = \sum_{\nu=0}^{\Lambda}(c\nu+g)\log(\nu+\alpha)(\nu+t) &= c\Lambda^2\log\Lambda\\ &\quad +(c+2g)\Lambda\log\Lambda - \tfrac{1}{2}c\Lambda^2 - (2g-c(\alpha+t))\Lambda\end{aligned}$$

$$+\left(\tfrac{1}{2}c\left(\tfrac{1}{3}-\alpha^2-t^2\right)+g(1+\alpha+t)\right)\log\Lambda$$
$$-c\zeta_R'(-1,\alpha)-c\zeta_R'(-1,\beta)-(g-c\alpha)\zeta_R'(0,\alpha)-(g-ct)\zeta_R'(0,\beta)$$
$$+\tfrac{1}{2}c\left(\alpha^2+\alpha+t^2+t+\tfrac{1}{3}\right)-\sum_{\nu=0}^{t_0-1}(c\nu+g-ct_0)\log(\nu+\beta)\,.$$

We have used the formula

$$\sum_{\nu=0}^{\Lambda}(\nu+\alpha)^{-s}=\frac{(\Lambda+\alpha)^{1-s}}{1-s}+\tfrac{1}{2}(\Lambda+\alpha)^{-s}+\zeta_R(s,\alpha)$$
$$-\sum_{k=1}^{\infty}(-1)^k\frac{B_k}{(2k)!}s(s+1)\ldots(s+2k-\alpha)(\Lambda+2)^{1-2k-s}\,,$$

where $\zeta_R(s,\alpha)$ is the generalized Riemann zeta-function

$$\zeta_R(s,\alpha)=\sum_{\nu=0}^{\infty}(\nu+\alpha)^{-s}\,,$$

and B_k are Bernoulli numbers defined by the formula

$$x(e^x-1)^{-1}=1-\tfrac{1}{2}x-\sum_{k=1}^{\infty}(-1)^kB_k\frac{x^k}{(2k)!}\,,$$
$$B_1=\tfrac{1}{6},\ B_2=\tfrac{1}{30},\ \ldots\,.$$

To calculate the terms D_j with $j\geqslant 1$ we use the Euler–Maclauren formula

$$\sum_{\nu=p+1}^{\Lambda}F(\nu)=\int_p^{\Lambda}F(x)\,dx+\left[\tfrac{1}{2}F(x)-\sum_{k=1}^{\infty}(-1)^k\frac{B_k}{(2k)!}\frac{d^k}{dx^k}F(x)\right]_p^{\Lambda}\,. \tag{B.3}$$

Let us take $p=-1$;

$$F(\nu)=(c\nu+g)\log\left((\nu+\alpha)(\nu+t)+M_j^2\right)\,,$$

so that

$$\sum_{\nu=0}^{\Lambda}F(\nu)=D_j\,.$$

Then we obtain

$$\int_{-1}^{\Lambda}F(x)\,dx=c\Lambda^2\log\Lambda-2g\Lambda\log\Lambda-\tfrac{1}{2}c\Lambda^2-2\left(g-\tfrac{1}{2}c(t+\alpha)\right)\Lambda$$
$$+\left(cM_j^2-\tfrac{1}{2}c(\alpha^2+t^2)+g(\alpha+t)\right)\log\Lambda-\tfrac{1}{2}cM_j^2\log M_j^2+\left(g-\tfrac{1}{2}c(t+\alpha)\right)\pi M_j$$
$$-\left(\tfrac{1}{2}c(2-\alpha^2-t^2)+g(\alpha+t-2)\right)\log M_j+\tfrac{1}{2}(t+\alpha)^2+O(\Lambda^{-1})+O(M_j^{-2})\,,$$

$$\tfrac{1}{2}F(x)|_{-1}^{\Lambda} = c\Lambda \log \Lambda + g \log \Lambda + \tfrac{1}{2}c(t+\alpha) + (c-g)\log M_j + \mathrm{O}(\Lambda^{-1}) + \mathrm{O}(M_j^{-1}),$$

$$\tfrac{1}{6}F'(x)|_{-1}^{\Lambda} = \tfrac{1}{6}c(\log(\Lambda/M_j)+1) + \mathrm{O}(\Lambda^{-1}) + \mathrm{O}(M_j^{-1}).$$

All other terms in the expansion (B.3) are negligible in our case. Thus we can obtain D_j. The sum over j in (B.2) can be easily taken and we obtain the formula (B.1).

This method can be applied in more general cases when the eigenvalues and their multiplicities are polynomial in ν.

Appendix C

We will obtain here the undetermined factor before expression (3.10). The result is

$$K_q = L^q, \qquad L = 2^6\, \mathrm{e}^{\gamma-2}. \tag{C.1}$$

To obtain (C.1) it is convenient to consider the instanton

$$v_0 = \frac{z^q - 1}{z^q + 1}. \tag{C.2}$$

This is the solution (1.8) with the parameters

$$c = 1, \qquad a_j = \exp(2\pi j i), \qquad b_j = \exp((2j+1)\pi i).$$

We will calculate the value

$$Q = \det{}' \Delta_{v_0} / \det{}' \Delta_0,$$

where Δ_{v_0} is the operator (2.4) with v_0 given by (C.2).

We introduce new variables $\eta = (1+|v_0|^2)^{-1}$, $\alpha = \arg z$. Using these variables we can write

$$\Delta_{v_0} = \sigma(z) g^{-1/2} T, \qquad \Delta_0 = \sigma(z) g^{-1/2} T_0,$$

where $\sigma(z) = (1+|z|^{2q})^{-2} 4q^2 |z|^{2q-1}$ (this value coincides with the density of the action);

$$T = -\eta^{-1}\partial_\eta \eta^3(1-\eta)\partial_\eta \eta^{-1} + \frac{1}{iq\eta}\partial_\alpha - \frac{1}{4q^2\eta(1-\eta)}\partial_\alpha^2,$$

$$T_0 = -\partial_\eta \eta(1-\eta)\partial_\eta - \frac{1}{4q^2\eta(1-\eta)}\partial_\alpha^2. \tag{C.3}$$

We shall first calculate the value

$$Q_1 = \det{}' T / \det{}' T_0.$$

The connection of Q_1 with Q will be established at the end of this section. We will prove the formula

$$\begin{aligned}\tfrac{1}{2}\log Q_1 &= -q\gamma + 2\log\left(\Gamma(q)\Gamma(2q)q^{-3q+1}\right) + 2(q-1)\log\pi \\ &\quad -(2q+1)\log 3 + 4q - \log 2\,. \end{aligned} \tag{C.4}$$

First we find the eigenvalues of operators T and T_0. Let us seek the eigenfunctions ψ, ψ_0 of these operators in the form

$$\psi = e^{im\alpha}\eta^{\delta}(1-\eta)^{\beta}\varphi(\eta)\,,$$

$$\psi_0 = e^{im_0\alpha}\eta^{\delta_0}(1-\eta)^{\beta_0}\varphi_0(\eta)\,,$$

where m, m_0 are integer and the coefficients β, δ, β_0, δ_0 must be found from the regularity condition. We obtain

$$\beta = \frac{m}{2q}\,,\qquad \delta = \frac{|m+2|}{2q}\,,\qquad \beta_0 = \delta_0 = \frac{|m_0|}{2q}\,.$$

We see that the eigenvalues satisfy the following equations of hypergeometrical type

$$\eta(1-\eta)\partial_\eta^2\varphi + (2\delta - 2(\beta+\delta+1)\eta)\partial_\eta\varphi - ((\beta+\delta)^2 + \beta + \delta - 1 - \lambda)\varphi = 0\,,$$

$$\eta(1-\eta)\partial_\eta^2\varphi_0 + (1 + 2\beta_0 - (4\beta_0+2)\eta)\partial_\eta\varphi - (4\beta_0^2 + 2\beta_0 - \lambda^{(0)})\varphi = 0.$$

Therefore the eigenvalues are given by the formulae

$$\lambda_{j_0}^{(0)} = j_0(j_0+1)\,,\qquad \lambda_j = j(j+1) - 2\,,$$

where $j_0 = 2\beta_0 + n$, $j = \beta + \delta + n$ and n is a non-negative integer number. Each $j(j_0)$ can be represented in the form

$$j(j_0) = \nu + \frac{r}{q}\,,\qquad r = 0, 1, \ldots, q-1\,,\qquad \nu = 0, 1, \ldots\,.$$

It is easy to obtain the multiplicity d_j of each eigenvalue

$$r \neq 0:\ d_j^0 = 4(\nu+1),\qquad d_j = 4\omega\,,$$

$$r = 0:\ d_j^0 = 2(2\nu+1)\,,\qquad d_j = 2(2\nu + 2q - 1)\,.$$

The determinants of T and T_0 can now be calculated by means of the results of appendix B. We obtain

$$\log\det{}' T = 2Z(-1, 2, 2, 2q-1) + 2\sum_{r=1}^{q-1} Z\left(-1 + \frac{r}{q}, 2 + \frac{r}{q}, 2, 0\right),$$

$$\log\det{}' T_0 = 2Z(0, 1, 2, 1) + 2\sum_{r=1}^{q-1} Z\left(\frac{r}{q}, 1 + \frac{r}{q}, 2, 2\right),$$

where the function Z is given by the formula (B.1). Now it is easy to obtain formula (C.4) using the equalities [12]

$$\zeta'_R(0,\alpha) = \log\Gamma(\alpha) - \tfrac{1}{2}\log 2\pi\,,$$

$$\prod_{r=0}^{m-1} \Gamma\left(z+\frac{r}{m}\right) = (2\pi)^{(m-1)/2} m^{1/2-mz}\Gamma(zm)\,, \qquad m=2,3,4,\dots\,.$$

There remains the only problem of connection of Q_1 with Q. To find this connection we introduce the operators $T(s) = \varkappa(z,s)T$ and $T_0(s) = \varkappa(z,s)T_0$, where $\varkappa(z,s) = s + (s-1)g^{-1/2}\sigma(z)$, $0 \leq s \leq 1$. It is evident that $T(0) = T$, $T_0(0) = T_0$; $T(1) = \Delta_{v_0}$, $T_0(1) = \Delta_0$. Using the methods of the sect. 3 we can find the dependence of the expression $\det T(s)/\det' T_0(s)$ on s and hence establish the connection of Q_1 with Q. We obtain

$$Q^{-1/2}\frac{\det M}{V} = e^{-P} Q_1^{-1/2}\frac{\det M_1}{V_1}\,, \tag{C.5}$$

where

$$\det M_1 = \det\left(\int \sigma\rho^{-2}\bar{z}^k z^j\, d^2x\right) = \pi^{4q-1}2^{-1}3^{-2q-1}q^{-4q+3}\Gamma^2(q)\Gamma^2(2q)\,,$$

$$V_1 = \int \sigma\, d^2x = 4\pi q\,,$$

$$P = \frac{1}{4\pi}\int \log(\sigma g^{-1/2})\sigma\, d^2x \simeq -2q + 2q\log 2q\,, \qquad \text{if } R\to\infty\,.$$

Combining formulae (2.7), (2.8), (2.14), (C.4), (C.5) we obtain

$$d\mu(v_0) = 2^{2q-2}\pi^{-1}\, e^{(\gamma-2)q} \prod_{k>j} |a_k - a_j|^2 |b_k - b_j|^2 \prod_{l,m} |b_l - a_m|^2 \prod_j d^2a_j\, d^2b_j\, d^2c\,.$$

To compare this expression with (3.10) we have to note that for instanton (C.2) $1 + |c|^2 = 2$; $\prod_{i,j}(a_i - b_j) = 2^q$. Thus we obtain the value of the constant (C.1).

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