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CONFORMAL ALGEBRA AND MULTIPOINT CORRELATION FUNCTIONS IN 2D STATISTICAL MODELS

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Based on the conformal algebra approach, a general technique is given for the calculation of multipoint correlation functions in 2D statistical models at the critical point. Particular conformal operator algebras are found for operators of the 2D q-component Potts model ($1 < q < 4$), and the $O(N)$ model ($0 < N < 2$) at the critical point. A number of four-point correlation functions are calculated for these models.

1. Introduction

In a recent paper by Belavin, Polyakov and Zamolodchikov [1] a new approach has been initiated for 2D conformal invariant theories. In that paper a full classification was given of the operators in such theories. It was also shown that there is a special class of 2D conformal invariant theories, the so-called degenerate conformal theories, in which all the anomalous dimensions of the operators are given by a simple algebraic equation. All the correlators in such theories satisfy special linear differential equations.

In this new approach the different 2D conformal invariant theories arise as particular representations of the general conformal operator algebra.

In ref. [1] still more special conformal theories were found in which the operator algebra consists of only a finite number of basic conformal operators. It was argued in [1] that such minimal theories should describe critical fluctuations in 2D statistical models at the second-order phase transition point. To back this idea it was shown that the simplest minimal operator algebra corresponds to the algebra of ε (energy) and σ (spin) operators of the 2D Ising model at the critical point.

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This idea was pursued further in ref. [2], where it was shown that degenerate conformal theories provide operator algebras which describe critical fluctuations in 2D q-component Potts models. In ref. [2] a detailed study was made of a particular operator algebra generated by the energy and spin operators of the Z_3 model. In this algebra, the same as in the Ising (Z_2) model, there are only a finite number of basic operators (five, instead of two as in the Ising model). But, in general, the Potts model with general $q \leqslant 4$ contains an infinite number of basic operators, which are generated by taking operator products of energy (ε) and spin (σ) operators.

In theories with scaling invariance the two-point correlation function has the following simple form:

$$\langle A(z)A(z')\rangle \sim \frac{1}{|z-z'|} 2\Delta . \qquad (1.1)$$

Here $A(z)$ is some basic operator of the theory and Δ is its scaling dimension. Some years ago Polyakov had suggested [3] that scaling invariance generalizes to conformal invariance. In particular, it was suggested that critical fluctuations are in fact conformal invariant. As was shown in [3], in conformal invariant theory the three-point correlators also have a simple form:

$$\langle A(z_1)B(z_2)C(z_3)\rangle \sim \frac{1}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3}|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}|z_{13}|^{\Delta_1+\Delta_3-\Delta_2}} . \qquad (1.2)$$

Nontrivial functions may appear in four and more point correlators. For example, the four-point correlator has the following general form [3]:

$$\langle A(z_1)B(z_2)C(z_3)D(z_4)\rangle \sim \frac{f(\eta)}{\prod_{i<j}|z_{ij}|^{\Delta_i+\Delta_j-\Delta/3}}, \qquad \Delta = \sum_i \Delta_i . \qquad (1.3)$$

Here $\eta = z_{12}z_{34}/z_{13}z_{24}$, and $f(\eta)$ is, in general, a nontrivial function, which is not fixed by the special (finite parameter) conformal group (projective group in 2D). Recent development of the 2D conformal theory in [1], which uses the general (infinite parameter) conformal group in 2D, makes it possible to find this function too. In particular, in degenerate conformal theories it is found as a quadratic form of solutions of the conformal linear differential equations. (It is this which we imply by saying sometimes that the correlators in degenerate conformal theories satisfy linear differential equations. The solutions of these equations are not really the correlators themselves but they are the conformal functions out of which the correlator is made. In [1] these functions are called "conformal blocks".)

In [1] the 4-spin correlator of the Ising model $\langle\sigma\sigma\sigma\sigma\rangle$ has been found using the conformal theory approach, and the result agreed with that of Kadanoff [4], and Luther and Peschel [5].

In [2] several 4-point correlation functions of the Z_3 model have been found. Calculations in [2] were restricted to correlators in which the conformal functions, which make the function $f(\eta)$ in (1.3), are solutions of the second-order equation. In the Z_3 model these are the correlators which contain the energy operator ε.

Recently Feigin and Fuchs developed a special integral representation for the conformal functions (conformal blocks) in the general conformal theory [6]. In particular, these integrals provide a convenient form for conformal solutions in the degenerate conformal theories.

In the present paper and in our next paper [7] we shall define, by using the Feigin-Fuchs integral representation, the correlation functions (or amplitudes, in the terminology of the general theory in [1]) for the operators in the degenerate conformal theories. For statistical physics it provides correlation functions, at the phase transition point, for all the operators of a particular statistical model for which the conformal algebra is known.

At present we have found two series of such particular algebras. More precisely, what we have found is a proper identification of conformal operators with the physical ones of spin and energy. The first is the Potts model with a continuous number of components $q \leqslant 4$ [8], and the second is the $O(n)$ model with continuous $0 \leqslant n \leqslant 2$ [9]. These are described in sect. 2.

In sect. 3 the Feigin-Fuchs integrals are introduced, and the relation of these integrals to the operator algebra of the degenerate conformal operators is described. In sect. 4 the monodromy problem, which is involved in constructing the physical correlators out of conformal functions, is introduced. And in this section we deal with the simplest case of nontrivial correlators – the 4-point correlators made of the second-order functions (hypergeometric functions).

In sect. 5 we start with higher-order functions. In this section we calculate 4-point correlators for the third-order conformal operator. In the $O(n)$ model this is the energy operator, see sect. 2.

Generalization to still higher operators is then straightforward, but technically involved. The first remarks on the conformal functions for the general degenerate operators are given in sect. 3. Explicit formulas will be worked out in our next paper [7].

Discussion of the results is given in sect. 6.

2. Conformal algebra of the Potts and O(n) models, generated by the energy and spin operators

The identification of the energy and spin operators of the general Potts model as the basic operators of the degenerate conformal algebra has already been found in [2]. We shall review this in short again, and then we shall give the identification for the energy and spin operators of the $O(n)$ model, with $0 \leqslant n \leqslant 2$. This model has been considered recently by Nienhuis [9].

The basic operators of the degenerate conformal algebra are classified by two integer numbers as $\phi_{n,m}(z)$, n, m being integers [1]. Conformal dimensions $\Delta_{n,m}$ of these operators are given by the Kac formula [10]:

$$\Delta_{n,m} = \tfrac{1}{4}\left[(\alpha_- n + \alpha_+ m)^2 - (\alpha_+ + \alpha_-)^2\right]. \tag{2.1}$$

Here

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}\,, \tag{2.2}$$

and α_0 is related to the anomaly number C of the conformal algebra as

$$C = 1 - 24\alpha_0^2, \tag{2.3}$$

see also sect. 3. The anomaly number C is a continuous parameter, and it determines a one-parameter set of degenerate conformal theories. We shall show shortly that in the operator algebras for the Potts and O(n) models the number C is a function of q and n, respectively.

In [1,2] it was found that the energy operator ε in the Ising ($q = 2$) and Z_3 ($q = 3$) models corresponds to the operator $\phi_{1,2}(z)$ of the conformal algebra. The number $(\alpha_-)^2 = (\alpha_+)^{-2}$ in (2.1) for these two models was found to be $\frac{3}{4}$ and $\frac{5}{6}$ respectively. They both correspond to the finite, or minimal, algebras.

The general rule is that the degenerate conformal algebra becomes finite if the number $(\alpha_-)^2$ is rational, i.e.

$$(\alpha_-)^2 = p/k. \tag{2.4}$$

Here p, k are integers. The main series of minimal theories found in [1] corresponds to $k - p = 1$. The Ising and Z_3 Potts models are two such theories. Now let us look at the whole series. We define

$$k = 2N, \qquad p = 2N - 1, \tag{2.5}$$

where N runs over integer and half-integer values.

For minimal theories the Kac formula (2.1) reads

$$\Delta_{n,m} = \frac{(pn - km)^2 - (k-p)^2}{4pk}. \tag{2.6}$$

As was said above, in Ising and Z_3 models the energy operator ε is represented, in the corresponding conformal algebras, by the operator $\phi_{1,2}$, with conformal dimension $\Delta_{1,2}$. Now we make a conjecture that this is the case for the general Potts model. For k, p defined as in (2.5) we find

$$\Delta_{1,2}=\frac{(2N-1-2N\times 2)^2-1}{4\times 2N\times(2N-1)}=\frac{N+1}{2(2N-1)}\,. \tag{2.7}$$

Now we recall that the physical critical dimensions of statistical operators are two times larger than their dimensions in the conformal algebra, see [2]. So we obtain from (7)

$$(\Delta_\varepsilon)_{\mathrm{ph}}=2\Delta_{1,2}=\frac{1+y}{2-y}\,,\qquad y=\frac{1}{N}\,. \tag{2.8}$$

The last formula is precisely that of den Nijs [11, 12]. This is the first thermal exponent X_{T_1} in his notation. Yet, relation (2.8) can be viewed here just as a definition of the parameter y. Consistency is established by calculating the second thermal exponent X_{T_2} using the same parameter $y=1/N$. In terms of the conformal theory the second thermal exponent corresponds to the operator $\phi_{1,3}$ which is created in the product of two energy operators:

$$\begin{aligned}\varepsilon(z)\varepsilon(z')&\sim\phi_{1,2}(z)\phi_{1,2}(z')\\&\sim\frac{1}{(z-z')}2\Delta_{12}I+\frac{1}{(z-z')^{2\Delta_{12}-\Delta_{13}}}\phi_{1,3}(z')\,,\end{aligned} \tag{2.9}$$

see [1, 2]. In the following we shall write such local operator expansions in a compact form, skipping the standard scaling factors, and indicating only the basic conformal operators in the expansion

$$\phi_{1,2}\phi_{1,2}\sim I+\phi_{1,3}\,. \tag{2.9$'$}$$

Here I is an identity operator of the algebra, and $\phi_{1,3}$ is the next-to-leading thermal operator. Its dimension is found from (2.6)

$$\begin{aligned}\Delta_{13}&=\frac{(2N-1-2N\times 3)^2-1}{4\times 2N\times(2N-1)}=\frac{4+2y}{2(2-y)}\,,\\X_{\mathrm{T}_2}&=2\Delta_{13}=\frac{4+2y}{2-y}\,,\end{aligned} \tag{2.10}$$

and it gives correctly the second thermal exponent, cf. [12].

In the leading series of the minimal conformal theories with $k=2N$, $p=2N-1$, and N an integer, the spin operator is placed right in the center of a table containing the operators $\phi_{n,m}$ (see [2]), and it corresponds to the operator $\phi_{N,N-1}$. This is a unique place in the tables for finite theories in the sense that only this operator is created again in the product with the energy operator

$$\varepsilon S \sim S + \cdots . \tag{2.11}$$

Anyway, in the Ising and Z_3 model the spin operator does correspond to the conformal operator $\phi_{N,N-1}$, and we assume that this is the case in general. Then the first and the second magnetic exponents, which correspond to the operators $\phi_{N,N-1}$, $\phi_{N,N-2}$, are found to be

$$X_{H_1} = 2\Delta_{N,N-1} = \frac{N^2-1}{4N(2N-1)} = \frac{1-y^2}{4(2-y)}, \tag{2.12}$$

$$X_{H_2} = 2\Delta_{N,N-2} = \frac{9N^2-1}{4N(2N-1)} = \frac{9-y^2}{4(2-y)}. \tag{2.13}$$

These are just the relations which were conjectured by Nienhuis et al. [13] and Pearson [14], and derived in terms of a Coulomb gas by den Nijs [15].

In general, the thermal and magnetic series of exponents for the Potts model are given by the formulas

$$X_{T_n} = 2\Delta_{1,n+1} = \frac{n^2+ny}{2-y}, \tag{2.14}$$

$$X_{H_n} = 2\Delta_{N,N-1-n} = \frac{(2n-1)^2-y^2}{4(2-y)}. \tag{2.15}$$

The parameter $y = 1/N$ above was restricted so far to a discrete set of values. The spin operator, which is the operator $\phi_{N,N-1}$, is contained in the set of degenerate conformal operators $\{\phi_{n,m}\}$ only for N being integer.

Also one can check that the set of conformal dimensions $\{\Delta_{n,m}\}$, given by the Kac formula (2.1), does not contain negative dimensions (which, of course, are physically unacceptable) only for the main series of minimal conformal theories. These correspond to N being integer, or half-integer. Note that this set of theories was also singled out in a different approach by Friedan, Qiu and Shenker in their recent paper [18] using the requirement of the absence of the negative norm Virassoro algebra states.

Yet the point is that even in the theories which do contain some of the operators $\{\phi_{n,m}\}$ with negative dimensions, it is possible to make a reduction of the whole operator theory to certain subalgebras, which do not contain pathological operators. Such subalgebras are well defined, at least from the statistical theory point of view.

We have not analyzed in much detail all the possible subalgebras which can be selected from other finite theories, not in the main series ($(\alpha_-)^2 = p/k$, but $k-p > 1$). And anyway, as the operator $\phi_{N,N-1}$ is contained in the set of degenerate conformal operators only for integer N, we take it that the spin operator in the Potts series of models is defined in our theory only for the discrete, but infinite, set of values of the parameter $y = 1/N$ above.

But the "thermal" operator subalgebra, which is generated by taking products of the energy operator $\varepsilon \sim \phi_{1,2}$, and which contains conformal operators $\{\phi_{1,n}\}$, does not include pathological operators for the whole continuous range of values of the parameter y. It is in this sense, if only the thermal algebra and the corresponding thermal critical exponents are considered, that we say the conformal theory of the Potts model is well defined for the continuous range of the parameter y or q.

We remark finally that the parameter q of the Potts model is related to the parameter y by the formula [16, 17]

$$\sqrt{q} = 2\cos\left(\tfrac{1}{2}\pi y\right). \tag{2.16}$$

On the other hand, for a continuous $(\alpha_-)^2$ the parameter $y = 1/N$ becomes

$$y = \frac{1}{N} = 2\left(1-\alpha_-^2\right) = 4\alpha_0\left(\sqrt{\alpha_0^2+1}-1\right), \qquad \alpha_0 = \sqrt{\tfrac{1}{24}(1-C)}\,. \tag{2.17}$$

In this way the "number of components" parameter q of the Potts model becomes related to the numbers of the conformal algebra $(\alpha_-)^2, C$.

We note here that the $q=4$ model is obtained in the limit $N \to \infty, (\alpha_-)^2 \to 1$. This algebra has the anomaly number $C=1$, and this is a trivial case of the conformal theory in the sense that all the multipoint correlators in this theory are given by simple algebraic functions. Some examples of that are given in appendix B.

Now we are in a position to find the conformal algebra for the O(n) model. This model has been considered recently by Nienhuis for a continuous range of the parameter n: $-2 \leqslant n \leqslant 2$ [9]. Under some plausible assumptions he derived the following relations for the thermal exponents of this model:

$$X_{T_1} = 2 - Y_{T_1} = 2t - 2, \tag{2.18}$$

$$X_{T_2} = 6t - 4, \tag{2.19}$$

with

$$n = -2\cos\frac{2\pi}{t}\,. \tag{2.20}$$

And he conjectured a relation for the magnetic exponent:

$$X_{H_1} = 1 - \frac{3}{4t} - \frac{t}{4}\,. \tag{2.21}$$

Relation (2.18) has been conjectured earlier by Cardy and Hamber [19].

We find the realization of the conformal algebra for this model by the following line of arguments. First, it is easy to convince oneself that exponents X_{T_1} and X_{T_2} in (2.18), (2.19) are essentially different from those for the Potts model, eqs. (2.8), (2.10). Even if we knew nothing of (2.20) and just adjusted the parameter t in (2.18) to the parameter y in the Potts model in order to make X_{T_1} in (2.8) and in (2.18) the same, the exponent X_{T_2} expressed in this parameter would be different.

Next we know that the two models must coincide for $q = 2$ and $n = 1$ – which is the Ising model. In particular, the exponent X_{T_1} which is given by the conformal dimension of the energy operator Δ_ε, should be the same at this crossing point of the two models. In minimal conformal theories, to which the Ising model belongs, each physical operator has two "representatives" among the conformal operators $\phi_{n,m}$, see [1,2]. In the Ising model algebra [1] the second partner of the energy operator $\varepsilon \sim \phi_{1,2}$ is $\phi_{3,1}$. One can check e.g. that for the Ising model value of $(\alpha_-)^2 = \frac{3}{4}$ in (2.1), $\Delta_{1,2} = \Delta_{3,1} = \frac{1}{2}$ ($(\Delta_\varepsilon)_{\rm ph} = 2\Delta_{-,2} = 1$ as it should be). It is only at this point ($(\alpha_-)^2 = \frac{3}{4}$) that the operator $\phi_{3,1}$ is equivalent to the energy operator of the Potts model $\phi_{1,2}$. Now it is natural to assume that the operator $\phi_{3,1}$ should be the energy operator of the O(n) model. Let us now check this assumption. We calculate $\Delta_{3,1}$ using the same parametrization as in the Potts model: $(\alpha_-)^2 = p/k$, $k = 2N$, $p = 2N - 1$, $y = 1/N$. We obtain

$$\Delta_{3,1} = \frac{(3p-k)^2 - (p-k)^2}{4pk} = 1 - \frac{1}{N} = 1 - y,$$

$$X_{T_1} = 2\Delta_{3,1} = 2 - 2y. \tag{2.22}$$

Now we adjust this equation to (2.18) and find the relation between the parameters:

$$y = 2 - t. \tag{2.23}$$

The next-to-leading thermal exponent, relative to the operator $\phi_{3,1}$, corresponds to the conformal operator which is created in the product of two energy operators. According to the general theory [1]

$$\phi_{3,1}\phi_{3,1} \sim I + \phi_{3,1} + \phi_{5,1}. \tag{2.24}$$

So the next-to-leading thermal operator is $\phi_{5,1}$. We calculate

$$\Delta_{5,1} = \frac{(5p-k)^2 - 1}{4pk} = 4 - 3y,$$

and we use relation (2.23) to find

$$X_{T_2} = 2\Delta_{5,1} = 6t - 4. \tag{2.25}$$

This agrees with (2.19). Finally, if we assume that the spin operators in both Potts and O(n) models correspond to the same conformal operator $\phi_{N,N-1}$, then we find

$$X_{H_1} = 2\Delta_{N,N-1} = \frac{1-y^2}{4(2-y)} = 1 - \frac{3}{4t} - \frac{t}{4}\,. \tag{2.26}$$

This is the Nienhuis' conjecture (2.21).

Thus we have established the following identifications: the energy operators of the Potts and O(n) models correspond respectively to the basic operators $\phi_{1,2}$ and $\phi_{3,1}$ of the degenerate conformal theory; the spin operators in both models correspond to the same conformal operator $\phi_{N,N-1}$.

We remark finally that one more infinite series of models is obtained if we choose $\phi_{2,1}$ to be the energy operator ε. It has been found by Friedan, Qiu and Shenker [18] that for this choice the tricritical Ising (Z_2) and Z_3 Potts models result. As they have found, the spin operator is again placed in the center of the table. By taking $y = 1/N$, and N running in this case over half-integer values, one gets an infinite set of tricritical Potts models. The thermal subalgebra of the operators $\phi_{2n+1,1}$ ($n = 0, 1, 2\dots$) is well defined for the continuous range of y.

3. Integral representation for conformal correlators

In this section it will be shown that the correlators of the general conformal theory in 2D [1] can be represented by averages of vertex operators in a Coulomb-like system with special boundary conditions (BC).

In this language the basic conformal operators are represented as exponents of a free field $\varphi(z,\bar{z})$:

$$V_\alpha(z,\bar{z}) = e^{i\alpha\varphi(z,\bar{z})}, \tag{3.1}$$

and the dynamics of the field φ are defined by the action

$$A[\varphi] \sim \int dz\, d\bar{z}\, \partial_z\varphi\, \partial_{\bar{z}}\varphi\,. \tag{3.2}$$

For the usual BC on φ at infinity the correlators of the theory (3.1), (3.2) are trivial:

$$\begin{aligned}
\langle V_\alpha(z) V_{-\alpha}(z')\rangle &= \frac{\int D\varphi\, e^{i\alpha\varphi(z)} e^{-i\alpha\varphi(z')} e^{-A[\varphi]}}{\int D\varphi\, e^{-A[\varphi]}} \\
&= \exp\left\{-\tfrac{1}{2}\alpha^2\left[2\langle\varphi^2\rangle - 2\langle\varphi(z)\varphi(z')\rangle\right]\right\} \\
&= \exp\left\{-4\alpha^2\left(\ln\frac{R}{a} - \ln\frac{R}{|z-z'|}\right)\right\} \\
&= \left|\frac{a}{z}\right|^{4\alpha^2} \sim \frac{1}{|z|}4\alpha^2.
\end{aligned} \tag{3.3}$$

In the same way we find

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)\rangle \sim \prod_{i<j}|z_i - z_j|^{4\alpha_i\alpha_j}, \qquad \sum\alpha_i = 0. \tag{3.4}$$

Here we used $\langle\varphi(z)\varphi(z')\rangle = 4\ln(R/|z-z'|)$, a is a cut-off scale at small distances (lattice spacing), R is the size of the system, $R \to \infty$. The correlators for vertices (3.1) are nonzero only if R cancels out, which imposes the usual neutrality condition $\Sigma\alpha_i = 0$.

Expression (3.3) shows that the scaling dimension of the operator $V_\alpha(z)$ is $2\alpha^2$.

The energy momentum (EM) tensor in this theory is

$$T_{zz} = -\tfrac{1}{4}\partial_z\varphi\partial_z\varphi, \qquad T_{\bar{z}\bar{z}} = -\tfrac{1}{4}\partial_{\bar{z}}\varphi\partial_{\bar{z}}\varphi,$$

$$T(z) \equiv T_{zz}(z) = -\tfrac{1}{4}:\partial_z\varphi\partial_z\varphi:. \tag{3.5'}$$

This theory, with operators (3.1) and the EM tensor (3.5), is a simple case of a 2D conformal invariant operator system, to which the general theory of [1] also applies.

In particular, it is shown that

$$\langle T_{z\bar{z}}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\dots\rangle = 0, \tag{3.6}$$

$$\partial_{\bar{z}}\langle T_{zz}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\phi_2(z_2\bar{z}_2)\dots\rangle = 0, \tag{3.7}$$

($\phi_1, \phi_2 \dots$ are any basic conformal invariant operators) which implies that the z and $\bar{z}$ dynamics decouple (see [1,2]), i.e. we can consider all operators and their correlators as dependent only on z_i, keeping all $\bar{z}_i$'s formally fixed. Dependence on $\{\bar{z}_i\}$ will be restored later by using certain symmetry requirements on physical correlators, see sects. 4 and 5.

So for the time being we shall ignore $\bar{z}$ dependencies altogether. Then the expressions (3.3) and (3.4) become

$$\langle V_\alpha(z)V_{-\alpha}(z')\rangle \sim \frac{1}{(z-z')}2\alpha^2, \tag{3.3'}$$

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)\rangle \sim \prod_{i<j}(z_i - z_j)^{2\alpha_i\alpha_j}, \tag{3.4'}$$

and the relevant component of the EM tensor is

$$T(z) \equiv T_{zz}(z) = -\tfrac{1}{4}:\partial_z\varphi\partial_z\varphi:. \tag{3.5'}$$

It follows from (3.3′) that the conformal dimension of the operator $V_\alpha(z)$ is

$$\Delta_\alpha = \alpha^2,$$

which is one-half of its real scaling dimension.

The product of fields in (3.5′) is "normal ordered", which implies that we extract an irrelevant divergence from the product of fields that would occur if we put this product inside some correlator, i.e.

$$:\partial_z\varphi\partial_z\varphi: = \lim_{z,z'\to(z+z')/2} \{\partial_z\varphi(z)\partial_{z'}\varphi(z') - \langle\partial_z\varphi(z)\partial_{z'}\varphi(z')\rangle\}.$$

The vertex operator (3.1) is also "normal ordered", which again implies that if we put it inside some correlator and expand the exponent in powers of φ

$$\exp(i\alpha\varphi(z)) = 1 + i\alpha\varphi + \tfrac{1}{2}(i\alpha)^2\varphi\varphi + \cdots,$$

then, under the functional averaging, the Wick pairings of φ's in the expansion should be made only with the outside fields. All internal pairings are extracted. This results in an overall divergent factor:

$$V_\alpha(z) = :\exp(i\alpha\varphi(z)): \sim \frac{1}{a^{\alpha^2}}\exp(i\alpha\varphi(z)). \tag{3.8}$$

Eq. (3.8) shows explicitly that the quantum dimension of the vertex $V_\alpha(z)$ is $\Delta_\alpha = \alpha^2$.

The Coulomb system (3.1)–(3.5) gets modified if we assume that there is a fixed charge $-2\alpha_0$, placed at ∞, and resulting in a modified BC at ∞ on the field $\varphi(z)$. Now the neutrality condition becomes

$$\sum\alpha_i = 2\alpha_0. \tag{3.9}$$

Only correlators which satisfy (3.9) are nonzero in this theory. The two-point correlator becomes

$$\langle V_\alpha(z)V_{2\alpha_0-\alpha}(z')\rangle \sim \frac{1}{(z-z')^{2\alpha(\alpha-2\alpha_0)}}. \tag{3.10}$$

Here $V_{2\alpha_0-\alpha}$ is a sort of conjugate to V_α (instead of $V_{-\alpha}$, if $\alpha_0 = 0$) and the conformal dimension becomes

$$\Delta_\alpha = \Delta_{2\alpha_0-\alpha} = \alpha^2 - 2\alpha\alpha_0. \tag{3.11}$$

Because of modified BC the EM tensor $T(z)$ gets an additional term (cf. (3.5′)):

$$T(z) = -\tfrac{1}{4}:\partial_z\varphi\partial_z\varphi: + A\partial_z^2\varphi. \tag{3.12}$$

The coefficient at $\partial_z^2\varphi$ is fixed by the conformal Ward identity (WI) [1]:

$$\langle T(z)\phi_1(z_1)\phi_2(z_2)\dots\rangle = \sum_i\left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i\right)\langle\phi_1(z_1)\phi_2(z_2)\dots\rangle. \tag{3.13}$$

Here $\{\phi_i\}$ are conformal operators with dimensions $\{\Delta_i\}$. The WI (3.13) fixes the first two terms in the operator expansion of the product $T(z)V_\alpha(z')$:

$$T(z)V_\alpha(z') = \frac{\Delta_\alpha}{(z-z')^2}V_\alpha(z') + \frac{1}{z-z'}\partial_{z'}V_\alpha(z') + \cdots . \tag{3.14}$$

On the other hand we can derive this expansion explicitly:

$$\begin{aligned} T(z)V_\alpha(z') &= :\left(-\tfrac{1}{4}\partial_z\varphi\partial_z\varphi + A\partial^2\varphi\right): :e^{i\alpha\varphi(z')}: \\ &= \frac{\alpha^2 + 2i\alpha A}{(z-z')^2} :e^{i\alpha\varphi(z)}: + \frac{i\alpha}{z-z'} :\partial_z\varphi(z)\, e^{i\alpha\varphi(z')}: \\ &\quad + :\left(-\tfrac{1}{4}\partial_z\varphi\partial_z\varphi + A\partial^2\varphi(z)\right)e^{i\alpha\varphi(z')}: \\ &= \frac{\alpha^2 + 2i\alpha A}{(z-z')^2}V_\alpha(z') + \frac{1}{z-z'}\partial_{z'}V_\alpha(z') + \cdots . \end{aligned} \tag{3.15}$$

Here we have used $\langle\varphi(z)\varphi(z')\rangle = 2\ln(R/(z-z'))$. Now comparing Δ_α in (3.11) with (3.14) and (3.15) we find $A = i\alpha_0$, so that

$$T(z) = -\tfrac{1}{4}:\partial_z\varphi\partial_z\varphi: + i\alpha_0\partial_z^2\varphi. \tag{3.16}$$

Transformation of the field $\varphi(z)$ under conformal transformations also changes. The basic conformal operator $\phi(z)$ with dimension Δ transforms, by definition, as follows (see [1]):

$$z \to f(z) \approx z + \varepsilon(z), \tag{3.17}$$

$$\begin{aligned} \phi(z) &\to (f'(z))^\Delta \phi(f(z)) \\ &\approx \phi(z) + \left(\Delta\varepsilon'(z) + \varepsilon(z)\partial_z\right)\phi(z). \end{aligned} \tag{3.18}$$

In the Coulomb theory (3.1), (3.2) with $\alpha_0 = 0$ only the variable z of the $\varphi(z)$ changes under the conformal transformations:

$$\varphi(z) \to \varphi(f(z)), \tag{3.19}$$

and the vertex $V_\alpha(z)$ acquires an additional scaling factor only because of its dependence on the cut-off scale a, see (3.8):

$$V_\alpha(z) = :e^{i\alpha\varphi(z)}: \to (f'(z))^{\Delta_\alpha}:e^{i\alpha\varphi(f(z))}:, \qquad \Delta_\alpha = \alpha^2. \tag{3.20}$$

In the Coulomb theory with $\alpha_0 \neq 0$ the transformation of $\varphi(z)$ has to be modified. This is due to special BC on $\varphi(z)$ at ∞. The easiest way to find this additional piece in $\delta\varphi(z)$ is by noticing that the dimension Δ_α in the transformation (3.18) of the vertex $V_\alpha(z)$ should be the same as that given by the two-point correlator (3.10). This requires the following transformation of the scalar field $\varphi(z)$:

$$\begin{aligned}\varphi(z) &\to \varphi(f(z)) + 2i\alpha_0 \ln f'(z)\\ &\approx \varphi(z) + \varepsilon(z)\,\partial_z\varphi(z) + 2i\alpha_0\varepsilon'(z).\end{aligned} \tag{3.21}$$

By using (3.21) and (3.8) we then find

$$\begin{aligned}V_\alpha = {:}\mathrm{e}^{i\alpha\varphi(z)}{:} &\to \left(f'(z)\right)^{\alpha^2} {:}\exp\{i\alpha\varphi(f(z)) - 2\alpha\alpha_0 \ln f'(z)\}{:}\\ &= \left(f'(z)\right)^{\alpha^2-2\alpha\alpha_0} V_\alpha(f(z)),\end{aligned} \tag{3.22}$$

in agreement with $\Delta_\alpha = \alpha^2 - 2\alpha\alpha_0$, see (3.11).

Another useful relation is that of the "vacuum" charge α_0 to the anomaly number C of the general conformal algebra [1]. The two-point correlator for $T(z)$ has the form:

$$\langle T(z)T(z')\rangle = \frac{\frac{1}{2}C}{(z-z')^4},$$

where C is the central charge of the Virasoro algebra for L_n components of $T(z)$:

$$\begin{aligned}T(z) &= \sum_n \frac{L_n(z_1)}{(z-z_1)^{n+2}},\\ [L_n, L_m] &= (n-m)L_{n+m} + \tfrac{1}{2}Cn(n^2-1)\,\delta_{n,-m},\end{aligned} \tag{3.23}$$

see [1,2]. We can now find the correlator $\langle T(z)T(z')\rangle$ explicitly:

$$\begin{aligned}\langle T(z)T(z')\rangle &= \langle{:}\left(-\tfrac{1}{4}\partial\varphi(z)\,\partial\varphi(z) + i\alpha_0\partial^2\varphi(z)\right){:}\\ &\quad\times{:}\left(-\tfrac{1}{4}\partial\varphi(z')\,\partial\varphi(z') + i\alpha_0\partial^2\varphi(z')\right){:}\rangle\\ &= \frac{1}{(z-z')^4}\,\tfrac{1}{2}\left(1-24\alpha_0^2\right),\end{aligned} \tag{3.24}$$

which gives

$$C = 1 - 24\alpha_0^2. \tag{3.25}$$

Now we shall start deriving an integral representation for multipoint conformal correlators, using the relation, established above, of the Coulomb system with $\alpha_0 \neq 0$ to the general conformal theory.

First we shall specify which particular operator theory we should like to obtain. We shall look at properties of operators $\{V_\alpha\}$ of the Coulomb system and impose certain requirements which should hold if these operators are to be identified with physical operators of some statistical theory.

The first requirement is that the 4-point correlator for any operator of the theory should be nonzero:

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle \neq 0. \tag{3.26}$$

In our Coulomb system we can construct the following function:

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle, \tag{3.27}$$

with $\Sigma\alpha_i = 2\alpha_0$. Next we want all the Coulomb operators in this function to have the same conformal dimension Δ, if they are to be identified with the 4-point correlator for a single physical operator, as in (3.26). This leaves us with the choice between V_α and $V_{2\alpha_0-\alpha}$ for any of V_{α_i} in (3.27). If $\alpha_0 = 0$ such a function is easy to find:

$$\langle V_\alpha V_\alpha V_{-\alpha} V_{-\alpha} \rangle .$$

In the case of $\alpha_0 \neq 0$ it seems that there is no way to meet the two requirements: $\Sigma_i \alpha_i = 2\alpha_0$, and $\alpha_i = \alpha$ or $2\alpha_0 - \alpha$. Functions like

$$\langle V_\alpha V_\alpha V_{2\alpha_0-\alpha} V_{2\alpha_0-\alpha} \rangle, \tag{3.28}$$

$$\langle V_\alpha V_\alpha V_\alpha V_{2\alpha_0-\alpha} \rangle, \tag{3.29}$$

are zero in our Coulomb theory because $\Sigma\alpha_i \neq 2\alpha_0$.

Yet there is, in fact, a way to make the correlators like (3.29) nonzero. In the Coulomb theory with $\alpha_0 \neq 0$ there are two nontrivial operators which can "screen" an additional charge. Such screening operators should be conformal invariant (have conformal dimension $\Delta = 0$) so that they do not change the conformal properties of the correlator. A local operator with $\Delta = 0$ is an identity operator of the algebra. In Coulomb theory it has two representatives: $V_{\alpha=0}(z)$ and $V_{2\alpha_0}(z)$. Neither of these operators would provide the necessary screening. There remains the possibility of the integral operators like

$$Q = \oint_C \mathrm{d}z\, J(z). \tag{3.30}$$

For the operator Q to be conformal invariant, the operator $J(z)$ must have $\Delta = 1$. We take the vertex $V_\alpha(z)$ and impose the condition

$$\Delta_\alpha = \alpha^2 - 2\alpha\alpha_0 = 1. \qquad (3.31)$$

There are two solutions to it:

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}, \qquad (3.32)$$

and so there are two screening operators

$$Q_\pm = \oint_C \mathrm{d}z J_\pm(z), \qquad J_\pm(z) = V_{\alpha\pm}(z). \qquad (3.33)$$

Now we can put operators (3.33) inside the Coulomb average in any numbers. They will not effect the conformal properties, which will be defined only by operators $V_{\alpha_i}(z_i)$ inside the correlator. And yet operators $Q_\pm$ will shift the balance of the charges $\{\alpha_i\}$.

Let us look now at the 4-point correlators (3.28), (3.29). For (3.28) $\Sigma\alpha_i - 2\alpha_0 = 2\alpha_0$. This surplus of $2\alpha_0$ cannot be cancelled by the charges $\alpha_\pm$ of $J_\pm$. But in the case of (3.29) we have $\Sigma\alpha_i - 2\alpha_0 = 2\alpha$, and this can be cancelled by adding $Q_\pm$, if α is quantized:

$$2\alpha = -\tilde{n}\alpha_- - \tilde{m}\alpha_+,$$

or

$$\alpha = \alpha_{n,m} = \tfrac{1}{2}(1-n)\alpha_- + \tfrac{1}{2}(1-m)\alpha_+. \qquad (3.34)$$

Here $\tilde{n}, \tilde{m}$ are positive integers, which are shifted in (3.34) for later convenience.

Thus we can build 4-point correlators with the properties natural for statistical theory (nonzero, if all 4 operators are the same) out of Coulomb vertices $V_\alpha(z)$ only if the parameter α is restricted to the discrete set of values $\alpha_{n,m}$, eq. (3.34). (We notice here that the set (3.34) is not unique. Another choice is to start with the Coulomb average $\langle V_\alpha V_\alpha V_\alpha V_\alpha \rangle$ and obtain the condition $4\alpha = 2\alpha_0 - n\alpha_- - m\alpha_+$, $\alpha = \tfrac{1}{4}(1-n)\alpha_- + \tfrac{1}{4}(1-m)\alpha_+$ [20]. In this paper we restrict ourselves to the conformal algebra related to the set (3.34).) The corresponding conformal dimensions are given by

$$\Delta_{n,m} = \alpha_{n,m}^2 - 2\alpha_{n,m}\alpha_0 = \tfrac{1}{4}\left[(\alpha_- n - \alpha_+ m)^2 - (\alpha_+ + \alpha_-)^2\right], \qquad (3.35)$$

which is precisely the Kac formula for the degenerate conformal operators [10], see also eq. (2.1), and eqs. (32), (33) in [2]. In general, the 4-point function will have the following form:

$$\langle\phi_{n,m}(z_1)\phi_{n,m}(z_2)\phi_{n,m}(z_3)\phi_{n,m}(z_4)\rangle$$

$$\sim\oint_{C_1}\mathrm{d}u_1\ldots\oint_{C_{n-1}}\mathrm{d}u_{n-1}\oint_{S_1}\mathrm{d}v_1\ldots\oint_{S_{m-1}}\mathrm{d}v_{m-1}\langle V_{\alpha_{nm}}(z_1)V_{\alpha_{nm}}(z_2)$$

$$\times V_{\alpha_{nm}}(z_3)V_{\alpha_{nm}}(z_4)J_-(u_1)\ldots J_-(u_{n-1})J_+(v_1)\ldots J_+(v_{m-1})\rangle. \quad (3.36)$$

As all the integrations are analytic, the integral does not depend on the precise form of the contours $\{C_1,\ldots C_{n-1},S_1,\ldots S_{m-1}\}$. But they have to be chosen carefully, winding around points z_1, z_2, z_3, z_4, so that they do not shrink to a point, resulting in the integral being zero.

This integral representation for the conformal functions has been found recently by Feigin and Fuchs [6]. Here it was described, perhaps, in a somewhat different language.

In the remainder of this section we shall start more specific studies of the conformal functions provided by the integrals (3.36).

Let us look first at a particular correlator:

$$\langle\phi_{n,m}\phi_{1,2}\phi_{1,2}\phi_{n,m}\rangle. \quad (3.37)$$

We assume that $(n,m)\geqslant(1,2)$. The corresponding integral with the minimal number of J's has the form

$$\oint_C \mathrm{d}v\langle V_{\alpha_{nm}}(z_1)V_{\alpha_{12}}(z_2)V_{\alpha_{12}}(z_3)V_{2\alpha_0-\alpha_{nm}}(z_4)J_+(v)\rangle$$

$$\sim(z_{12}z_{13})^{2\alpha_{12}\alpha_{nm}}(z_{23})^{2\alpha_{12}^2}(z_{24})^{2\alpha_{nm}(2\alpha_0-\alpha_{nm})}(z_{24}z_{34})^{2\alpha_{12}(2\alpha_0-\alpha_{nm})}$$

$$\times\oint_C\mathrm{d}v(v-z_1)^{2\alpha_+\alpha_{nm}}((v-z_2)(v-z_3))^{2\alpha_+\alpha_{12}}(v-z_4)^{2\alpha_+(2\alpha_0-\alpha_{nm})}. \quad (3.38)$$

This integral gives the functions of z_1, z_2, z_3, z_4, which is projectively invariant. Using this invariance we can fix three points at arbitrary values. The standard choice is $z_1=0$, $z_2=z$, $z_3=1$, $z_4\to\infty$. Then the integral takes the form

$$z^{2\alpha_{12}\alpha_{nm}}(1-z)^{2\alpha_{12}^2}\int_C\mathrm{d}v\,v^{2\alpha_+\alpha_{nm}}(v-1)^{2\alpha_+\alpha_{12}}(v-z)^{2\alpha_+\alpha_{12}}. \quad (3.39)$$

The integral here is the hypergeometric function which is a solution of the second-order differential equation.

There are two independent choices of the contour C in (3.39), fig. 1. They correspond to two independent solutions of the hypergeometric differential equation:

$$I_1(a,b,c;z)=\int_1^{\infty} \mathrm{d}v\, v^a(v-1)^b(v-z)^c$$

$$=\frac{\Gamma(-a-b-c-1)\Gamma(b+1)}{\Gamma(-a-c)}F(-c,-a-b-c-1;-a-c;z),$$

$$I_2(a,b,c;z)=\int_0^{z} \mathrm{d}v\, v^a(1-v)^b(z-v)^c$$

$$=z^{1+a+c}\int_0^1 \mathrm{d}v\, v^a(1-v)^c(1-zv)^b$$

$$=z^{1+a+c}\frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)}F(-b,a+1,a+c+2;z). \qquad (3.40)$$

Here $a=2\alpha_+\alpha_{nm}$, $b=c=2\alpha_+\alpha_{12}$ and $F(\alpha,\beta;\gamma;\ )$ is the hypergeometric function. It is assumed here that powers of $v, v-1, v-z$ in these integrals are such that the integrals are convergent. Otherwise the contours of integration should be taken winding around the corresponding points, or, equivalently but technically simpler, the functions (3.40) can be assumed as an analytic continuation from values of the parameters for which the integrals are convergent. This analytic continuation simplifies essentially the handling with higher integrals (with the number of integrations more than one), see sect. 5 and the coming paper [7].

In a similar way, the integral for the correlator

$$\langle\phi_{nm}(0)\phi_{31}(z)\phi_{31}(1)\phi_{nm}(\infty)\rangle, \qquad (3.41)$$

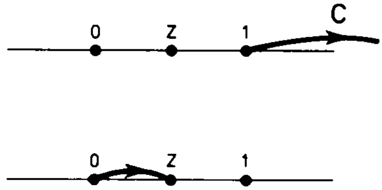

Fig. 1.

takes the form

$$z^{2\alpha_{31}\alpha_{nm}}(1-z)^{2\alpha_{31}^2}\int_{C_1} du_1 \int_{C_2} du_2\, u_1^a(u_1-1)^b(u_1-z)^c$$

$$\times u_2^a(u_2-1)^b(u_2-z)^c(u_1-u_2)^g. \quad (3.42)$$

Here $a=2\alpha_-\alpha_{nm}, b=c=2\alpha_-\alpha_{31}, g=2\alpha_-^2$. In this case we choose the set of contours shown in fig. 2. The corresponding integrals are studied in detail in sect. 5. Here we only note that in this case there are three independent configurations of contours C_1, C_2 – those in fig. 2. The corresponding integrals provide three independent solutions to the third-order differential equation of the general theory [1].

For the correlator

$$\langle \phi_{nm}(0)\phi_{22}(z)\phi_{22}(1)\phi_{nm}(\infty)\rangle, \quad (3.43)$$

the Coulomb average will contain both $J_-(u)$ and $J_+(v)$. The integral will take the form:

$$z^{2\alpha_{22}\alpha_{nm}}(1-z)^{2\alpha_{22}^2}\int_C du \int_S dv\, u^a(u-1)^b(u-z)^c$$

$$\times v^{a'}(v-1)^{b'}(v-z)^{c'}(u-v)^g. \quad (3.44)$$

Here $a=2\alpha_-\alpha_{nm}$, $b=c=2\alpha_-\alpha_{22}$, $a'=2\alpha_+\alpha_{nm}$, $b'=c'=2\alpha_+\alpha_{22}$, $g=2\alpha_+\alpha_- = -2$.

In this case there are four independent configurations of contours, those in fig. 3, which provide four solutions of the fourth-order differential equation.

Now the generalization is straightforward. It is easy to check that for the correlator

$$\langle \phi_{nm}(0)\phi_{kl}(z)\phi_{kl}(1)\phi_{nm}(\infty)\rangle, \quad (3.45)$$

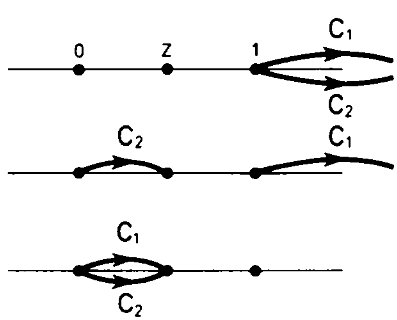

Fig. 2.

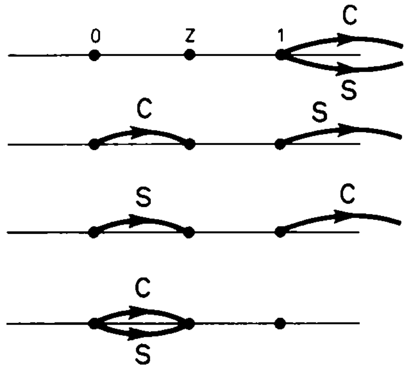

Fig. 3.

there are $k \times l$ independent integrals (we assume that $k \times l < n \times m$). In this way we recover the result of the general theory that the conformal correlators containing the degenerate operator $\phi_{k,l}$ satisfy the $(k \times l)$th-order differential equation. By using the Coulomb representation described above we can find directly all the solutions to those equations.

We should like to remark finally on the apparent ambiguity in the choice of integrals for a particular correlator. For example, for the correlator (3.41) we could equally well write the following integral:

$$\int_{C_1} du_1 \dots \int_{C_{n-1}} du_{n-1} \int_{S_1} dv_1 \dots \int_{S_{m-1}} dv_{m-1} \langle V_{\alpha_{31}}(0) V_{\alpha_{nm}}(z) V_{\alpha_{nm}}(1)$$
$$\times V_{2\alpha_0 - \alpha_{31}}(\infty) J_-(u_1) \dots J_-(u_{n-1}) J_+(v_1) \dots J_+(v_{m-1}) \rangle .$$

Taking all the independent configurations of contours we would obtain $n \times m$ functions, which are solutions of the differential equation corresponding to the operator $\phi_{n,m}$. The point is that among those $n \times m$ functions there are three which are also solutions of the third-order equation (corresponding to the operator $\phi_{3,1}$ and given by the integrals (3.42)), while the others are redundant.

Particular examples of how this technique enables us actually to find the physical correlators in 2D statistical models are described in detail in the next sections.

4. Monodromy problem involved in the calculation of the conformal operators. The second-order correlator

In this section we shall present our technique as applied to the simplest nontrivial case – the 4-point correlator which contains the second-order conformal operator ϕ_{12}.

As explained in the preceding section the conformal functions related to the correlator

$$\langle \phi_{n_1 m_1}(0)\phi_{12}(z)\phi_{n_3 m_3}(1)\phi_{n_4 m_4}(\infty)\rangle, \tag{4.1}$$

are given by the integral

$$\int_C \mathrm{d}t \langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)J_+(t)\rangle$$

$$= z^{2\alpha_1\alpha_2}(1-z)^{2\alpha_2\alpha_3}\int_C \mathrm{d}t\, t^a(t-1)^b(t-z)^c, \tag{4.2}$$

with the two choices of C, those in fig. 1. Here $a = 2\alpha_1\alpha_+$, $b = 2\alpha_3\alpha_+$, $c = 2\alpha_2\alpha_+$ and $\alpha_i \equiv \alpha_{n_i m_i}$ are given by (3.34), in particular $\alpha_2 \equiv \alpha_{12} = -\frac{1}{2}\alpha_+$. And $\{\alpha_i\}$ are subject to the neutrality condition (3.9), so that $\alpha_4 = 2\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_+$.

For the two choices of the contour C shown in fig. 1 we shall have the two integrals $I_1(z)$ and $I_2(z)$, see (3.40). To form the physical correlator we have to restore the dependence on $\bar{z}$. The physical correlator should be of the following form:

$$G(z,\bar{z}) = \sum X_{ij} I_i(z)\overline{I_j(z)}\,. \tag{4.3}$$

The function $I_i(z)$ has the singular points $0, 1, \infty$. If we continue z analytically along a closed contour C_0 or C_1 encircling the point $z=0$ or $z=1$ (fig. 4), the function $I_i(z)$ transforms as

$$I_i(z) \overset{C_0}{\rightarrow} (g_0)_{ij} I_j(z),$$

$$I_i(z) \overset{C_1}{\rightarrow} (g_1)_{ij} I_j(z). \tag{4.4}$$

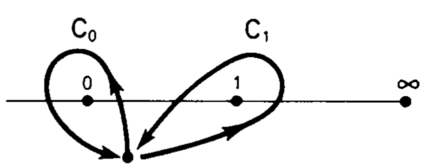

Fig. 4.

Here g_0 and g_1 are 2×2 matrices. They are the generating elements of the monodromy group transformations*. If we continue $I_i(z)$ and $\overline{I_j(z)}$ in (4.3) simultaneously along C_0 or C_1, the function $G(z,\bar{z})$ will change, in general. Obviously the physical correlator for the operators like energy or spin should be monodromy invariant.

From (3.40) it follows that g_0 has a diagonal form:

$$g_0=\begin{pmatrix}1 & 0\\ 0 & \exp(2\pi i(1+a+c))\end{pmatrix}. \tag{4.5}$$

$G(z,\bar{z})$ will be invariant with respect to the g_0 transformation if it has a diagonal form, too. So we are led to the expression

$$G(z,\bar{z})=X_1I_1(z)\overline{I_i(z)}+X_2I_2(z)\overline{I_2(z)}. \tag{4.6}$$

There remains the transformation g_1. This matrix can be made unitary by rescaling I_1 and I_2 like

$$\hat{I}_1(z)=I_1(z),\qquad \hat{I}_2(z)=CI_2(z), \tag{4.7}$$

see [2]. When both g_0 and g_1 are unitary the quadratic form

$$G(z_1\bar{z})\sim\hat{I}_1(z)\overline{\hat{I}_1(z)}+\hat{I}_2(z)\overline{\hat{I}_2(z)} \tag{4.8}$$

will be monodromy invariant.

But in this paper we shall take the slightly different route of constructing the invariant function $G(z,\bar{z})$, which is shorter, and is easier to generalize to higher correlators.

We rewrite (4.6) in a compact form as

$$G(z,\bar{z})\sim\sum X_iI_i(z)\overline{I_i(z)}. \tag{4.9}$$

As was said above, this function is explicitly g_0-invariant. Now for the integrals $I_i(z)$, which are the basic solutions of the differential equation, there is the following expansion [21]:

$$I_i(z)=\sum_j\alpha_{ij}\tilde{I}_j(1-z). \tag{4.10}$$

The set of integrals $I_i(z)$ is said to be a canonical one for the point $z=0$, because each integral has no more than one singularity as $z\to 0$. The functions $\tilde{I}_j(1-z)$

* See any textbook on linear differential equations.

form a canonical set for the point $z = 1$. In particular, for the base $\{\tilde{I}_j(1-z)\}$ the matrix g_1 has a diagonal form like (4.5). If we put (4.10) into (4.9) we obtain

$$G(z,\bar{z}) = \sum_i X_i \alpha_{ik} \alpha_{il} \tilde{I}_k(1-z) \overline{\tilde{I}_l(1-z)} . \tag{4.11}$$

The function $G(z,\bar{z})$ will be invariant with respect to g_1 if the quadratic form on the r.h.s. of (4.11) is diagonal, i.e.

$$\sum X_i \alpha_{ik} \alpha_{il} = 0, \qquad k \neq l. \tag{4.12}$$

If the matrix α_{ij} of (4.10) is known, the coefficients X_1, X_2 in the quadratic form (4.7) can be found easily from (4.12) (their relation X_1/X_2 in fact).

So the remaining problem is to find the matrix α_{ij}. For the second-order case, in which $\{I_i\}$ are hypergeometric integrals, the expansion (4.10) is well known of course. But we shall derive it again here by the technique which can be easily generalized to the higher-order cases.

Let us take the integral $I_1(z)$, eq. (3.10), and continue analytically the contour of the integration in the two different ways, as shown in fig. 5. Then we multiply the two resulting integrals by the phase factors shown in fig. 5a in order to cancel the phases in the interval $(0, z)$. And then we subtract them one from another. In this

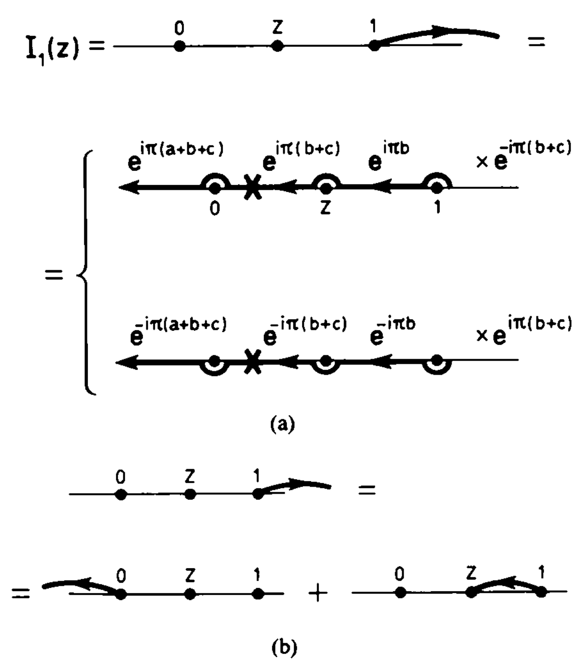

Fig. 5.

way we find the relation (see also fig. 5b, where the coefficients are omitted):

$$I_1(a,b,c;z) = \frac{s(a)}{s(b+c)} I_1(b,a,c;1-z) - \frac{s(c)}{s(b+c)} I_2(b,a,c;1-z). \tag{4.13}$$

Here $s(A) \equiv \sin(\pi A)$, and so on. A similar expansion for the function $I_2(z)$ is found as shown in fig. 6. We obtain

$$I_2(a,b,c;z) = -\frac{s(a+b+c)}{s(b+c)} I_1(b,a,c;1-z) - \frac{s(b)}{s(b+c)} I_2(b,a,c;1-z). \tag{4.14}$$

So, we find

$$\alpha_{11} = \frac{s(a)}{s(b+c)}, \qquad \alpha_{12} = -\frac{s(c)}{s(b+c)},$$

$$\alpha_{21} = -\frac{s(a+b+c)}{s(b+c)}, \qquad \alpha_{22} = -\frac{s(b)}{s(b+c)}. \tag{4.15}$$

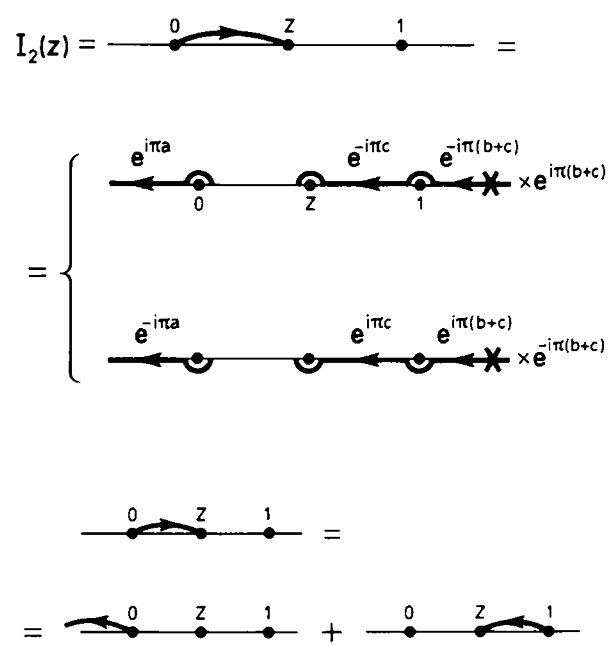

Fig. 6.

Now, from the eq. (4.12) we find

$$\frac{X_1}{X_2} = -\frac{\alpha_{21}\alpha_{22}}{\alpha_{11}\alpha_{22}} = \frac{s(a+b+c)s(b)}{s(a)s(c)}. \tag{4.16}$$

The invariant function $G(z, \bar{z})$ in (4.6) is found to be

$$G(z,\bar{z}) \sim s(a+b+c)s(b)|I_1(a,b,c;z)|^2 + s(a)s(c)|I_2(a,b,c;z)|^2. \tag{4.17}$$

Restoring all the standard scaling factors and returning to the gauge in which z_1, z_3 and z_4 are arbitrary, we find the following expression for the general second-order correlator (up to an overall normalization):

$$\begin{aligned}
&\langle \phi_{n_1m_1}(z_1)\phi_{n_2m_2}(z_2)\phi_{n_3m_3}(z_3)\phi_{n_4m_4}(z_4)\rangle \\
&\sim \frac{|z_{13}|^{2[\Delta(\alpha_1+\alpha_3+\alpha_+)-\Delta_1-\Delta_3+2\alpha_+\alpha_2]}|z_{24}|^{2[\Delta(\alpha_2+\alpha_4)-\Delta_2-\Delta_4+2\alpha_+\alpha_2]}}{|z_{12}|^{-2[\Delta(\alpha_1+\alpha_2)-\Delta_1-\Delta_2]}|z_{23}|^{-2[\Delta(\alpha_2+\alpha_3)-\Delta_2-\Delta_3]}} \\
&\qquad |z_{34}|^{-2[\Delta(\alpha_3+\alpha_4+\alpha_+)-\Delta_3-\Delta_4]}|z_{14}|^{-2[\Delta(\alpha_1+\alpha_4+\alpha_+)-\Delta_1-\Delta_4]} \\
&\quad \times \{ s(a+b+c)\times s(b)|I_1(a,b,c;\eta)|^2 \\
&\quad + s(a)s(c)|I_2(a,b,c;\eta)|^2 \}.
\end{aligned} \tag{4.18}$$

Here $\eta = z_{12}z_{34}/z_{13}z_{24}$; the parameters are defined as in (4.2), and the conformal dimensions $\Delta(\alpha_i)$ are related to the Coulomb parameters $\alpha_i \equiv \alpha_{n_im_i}$ of the operators $\phi_{n_im_i}$ by the Kac formula (3.35).

The integrals $I_i(\eta)$ here are proportional to the hypergeometric functions, see (3.40). When $\eta \to 0$ the expansion of $I_i(\eta)$ starts as

$$\begin{aligned}
I_1(\eta) &\approx N_1(1 + a_1\eta + \cdots), \\
I_2(\eta) &\approx N_2\eta^{1+a+c}(1 + b_1\eta + \cdots).
\end{aligned} \tag{4.19}$$

We may need to know the normalization numbers N_i of the integrals $I_i(\eta)$ because they determine the relative numerical values of the coefficients in the operator algebra expansions:

$$\phi_i(z_i)\phi_j(z_j) \sim \sum_p C_{ij}^p \frac{1}{|z_i - z_j|^{\Delta_i+\Delta_j-\Delta_p}} \phi_p(z_j). \tag{4.20}$$

These last can be derived from the 4-point correlator $\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle$ if we let some pairs of the points, e.g. z_1, z_2 and z_3, z_4, come close together, $z_1 \to z_2$, $z_3 \to z_4$, and examine the singularities which result in this limit. In this way the structure constants of the operator algebra C_{ij}^p become related to the normalization constants N_1 and N_2 in the expansions (4.19).

The general structure of the conformal operator algebra, as related to the properties of the 4-point correlator (or amplitude) are treated in detail in [1]. We shall come back to this point in our next paper.

For the Potts models eq. (4.18) provides 4-point correlation functions containing the energy operator ε. This operator corresponds to the second-order conformal operators $\phi_{1,2}$ in the Potts model, and $\phi_{2,1}$ in the tricritical Potts model, see sect. 2. The "symmetric" correlation functions $\langle\phi_{nm}\varepsilon\varepsilon\phi_{nm}\rangle$ of the Potts model have been calculated in [2]. The four-energy correlation function $\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle$ of the Potts model was also found by Kadanoff and Nienhuis [21]. Expression (4.18) also provides nonsymmetric correlation functions. For the particular representation of the Z_3 model some of these are listed in the summary section of [2].

For $O(n)$ series of statistical models the energy operator is the third-order conformal operator $\phi_{3,1}$ (see sect. 2). So in this case even the "thermal" correlators require the higher-order functions. In the next section, we shall calculate the third-order correlation functions, and, in particular, we shall find the thermal correlators of the $O(n)$ model.

5. The third-order correlation functions

The conformal functions related to the third-order correlator

$$\langle\phi_{n_1m_1}(0)\phi_{31}(z)\phi_{n_3m_3}(1)\phi_{n_4m_4}(\infty)\rangle, \tag{5.1}$$

are given by the integral (see sect. 3):

$$\int_{C_1}\mathrm{d}t_1\int_{C_2}\mathrm{d}t_2\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)J_-(t_1)J_-(t_2)\rangle$$

$$= z^{2\alpha_1\alpha_2}(1-z)^{2\alpha_2\alpha_3}\int_{C_1}\mathrm{d}t_1\int_{C_2}\mathrm{d}t_2\, t_1^a(t_1-1)^b(t_1-z)^c$$

$$\times t_2^a(t_2-1)^b(t_2-z)^c(t_1-t_2)^g. \tag{5.2}$$

Here $a=2\alpha_1\alpha_-$, $b=2\alpha_3\alpha_-$, $c=2\alpha_2\alpha_-$, $g=2\alpha_-^2$ and $\alpha_i\equiv\alpha_{n_im_i}$ are given by (3.34). In particular, $\alpha_{3,1}=-\alpha_-$, so that $c=2\alpha_{3,1}\alpha_-=-2\alpha_-^2=-g$. The parameters $\{\alpha_i\}$ are subject to the neutrality condition (3.9), so that $\alpha_4=2\alpha_0-\alpha_1-\alpha_2-\alpha_3-2\alpha_-$.

In this case there are three independent configurations of contours – they are shown in fig. 2. Accordingly, we define the following set of basic functions:

$$J_1(a,b,c;g;z)=\mathrm{e}^{-i\pi g/2}\int_1^\infty\mathrm{d}t_1\int_1^\infty\mathrm{d}t_2(\dots)$$

$$=2c(\tfrac{1}{2}g)\int_1^\infty\mathrm{d}t_1\int_1^\infty\mathrm{d}t_2(\dots)\equiv 2c(\tfrac{1}{2}g)I_1(a,b,c;g;z), \tag{5.3}$$

$$J_2(a,b,c;g;z)=\int_1^\infty \mathrm{d}t_1\int_0^z \mathrm{d}t_2(\ldots)$$

$$=z^{1+a+c}\int_1^\infty \mathrm{d}t_1\int_0^1 \mathrm{d}t_2\, t_1^a(t_1-1)^b(t_1-z)^c$$

$$\times\; t_2^a(1-t_2)^b(t_1-zt_2)^g\equiv I_2(a,b,c;g;z), \tag{5.4}$$

$$J_3(a,b,c;g;z)=\mathrm{e}^{-i\pi g/2}\int_0^z \mathrm{d}t_1\int_0^z \mathrm{d}t_2(\ldots)$$

$$=z^{2+2a+2c+g}2c(\tfrac{1}{2}g)\int_0^1 \mathrm{d}t_1\int_0^1 \mathrm{d}t_2\, t_1^a(1-t_1)^c(1-zt_1)^b$$

$$\times t_2^a(1-t_2)^c(1-zt_2)^b(t_1-t_2)^g\equiv 2c(\tfrac{1}{2}g)I_3(a,b,c;g;z) \tag{5.5}$$

Here $(\ldots)$ stands for the standard integrand, that in the integral (5.2); $c(\frac{1}{2}G)=\cos(\frac{1}{2}\pi G)$. Integration in the first expressions for the integrals (5.3), (5.5), in which (t_1-t_2) changes sign, is defined as in fig. 2 – one contour goes above the other.

In the integral (5.2) there are, in fact, only three parameters, since $g=-c$. We shall build the invariant form $G(z,\bar{z})$, eq. (4.9) (the correlation function itself) for a more general case, when g is not related to c. There is an additional reason for doing this. In the integral (5.2), related to the correlator (5.1), it is assumed that one of the operators is the third-order operator $\phi_{3,1}$. For this particular case the third-order integral (5.2) satisfied the linear differential equation of the general theory in [1], which is derived by using the third-level degeneracy of Virassoro states of the operator $\phi_{3,1}$. Yet the situation may be more general. In the next section an example of the correlator is given, which does not contain the third-order operators $\phi_{3,1}$ or $\phi_{1,3}$, and yet the correlation function is given by the third-order integrals (5.3)–(5.5), with $g\neq -c$. The differential equation for this more general case, when g is an independent parameter, is given in appendix A. One can check that for $g=-c$ this equation reduces to that given in [1].

We remark also that for the special case of "symmetric" correlators, like $\langle ABBA\rangle$, the situation will be the standard one: the order of the conformal functions will correspond to the order of the operators, and the differential equations will be those found in [1]. In sect. 3 we considered integrals only for this symmetric case.

We shall turn now to the construction of the invariant function $G(z,\bar{z})$, eq. (4.9).

The integrals $I_i(z)$, eqs. (5.3)–(5.5), form a full set of independent solutions of eq. (A.9), canonical for the point $z=0$. They have the following form:

$$I_i(z)=z^{\rho_i^{(0)}}f_i(z),$$

$$\rho_1^{(0)}=0,\qquad \rho_2^{(0)}=1+a+c,\qquad \rho_3^{(0)}=2+2a+2c+g. \tag{5.6}$$

Here $f_i(z)$ are analytic functions regular at $z=0$.

The integrals $I_i(z)$, the same as the second-order (hypergeometric) integrals in the preceding section, have three singular points: $0, 1, \infty$. Corresponding to an analytic continuation of these functions around $z=0$ and $z=1$ there are two generating elements of the monodromy group: g_0 and g_1 (see sect. 4). In this case they are 3×3 matrices. In particular, the matrix g_0 is diagonal – this follows from (5.6).

The function $G(z, \bar{z})$, eq. (4.9), in this case has the form

$$G(z, \bar{z}) = X_1 |I_1(z)|^2 + X_2 |I_2(z)|^2 + X_3 |I_3(z)|^2, \tag{5.7}$$

and it is explicitly g_0-invariant. The invariance with respect to a g_1-transformation is to be ensured by a proper choice of the coefficients X_i in (5.7). Just as in sect. 4, we can express the functions $I_i(z)$ as a linear combination of solutions $\tilde{I}_i(1-z)$ which are canonical for the point $z=1$. In the basis of the functions $\{\tilde{I}_i(1-z)\}$ the matrix g_1 will be diagonal. We have the relation

$$I_i(z) = \sum_j \alpha_{ij} \tilde{I}_j(1-z), \tag{5.8}$$

and we substitute it into (5.7) to find

$$G(z, \bar{z}) = \sum_i X_i \alpha_{ik} \alpha_{il} \tilde{I}_k(1-z) \overline{\tilde{I}_l(1-z)}\,. \tag{5.9}$$

The quadratic form $G(z, \bar{z})$ will be g_1-invariant if the matrix $\sum_i X_i \alpha_{ik} \alpha_{il}$ is diagonal. So we obtain the equation

$$\sum X_i \alpha_{ik} \alpha_{il} = 0, \qquad k \neq l. \tag{5.10}$$

which determines the coefficients X_i (their relations in fact) if the matrix α_{ij} is known.

We have to remark now that, apparently, there is a question of solubility of the system (5.10) with respect to $\{X_i\}$. The fact is that if the matrix α_{ij} is not arbitrary, but is related to the solutions of the differential equation, as in (5.8), then the system (5.10) is solvable. We have checked this by using the explicit form of the matrix α_{ij}, but we expect that there should be a general theorem on this.

The remaining problem is to find the matrix α_{ij}. It can be solved by using a technique similar to that which we have used in sect. 4 for the second-order integrals. Now we first transform only one of the contours. It is transformed in two different ways, as in fig. 7a. Then we multiply the two resulting integrals by the phase factors and subtract them one from another, so as to eliminate the unwanted piece of the first contour. After that the same is done with the second contour. This is shown in fig. 7b. As a result we obtain the first line of the relation (5.8). One of the manipulations with the contours needed for the integral $I_2(z)$ is shown in fig. 8. In

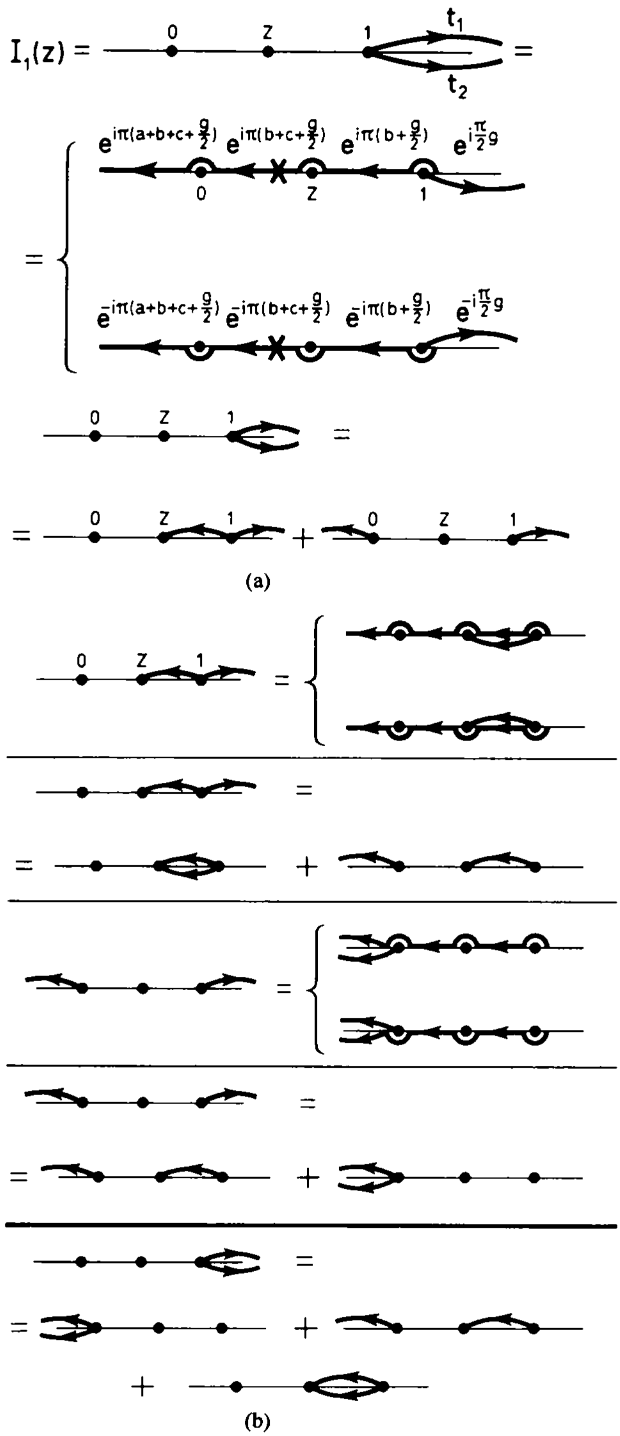
$I_1(z) =$
0
z
1
t_1
t_2
$e^{i\pi(a+b+c+\frac{g}{2})}$
$e^{i\pi(b+c+\frac{g}{2})}$
$e^{i\pi(b+\frac{g}{2})}$
$e^{i\frac{\pi}{2}g}$
$e^{-i\pi(a+b+c+\frac{g}{2})}$
$e^{-i\pi(b+c+\frac{g}{2})}$
$e^{-i\pi(b+\frac{g}{2})}$
$e^{-i\frac{\pi}{2}g}$
(a)
(b)

Fig. 7.

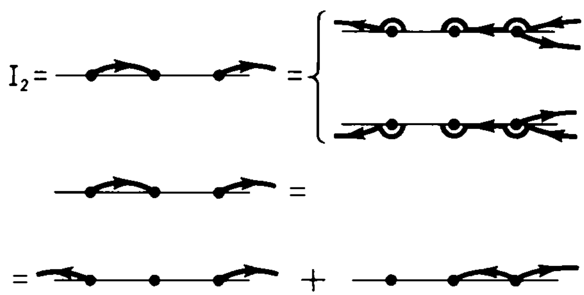

Fig. 8.

this way we obtain the following expressions for the matrix elements $\alpha_{ij}(a, b, c; g)$ in relation (5.8):

$$\alpha_{11} = \frac{s(a)s(a+\frac{1}{2}g)}{s(b+c)s(b+c+\frac{1}{2}g)}, \qquad \alpha_{12} = \frac{s(a)s(c)}{s(b+c)s(b+c+g)},$$

$$\alpha_{13} = \frac{s(c)s(c+\frac{1}{2}g)}{s(b+c+\frac{1}{2}g)s(b+c+g)},$$

$$\alpha_{21} = -\frac{s(a+b+c+\frac{1}{2}g)s(a+\frac{1}{2}g)}{s(b+c)s(b+c+\frac{1}{2}g)}2c(\tfrac{1}{2}g),$$

$$\alpha_{23} = \frac{s(b+\frac{1}{2}g)s(c+\frac{1}{2}g)}{s(b+c+\frac{1}{2}g)s(b+c+g)}2c(\tfrac{1}{2}g),$$

$$\alpha_{22} = \frac{s(a+b+c+\frac{1}{2}g)s(c)}{s(b+c)s(b+c+\frac{1}{2}g)} - \frac{s(b+\frac{1}{2}g)s(a)}{s(b+c+\frac{1}{2}g)s(b+c+g)},$$

$$\alpha_{31} = \frac{s(a+b+c+\frac{1}{2}g)s(a+b+c+g)}{s(b+c)s(b+c+\frac{1}{2}g)}, \qquad \alpha_{32} = \frac{s(a+b+c+g)s(b)}{s(b+c)s(b+c+g)},$$

$$\alpha_{33} = \frac{s(b)s(b+\frac{1}{2}g)}{s(b+c+\frac{1}{2}g)s(b+c+g)}. \tag{5.11}$$

Here $s(A) = \sin(\pi A)$, $c(A) = \cos(\pi A)$, and so on. The integrals $\tilde{I}_i(1-z)$ in relation (5.8) are given by

$$\tilde{I}_i(1-z) = I_i(b, a, c; g; 1-z), \tag{5.12}$$

with I_i defined by (5.3)–(5.5).

Now from (5.10) we find

$$\frac{X_1}{X_3}=\frac{\alpha_{33}\tilde{\alpha}_{31}}{\alpha_{13}\tilde{\alpha}_{33}}=\frac{s(a+b+c+g)s(a+b+c+\frac{1}{2}g)s(b)s(b+\frac{1}{2}g)s(a+c+g)}{s(a)s(a+\frac{1}{2}g)s(c)s(c+\frac{1}{2}g)s(a+c)},$$

$$\frac{X_2}{X_3}=\frac{\alpha_{33}\tilde{\alpha}_{32}}{\alpha_{23}\tilde{\alpha}_{33}}=\frac{s(a+b+c+g)s(a+c+\frac{1}{2}g)s(b)}{s(c+\frac{1}{2}g)s(a+\frac{1}{2}g)s(a+c)2c(\frac{1}{2}g)}. \tag{5.13}$$

Here $\tilde{\alpha}_{ij}$ is the matrix for the relation inverse to (5.12):

$$\tilde{I}_i(1-z)=\sum_j \tilde{\alpha}_{ij} I_j(z). \tag{5.14}$$

Obviously

$$\tilde{\alpha}_{ij}(a,b,c;g)=\alpha_{ij}^{-1}(a,b,c;g)=\alpha_{ij}(b,a,c;g). \tag{5.15}$$

Now, collecting all the factors and returning to the general values of z_1, z_3, z_4, we obtain the following expression for the third-order correlator (up to an overall constant):

$$\begin{aligned}&\langle\phi_{n_1m_1}(z_1)\phi_{n_2m_2}(z_2)\phi_{n_3m_3}(z_3)\phi_{n_4m_4}(z_4)\rangle\\ &\times\frac{|z_{13}|^{2[\Delta(\alpha_1+\alpha_3+2\alpha_-)-\Delta_1-\Delta_3+4\alpha_2\alpha_-]}|z_{24}|^{2[\Delta(\alpha_2+\alpha_4)-\Delta_2-\Delta_4+4\alpha_2\alpha_-]}}{|z_{12}|^{-2[\Delta(\alpha_1+\alpha_2)-\Delta_1-\Delta_2]}|z_{23}|^{-2[\Delta(\alpha_2+\alpha_3)-\Delta_2-\Delta_3]}}\\ &\qquad |z_{34}|^{-2[\Delta(\alpha_3+\alpha_4+2\alpha_-)-\Delta_3-\Delta_4]}|z_{14}|^{-2[\Delta(\alpha_1+\alpha_4+2\alpha_-)-\Delta_1-\Delta_4]}\\ &\times\{s(a+b+c+g)s(a+b+c+\tfrac{1}{2}g)s(b)s(b+\tfrac{1}{2}g)\\ &\times s(a+c+g)2c(\tfrac{1}{2}g)|I_1(a,b,c;g;\eta)|^2\\ &+s(a+b+c+g)s(a+c+\tfrac{1}{2}g)s(b)s(a)s(c)|I_2(a,b,c;g;\eta)|^2\\ &+s(a)s(a+\tfrac{1}{2}g)s(c)s(c+\tfrac{1}{2}g)s(a+c)2c(\tfrac{1}{2}g)|I_3(a,b,c;g;\eta)|^2.\end{aligned} \tag{5.16}$$

Here $\eta=z_{12}z_{34}/z_{13}z_{24}$; the parameters are defined as in (5.2), and the conformal dimensions $\Delta(\alpha_i)$ are related to the Coulomb parameters $\alpha_i=\alpha_{n_im_i}$ by the Kac formula (3.35).

For the reasons pointed out in sect. 4 it is convenient to introduce normalized functions $\mathcal{F}_i(a,b,c;g;z)$, such that for $z\to 0$ they have the following expansions:

$$\mathcal{F}_i(a,b,c;g;z)=\frac{1}{N_i}I_i(z)=z^{\rho_i^{(0)}}(1+\alpha_i z+\beta_i\tfrac{1}{2}z^2+\cdots),$$

$$\rho_1^{(0)}=0,\qquad \rho_2^{(0)}=1+a+c,\qquad \rho_3^{(0)}=2+2a+2c+g. \tag{5.17}$$

The normalization numbers $\{N_i\}$ here are given by

$$N_1 = \frac{\Gamma(g)}{\Gamma(\frac{1}{2}g)} \times \frac{\Gamma(-1-a-b-c-g)\Gamma(-1-a-b-c-\frac{1}{2}g)\Gamma(1+b)\Gamma(1+b+\frac{1}{2}g)}{\Gamma(-a-c)\Gamma(-a-c-\frac{1}{2}g)},$$

$$N_2 = \frac{\Gamma(-1-a-b-c-g)\Gamma(1+b)}{\Gamma(-a-c-g)} \frac{\Gamma(1+a)\Gamma(1+c)}{\Gamma(2+a+c)},$$

$$N_3 = \frac{\Gamma(g)}{\Gamma(\frac{1}{2}g)} \frac{\Gamma(1+a)\Gamma(1+a+\frac{1}{2}g)\Gamma(1+c)\Gamma(1+c+\frac{1}{2}g)}{\Gamma(2+a+c+\frac{1}{2}g)\Gamma(2+a+c+g)}. \tag{5.18}$$

Finally, we give two explicit expressions for the correlation functions of the O(n) model, which follow from the general formula (5.16). These are the four-energy correlator $\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle$ and the spin-energy correlator $\langle\sigma\varepsilon\varepsilon\sigma\rangle$. Using the parameter y defined in sect. 2 we get the following results:

$$\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle \sim |z_{12}z_{23}z_{34}z_{14}|^{4-2y}|z_{13}z_{24}|^{8y-12} \times \sum_i A_i|\mathcal{F}_i(y-2, y-2, y-2; 2-y; \eta)|^2. \tag{5.19}$$

Here

$$A_1 = 2c(\tfrac{1}{2}y)s(\tfrac{5}{2}y)s(2y)s^2(y)s(\tfrac{1}{2}y)(N_1)^2,$$

$$A_2 = s(2y)s(\tfrac{3}{2}y)s^3(y)(N_2)^2,$$

$$A_3 = 2c(\tfrac{1}{2}y)s(2y)s^2(y)s^2(\tfrac{1}{2}y)(N_3)^2,$$

$$N_1 = \frac{\Gamma(2-y)}{\Gamma(1-\frac{1}{2}y)} \frac{\Gamma(1-2y)\Gamma(4-\frac{5}{2}y)\Gamma(-1+y)\Gamma(\frac{1}{2}y)}{\Gamma(4-2y)\Gamma(3-\frac{3}{2}y)},$$

$$N_2 = \frac{\Gamma(1-2y)\Gamma(-1+y)}{\Gamma(2-y)} \frac{\Gamma^2(-1+y)}{\Gamma(-2+2y)},$$

$$N_3 = \frac{\Gamma(2-y)}{\Gamma(1-\frac{1}{2}y)} \frac{\Gamma^2(-1+y)\Gamma^2(\frac{1}{2}y)}{\Gamma(-1+\frac{3}{2}y)\Gamma(y)}. \tag{5.20}$$

A similar expression is found for the other correlator:

$$\langle\sigma\varepsilon\varepsilon\sigma\rangle \sim |z_{12}z_{34}|^{1+y}|z_{13}z_{24}|^{5y-8}|z_{23}|^{4-2y}|z_{14}|^{(31-40y+13y^2)/2(2-y)}$$

$$\times \sum_{i=1}^{3} B_i|\mathcal{F}_i(-\tfrac{1}{2}(1+y), y-2, y-2; 2-y; \eta)|^2. \tag{5.21}$$

The expressions for the coefficients B_i here follow from (5.16) and (5.18). As is found in sect. 2, the parameter y is related to the parameter n of the O(n) model by the formula (it follows from (2.20), (2.23)):

$$n = 2\cos\frac{\pi y}{2-y}. \tag{5.22}$$

6. Summary and discussion

In this paper we have described the technique of calculating the multipoint correlation functions in 2D statistical models at the critical point, which uses the conformal algebra approach initiated recently in the paper by Belavin, Polyakov and Zamolodchikov [1].

We have found the conformal representations for the Potts and O(n) series of models, see [2] and sect. 2 of this paper. Also Friedan, Qiu and Shenker have found the representations for the tricritical Potts models [18].

The 4-point correlation functions in these systems appear to be described by the higher analytic functions of the Fuchs kind (see previous footnote). The simplest of these are hypergeometric functions.

In this paper we have also studied some analytic properties of such functions, in particular their monodromy relations, which appear to be necessary for the calculation of correlation functions (Green functions, in the corresponding quantum field theories).

Detailed calculations have been presented in this paper only for the third-order functions. We have also found the general expressions for the 4-point Green functions in conformal theories of this kind (abelian theories). But this we shall leave until our next paper [7], because the calculations become much more complicated and require substantial space.

Now we shall make one or two remarks on the subject.

It should be noted that all the calculations of correlation functions in this paper deal with the cases where there is no degeneracy among the analytic functions. These may occur if the dimensions of the basic conformal operators differ by integers. More precisely, in terms of the operator algebra, the degeneracy occurs if the conformal dimensions of the operators, which appear in the intermediate channels of the 4-point Green functions, differ by integers. The integrals $I_i(z)$ provide independent solutions of the corresponding equations if this is not the case ($\rho_i^{(0)} - \rho_j^{(0)} \neq$ integer, in (5.17)).

Degeneracies result in zeroes or poles in the structural coefficients $\{X_i\}$ of the correlators (sects. 4, 5), as functions of parameters. In such anomalous cases one should be careful and define the necessary limits.

One check which we have carried out is the following. As was stated in sect. 2, in the IM algebra the operators $\phi_{1,2} \sim \varepsilon$ and $\phi_{3,1}$ are identical, see [1]. On the other hand we have found the general expression for the correlator $\langle \phi_{31}\phi_{31}\phi_{31}\phi_{31} \rangle$ which should reduce to $\langle \phi_{12}\phi_{12}\phi_{12}\phi_{12} \rangle$ at the IM point. We have checked that this is the case, but the limit is rather subtle. In particular, some coefficients A_i in

$$\langle \phi_{31}\phi_{31}\phi_{31}\phi_{31} \rangle \sim \sum_i A_i |\mathcal{F}_i(z)|^2 \tag{6.1}$$

go to zero, when we approach the IM point. But some of the coefficients of the expansion of functions $\mathcal{F}_i(z)$ themselves become divergent, providing a finite result.

Another point is that in the $C = 1$ conformal theory the correlation functions reduce to simple algebraic functions. For Potts and O(n) models $C \to 1$ corresponds to $q \to 4, n \to 2$ (see sect. 2). As a result, all the multipoint correlation functions in $q = 4$ and $n = 2$ models have a simple algebraic form. We demonstrate this reduction for the Potts model correlators $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$ and $\langle \sigma\varepsilon\varepsilon\sigma \rangle$ in appendix B. The same can also be checked for the third-order integrals of sect. 5.

And our last remark is on the existence of correlation functions which do not follow the standard rules. Here we just give an example of the following correlator:

$$\langle \phi_{12}\phi_{14}\phi_{14}\phi_{\overline{14}} \rangle . \tag{6.2}$$

It is easy to check that the corresponding integral is the following third-order function:

$$\int \mathrm{d}t_1 \int \mathrm{d}t_2 \langle V_{12}(0) V_{14}(z) V_{14}(1) V_{\overline{14}}(\infty) J_+(t_1) J_+(t_2) \rangle . \tag{6.3}$$

This correlator, being a third-order one, does not contain the third-order operators $\phi_{1,3}$ or $\phi_{3,1}$. Also, there are no second-order integrals for the set of operators in (6.2). Yet we have to include the correlator (6.2) in the theory, or otherwise we would get problems with the operator algebra relations.

Some remarks on this example have already been made in sect. 5. A detailed study of the operator algebra issues in this theory is left until our next paper [7].

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Appendix A

Let us consider the analytic function

$$I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} c\ b \\ c \end{matrix}; g; z\right) \equiv I(a,b,c;g;z) = \int_{C_1} dt_1 \int_{C_2} dt_2\, t_1^a (t_1-1)^b (t_1-z)^c$$

$$\times t_2^a (t_2-1)^b (t_2-z)^c (t_1-t_2)^g . \qquad \text{(A.1)}$$

It is assumed that we can integrate by parts, and the ends of the contours C_1 and C_2 do not contribute. In other words, it is assumed that the integrals are always convergent.

It is easy to check that the function

$$I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g; z\right)$$

has the following properties:

$$\left(\begin{matrix} a\ b \\ c+1 \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g-1\right) - I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} e\ f \\ c+1 \end{matrix}; g-1\right) = I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g\right), \qquad \text{(A.2)}$$

$$I\left(\begin{matrix} a\ b \\ c+1 \end{matrix}\middle|\begin{matrix} e\ f \\ c-1 \end{matrix}; g-1\right) - I\left(\begin{matrix} a\ b \\ c-1 \end{matrix}\middle|\begin{matrix} e\ f \\ c+1 \end{matrix}; g-1\right) = I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} e\ f \\ c-1 \end{matrix}; g\right) + I\left(\begin{matrix} a\ b \\ c-1 \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g\right),$$

$$\text{(A.3)}$$

$$zI\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g; z\right) = I\left(\begin{matrix} a+1, b \\ c \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g; z\right) - I\left(\begin{matrix} a\ b \\ c+1 \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g; z\right), \qquad \text{(A.4)}$$

$$(z-1)I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g; z\right) = I\left(\begin{matrix} a, b+1 \\ c \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g; z\right) - I\left(\begin{matrix} a\ b \\ c+1 \end{matrix}\middle|\begin{matrix} e\ f \\ c \end{matrix}; g; z\right),$$

$$\text{(A.5)}$$

$$aI\left(\begin{matrix} a-1, b \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g\right) + bI\left(\begin{matrix} a, b-1 \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g\right)$$

$$+ cI\left(\begin{matrix} a\ b \\ c-1 \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g\right) + gI\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g-1\right) = 0, \qquad \text{(A.6)}$$

$$dI\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d-1, e \\ f \end{matrix}; g\right) + eI\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d, e-1 \\ f \end{matrix}; g\right)$$

$$+ LI\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f-1 \end{matrix}; g\right) - gI\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g-1\right) = 0, \qquad \text{(A.7)}$$

$$I\left(\begin{matrix} a+1, b \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g-1\right) - I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d+1, e \\ f \end{matrix}; g-1\right) = I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} d\ e \\ f \end{matrix}; g\right). \qquad \text{(A.8)}$$

Using relations (A.2)–(A.8) it can be shown that the function

$$I\left(\begin{matrix} a\ b \\ c \end{matrix}\middle|\begin{matrix} a\ b \\ c \end{matrix}; g; z\right) \equiv I(a, b, c; g; z)$$

satisfies the following differential equation:

$$z^2(z-1)^2 I^{111}(z) + (K_1 z + K_2(z-1)) z(z-1) I^{11}(z)$$
$$+ \left(L_1 z^2 + L_2(z-1)^2 + L_3 z(z-1)\right) I^1(z)$$
$$+ (M_1 z + M_2(z-1)) I(z). \qquad \text{(A.9)}$$

The coefficients here are expressed through the parameters a, b, c and g of the function $I(a, b, c; g; z)$, eq. (A.1), by the following relations:

$$K_1 = -(g + 3b + 3c), \qquad K_2 = -(g + 3a + 3c),$$
$$L_1 = (b+c)(2b + 2c + g + 1), \qquad L_2 = (a+c)(2a + 2c + g + 1),$$
$$L_3 = (b+c)(2a + 2c + g + 1) + (a+c)(2b + 2c + g + 1)$$
$$+ (c-1)(a+b+c) + (3c+g)(a+b+c+g+1),$$
$$M_1 = -c(2b + 2c + g + 1)(2a + 2b + 2c + g + 2),$$
$$M_2 = -c(2a + 2c + g + 1)(2a + 2b + 2c + g + 2). \qquad \text{(A.10)}$$

Appendix B

We shall demonstrate the reduction of the correlators in the limit $C \to 1$ to simple algebraic functions by using a relatively simple example of the correlators $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$ and $\langle \sigma\varepsilon\varepsilon\sigma \rangle$ of the Potts model. For this model $C \to 1$ corresponds to $q \to 4$, see (2.16), (2.17).

From (3.40), (4.18), and in the gauge $z_1 = 0, z_2 = z, z_3 = 1, z_4 \to \infty$, we obtain the expressions

$$\langle \varepsilon(0)\varepsilon(z)\varepsilon(1)\varepsilon(\infty)\rangle \sim |z(1-z)|^{1+\delta}\Big\{ s(3+3\delta)s(1+\delta) \times \left|\frac{\Gamma(2+3\delta)\Gamma(-\delta)}{\Gamma(2+2\delta)}\right|^2 |F(1+\delta, 2+3\delta; 2+2\delta; z)|^2 + |z(1-z)|^{2(-1-2\delta)} s(1+\delta)s(1+\delta) \times \left|\frac{\Gamma(-\delta)\Gamma(-\delta)}{\Gamma(-2\delta)}\right|^2 |F(-1-3\delta, -\delta; -2\delta; z)|^2, \quad \text{(B.1)}$$

$$\langle \sigma(0)\varepsilon(z)\varepsilon(1)\sigma(\infty)\rangle \sim |z|^{-1/2-3\delta/2}|1-z|^{1+\delta} \times \left\{ s(-\tfrac{3}{2}-\tfrac{1}{2}\delta)s(\delta)\left|\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\delta)\Gamma(-\delta)}{\Gamma(\frac{1}{2}-\frac{1}{2}\delta)}\right|^2 |F(1+\delta, \tfrac{1}{2}; \tfrac{1}{2}-\delta; z)|^2 + |z|^{2(1/2+\delta/2)}|1-z|^{2(-1-2\delta)} s(\tfrac{1}{2}+\tfrac{3}{2}\delta)s(\delta) \times \left|\frac{\Gamma(\frac{3}{2}+\frac{3}{2}\delta)\Gamma(-\delta)}{\Gamma(\frac{3}{2}+\frac{1}{2}\delta)}\right|^2 |F(\tfrac{1}{2}-\tfrac{1}{2}\delta, -\delta; \tfrac{3}{2}+\tfrac{1}{2}\delta; z)|^2 \right\}. \quad \text{(B.2)}$$

Here we have used the following expressions for the parameters:

$$\varepsilon \sim \phi_{12}, \qquad \alpha_{12} = -\tfrac{1}{2}\alpha_+,$$
$$a = b = c = 2\alpha_{12}\alpha_+ = -\alpha_+^2 \approx -1-\delta,$$
$$\Delta(\alpha_i) + \Delta(\alpha_j) - \Delta(\alpha_i+\alpha_j) = -2\alpha_i\alpha_j,$$
$$4\alpha_{12}^2 = \alpha_+^2 \approx 1+\delta, \quad \text{(B.3)}$$

in the case of correlator (B.1), and

$$\varepsilon \sim \phi_{12}, \qquad \sigma \sim \phi_{N,N-1}, \qquad \alpha_{N,N-1} = \tfrac{1}{2}(1-N)\alpha_- + \tfrac{1}{2}(2-N)\alpha_+,$$
$$a = 2\alpha_{N,N-1}\alpha_+ \approx \tfrac{1}{2} + \tfrac{3}{2}\delta, \qquad b = c \approx -1-\delta,$$
$$4\alpha_{N,N-1}\alpha_{12} = -2\alpha_{N,N-1}\alpha_+ \approx -\tfrac{1}{2} - \tfrac{3}{2}\delta, \quad \text{(B.4)}$$

in the case of correlator (B.2). In both cases we defined $(\alpha_+)^{-2} = (\alpha_-)^2 = 1 - 1/2N \equiv 1 - \delta$.

The limit $C \to 1$ implies that $(\alpha_+)^2 \to 1$, i.e. $\delta \to 0$ (see (3.25), (3.32)). In this limit we obtain the results

$$\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle \sim \frac{1}{|z(1-z)|} + \frac{|z|}{|1-z|} + \frac{|1-z|}{|z|}, \tag{B.5}$$

$$\langle \sigma\varepsilon\varepsilon\sigma \rangle \sim \frac{1}{|z|^{1/2}|1-z|} + \frac{|z|^{1/2}}{|1-z|}. \tag{B.6}$$

It is easy to check that we would obtain the same results by just averaging the exponents of free fields $\varphi(z)$:

$$\langle \cos(\alpha_\varepsilon\varphi(0)) \cos(\alpha_\varepsilon\varphi(z)) \cos(\alpha_\varepsilon\varphi(1)) \cos(\alpha_\varepsilon\varphi(\infty)) \rangle,$$

$$\alpha_\varepsilon = \alpha_{12} \to -\tfrac{1}{2}, \qquad \delta = \frac{1}{2N} \to 0; \tag{B.7}$$

$$\langle \cos(\alpha_\sigma\varphi(0)) \cos(\alpha_\sigma\varphi(z)) \cos(\alpha_\sigma\varphi(1)) \cos(\alpha_\sigma\varphi(\infty)) \rangle,$$

$$\alpha_\sigma = \alpha_{N,N-1} \to \tfrac{1}{4}, \qquad \delta = \frac{1}{2N} \to 0. \tag{B.8}$$

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