International Journal of Modern Physics A
Vol. 31, Nos. 28 & 29 (2016) 1645038 (17 pages)

DOI: 10.1142/S0217751X1645038X

Minimal string theory and the Douglas equation

A. A. Belavin

L.D. Landau Institute for Theoretical Physics,
142432 Chernogolovka, Russia
and
Institute for Information Transmission Problems,
127994 Moscow, Russia
and
Moscow Institute of Physics and Technology,
141700 Dolgoprudny, Russia
belavin@itp.ac.ru

V. A. Belavin

P.N. Lebedev Physical Institute,
119991 Moscow, Russia
and
Institute for Information Transmission Problems,
127994 Moscow, Russia
belavin@lpi.ru

Published 19 October 2016

We use the connection between the Frobenius manifold and the Douglas string equation to further investigate Minimal Liouville gravity. We search for a solution of the Douglas string equation and simultaneously a proper transformation from the KdV to the Liouville frame which ensures the fulfilment of the conformal and fusion selection rules. We find that the desired solution of the string equation has an explicit and simple form in the flat coordinates on the Frobenius manifold in the general case of (p,q) Minimal Liouville gravity.

1. Introduction

The purpose of this talk[1] is to present the progress in the study of the Minimal Liouville Gravity (MLG)[2] using an approach based on the Douglas string equation.[3] This study is a continuation of earlier works.[4–7]

The Liouville Gravity represents a consistent example of the noncritical String theory. In the initial continuous approach the Liouville Gravity is formulated as a BRST invariant theory composed of the matter sector, the Liouville theory and the ghosts system. MLG represents the theory, where the matter sector is taken to be a (p, q) Minimal Model of CFT.[8] The main problem of MLG is to calculate correlation functions of BRST invariant observables, which are given by integrals over moduli of Riemannian surfaces. Usually they are called the correlation numbers. Numerous examples show that the solution of the problem is quite nontrivial within the framework of the continuous approach.

An alternative approach to MLG has grown up from the idea of triangulations of two-dimensional surfaces realized in terms of Matrix Models.[9–15] One of the most important points of the approach is the string equation which was derived by Douglas[3] in the Matrix Models approach to two dimensional gravity. The subject of the string equation is the generating function of the correlation numbers which depends on the parameters of the problem (the so called KdV times). In our work, following Refs. 5–7, we will conjecture that the Douglas equation is applicable to the Minimal Liouville gravity as well as to Matrix Models of 2D gravity.

This conjecture requires the following two questions to be answered: how to choose the desired solution of the Douglas string equation and an appropriate form of the so called resonance transformation[4] from the KdV times to the Liouville coupling constants. Once these two questions are answered, the generating function of the correlation functions in MLG is given explicitly as an integrated one-form defined uniquely for each (p, q) MLG model and coincides with a special choice of the tau-function of the dispersionless limit[16,17] of the generalized KdV hierarchy.

In this talk, using the connection[6] of the approach to MLG[5] based on the string equation with the Frobenius manifold structure, we find the necessary solution of the string equation. We also show that this solution together with the suitable chosen resonance transformation lead to the results which are consistent with the main requirements of (p, q) models of MLG (the so-called selection rules). It is remarkable that the needed solution of the Douglas equation has a very simple form in the flat coordinates on the Frobenius manifold in the general case of (p,q) Minimal Liouville as well as it has been found recently in the case of unitary models of MLG.[7]

2. Frobenius Manifolds

In this and the two next sections we give the definition and a short review of the main properties of the Frobenius manifolds needed for our purposes. Here we follow the paper by B. Dubrovin,[17] see also Ref. 6.

By definition a commutative associative algebra A with unity equipped with a nondegenerate invariant bilinear form (,) is called Frobenius algebra. The invariance of the bilinear form means that for any three vectors a, b, c in A:

$$(a \cdot b, c) = (a, b \cdot c). \tag{1}$$

Let M be n-dimensional manifold with a flat metric $\eta_{\alpha\beta} dv^\alpha dv^\beta$ which is constant in the flat coordinates v^α.

We introduce in the tangent space $T_{\mathbf{v}}M$ the structure of the Frobenius algebra by the following identification of the bases

$$\frac{\partial}{\partial v^\alpha} \to e_\alpha, \tag{2}$$

Thus, we can multiply tangent vectors at any point of M

$$e_\alpha e_\beta = C^\gamma_{\alpha\beta} e_\gamma. \tag{3}$$

The structure constants $C^\gamma_{\alpha\beta}$ may depend on v^α. Such a manifold M can be called quasi-Frobenius manifold.

Definition 2.1. *The manifold M is called Frobenius manifold if these two structures are adjusted with each other in such a way that*

(1) *the invariant bilinear form $(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta})$ is identical to the metric $\eta_{\alpha\beta}$;*
(2) *the structure of the Frobenius algebra at each point of M and the metric on M are constrained by the following relation*

$$\nabla_\rho C_{\alpha\beta\gamma} = \nabla_\alpha C_{\rho\beta\gamma}. \tag{4}$$

The last requirement is equivalent to the requirement that there exists a function F on M which is connected with the structure constants of the Frobenius algebra as

$$C_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma}, \tag{5}$$

where

$$C_{\alpha\beta\gamma} = \eta_{\alpha\rho} C^\rho_{\beta\gamma}. \tag{6}$$

The function F is called Frobenius potential. The consistency of this property with the associativity of the Frobenius algebra is known as WDVV condition[18]

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\rho}\, \eta^{\rho\lambda}\, \frac{\partial^3 F}{\partial v^\lambda \partial v^\mu \partial v^\nu} = \frac{\partial^3 F}{\partial v^\nu \partial v^\beta \partial v^\rho}\, \eta^{\rho\lambda}\, \frac{\partial^3 F}{\partial v^\lambda \partial v^\mu \partial v^\alpha}. \tag{7}$$

The following statement[17] follows from these properties of the Frobenius manifold M. There exist an one-parametric flat deformation $\widetilde{\nabla}_\alpha$ of the connection ∇_α

$$\widetilde{\nabla}_\alpha x^\gamma = \nabla_\alpha x^\gamma - z C^\gamma_{\alpha\beta} x^\beta, \tag{8}$$

or, equivalently,

$$[\widetilde{\nabla}_\alpha(z), \widetilde{\nabla}_\beta(z)] = 0. \tag{9}$$

The proof is based on the associativity of the Frobenius algebra and the equation (4). As a consequence of Eq. (9), there exist n linear independent solutions

$$\theta^\alpha(v,z) = \sum_{k=0}^{\infty} \theta^\alpha_k(v) z^k, \tag{10}$$

of the equation $\widetilde{\nabla}_\alpha d\theta^\lambda(v,z)=0$, which is equivalent to

$$\frac{\partial^2\theta^\lambda}{\partial v^\alpha \partial v^\beta}(v,z)=zC^\gamma_{\alpha\beta}\frac{\partial\theta^\lambda}{\partial v^\gamma}(v,z), \tag{11}$$

or

$$\frac{\partial^2\theta^\lambda_{k+1}}{\partial v^\alpha \partial v^\beta}(v)=C^\gamma_{\alpha\beta}\frac{\partial\theta^\lambda_k}{\partial v^\gamma}(v). \tag{12}$$

The functions $\theta^\alpha(v,z)$ can be considered as the flat coordinates of the deformed connection $\widetilde{\nabla}_\alpha(z)$. We choose $\theta^\lambda(v,z)$ so that $\theta^\lambda(v,0)=\theta^\lambda_0(v)=v^\lambda$. From Eq. (12) it follows, that

$$\nabla(\nabla\theta^\alpha(v,z_1),\nabla\theta^\beta(v,z_2))=(z_1+z_2)\nabla\theta^\alpha(v,z_1)\cdot\nabla\theta^\beta(v,z_2), \tag{13}$$

and, hence, the scalar product $(\nabla\theta^\alpha(v,z),\nabla\theta^\beta(v,-z))=Const(z)$ does not depend on v^α. For $z=0$ we find $Const(0)=\eta^{\alpha\beta}$. Equation (12) is invariant with respect to the transformation

$$\theta^\mu(v,z)\to A^\mu_\nu(z)\theta^\nu(v,z), \tag{14}$$

where $A^\mu_\nu(0)=\delta^\mu_\nu$. Using these transformations one can fix the normalization in such a way that

$$(\nabla\theta^\alpha(v,z),\nabla\theta^\beta(v,-z))=\eta^{\alpha\beta}. \tag{15}$$

3. Main Example: Frobenius Manifold of A_{q-1}-Type

Our main example is A_{q-1} Frobenius manifold.[18] Let $Q(y)$ be a polynomial of y

$$Q(y)=y^q+u_1y^{q-2}+\cdots+u_{q-1}, \tag{16}$$

and $\{u_\alpha\}$ represent some coordinates on M. We call $\{u_\alpha\}$ the canonical coordinates.

Definition 3.1. *A_{q-1} Frobenius algebra is the space of polynomials modulo polynomial $\frac{dQ}{dy}$:*

$$A_{q-1}(u)=\mathbb{C}[y]/\frac{dQ}{dy}. \tag{17}$$

The corresponding manifold M is called the Frobenius manifold of A_{q-1} type

The polynomials

$$P_\alpha(y)=\frac{\partial Q}{\partial u_\alpha}, \tag{18}$$

form a basis in the tangent space $T_{\mathbf{v}}M$. An invariant bilinear form (which is equivalent to the metric) is defined by

$$(P_\alpha,P_\beta)=\operatorname*{res}_{y=\infty}\left(\frac{P_\alpha(y)P_\beta(y)}{\frac{dQ}{dy}(y)}\right). \tag{19}$$

With this definition one can verify that the corresponding metric is flat and

$$C_{\alpha\beta\gamma} = \nabla_\alpha \nabla_\beta \nabla_\gamma F(u). \tag{20}$$

To this end we perform the transformation from the canonical coordinates $\{u_\alpha\}$ to the new coordinates $\{v^\alpha\}$ by means of the following relation

$$y = z - \frac{1}{q}\left(\frac{v^{q-1}}{z} + \frac{v^{q-2}}{z^2} + \cdots + \frac{v^1}{z^{q-1}}\right) + \mathcal{O}\left(\frac{1}{z^{q+1}}\right), \tag{21}$$

where $z^q = Q(y)$.

Some useful properties of the new coordinates are formulated in the following

Theorem 3.1. *From the transformation* (21) *it follows that*

(1) v^α *form flat coordinates, i.e., the metric in this coordinates is constant and*

$$\eta_{\alpha\beta} = -q\left(\frac{\partial Q}{\partial v^\alpha}, \frac{\partial Q}{\partial v^\beta}\right) = \delta_{\alpha+\beta,q}, \tag{22}$$

(2)

$$C_{\alpha\beta\gamma} = -q \operatorname*{res}_{y=\infty}\left(\frac{\frac{\partial Q}{\partial v^\alpha}\frac{\partial Q}{\partial v^\beta}\frac{\partial Q}{\partial v^\gamma}}{\frac{dQ}{dy}}\right) = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma}. \tag{23}$$

(3)

$$\theta_{\alpha,k} = -c_{\alpha,k} \operatorname*{res}_{y=\infty} Q^{k+\frac{\alpha}{q}}(y), \tag{24}$$

where

$$c_{\alpha,k} = \frac{\Gamma(\frac{\alpha}{q})}{\Gamma(\frac{\alpha}{q}+k+1)}. \tag{25}$$

To prove these statements it is convenient to use the basis elements of A_{q-1} in flat coordinates defined by $\Phi_\alpha(y) = \frac{\partial Q(y)}{\partial v^\alpha}$ which possess the following property

$$\Phi_\alpha(y) = \frac{1}{\alpha}\frac{d}{dy}\left(Q^{\frac{\alpha}{q}}\right)_+. \tag{26}$$

In what follows we use the following convention

$$\theta_{\mu,k} = \theta_{\mu - q\lfloor \mu/q \rfloor, k + \lfloor \mu/q \rfloor}, \tag{27}$$

where $\lfloor \mu/q \rfloor$ is the integer part of μ/q. It is clear that (27) agrees with the definition (24).

4. Frobenius Manifolds and Douglas String Equation

4.1. *Integrable hierarchies*

Let $\mathcal{M}$ be a space of functions of x taking values in M. Let I and J be functionals on $\mathcal{M}$. We define the Poisson bracket on $\mathcal{M}$ as

$$\{I, J\} = \int \frac{\delta I}{\delta v^\alpha(x)} \eta^{\alpha\beta} \frac{d}{dx} \frac{\delta I}{\delta v^\beta(x)} dx, \tag{28}$$

or

$$\{v^\alpha(x), v^\beta(y)\} = \eta^{\alpha\beta}\delta'(x-y), \tag{29}$$

where, as usual in the calculus of variations, the integrand is defined modulo total derivatives. The functionals

$$H_{\alpha,k} = \int \theta_{\alpha,k+1}(\vec{v}(x))dx, \quad \alpha = 1,\ldots,n, \quad k \geq 0, \tag{30}$$

mutually commute among themselves

$$\{H_{\alpha,k}, H_{\beta,l}\} = 0. \tag{31}$$

As a result, the Hamiltonian flows

$$\frac{\partial v^\mu}{\partial t_k^\alpha} = \{v^\mu, H_{\alpha,k}\} = \eta^{\mu\nu}\frac{\partial}{\partial x}\frac{\partial\theta_{\alpha,k+1}}{\partial v^\nu} = C_\lambda^{\mu\rho}\frac{\partial\theta_{\alpha,k}}{\partial v^\rho}\frac{\partial v^\lambda}{\partial x}. \tag{32}$$

commute, i.e.,

$$\frac{\partial}{\partial t_l^\beta}\frac{\partial\vec{v}}{\partial t_k^\alpha} = \frac{\partial}{\partial t_k^\alpha}\frac{\partial\vec{v}}{\partial t_l^\beta}. \tag{33}$$

It follows from Eq. (32) that $t_0^1 = x$.

4.2. *Douglas string equation*

Let us define a function $S(v,t)$ on M which depends on the additional parameters $\{t_k^\alpha\}$

$$S(v, t_k^\alpha) = \sum_{\alpha=1}^{n}\sum_{k\geq 0} t_k^\alpha \theta_{\alpha,k}(v). \tag{34}$$

The equation

$$\frac{\partial S}{\partial v^\alpha} = 0, \tag{35}$$

is called a string equation. In the case of Frobenius manifold of A_{q-1} type it is nothing but the Douglas string equation written in the form of the principle of least string action.[20] It can be shown that solutions $\vec{v}(t_k^\alpha)$ of the string equation (35) satisfy also Eq. (32).

4.3. *Equation for Tau-function*

We define the function $Z[t] = \log\tau(t)$, where

$$Z[t] = \frac{1}{2}\int_0^{v=v^*(t)} \Omega, \tag{36}$$

and

$$\Omega = C_\alpha^{\beta\gamma}(v)\frac{\partial S(v,t)}{\partial v^\beta}\frac{\partial S(v,t)}{\partial v^\gamma}dv^\alpha, \tag{37}$$

is the differential form and $v^*(t)$ is one of the solutions of the string equation (35). From the associativity of the algebra A_{q-1} and the equations (12) it follows that Ω is a closed one-form.

Lemma 4.1. *On the solution of the string equation*[a] *Z(t) satisfies*

$$\frac{\partial^2 Z(t)}{\partial t_k^\alpha \partial t_0^1} = \theta_{\alpha,k}(v(t)). \tag{38}$$

In particular,

$$v^\alpha(t) = \eta^{\alpha\beta} \frac{\partial^2 Z}{\partial t_0^\beta \partial t_0^1}, \tag{39}$$

and for $v^{q-1}(t) = u_1(t)$

$$\frac{\partial^2 Z}{\partial x^2} = u_1(t). \tag{40}$$

Proof. Differentiating with respect to t_k^α and t_0^1 and taking into account the string equation, we find

$$\frac{\partial^2 Z}{\partial t_k^\alpha \partial t_0^1} = \int_0^{v^*(t)} C_\lambda^{\beta\gamma} \frac{\partial \theta_{\alpha,k}}{\partial v^\beta} \frac{\partial \theta_{1,0}}{\partial v^\gamma} dv^\lambda = \int_0^{v^*(t)} \frac{\partial \theta_{\alpha,k}}{\partial v^\lambda} dv^\lambda = \theta_{\alpha,k}. \tag{41}$$

Here we used that $\theta_{1,0} = v_1 = v^{q-1}$, $C_\lambda^{\beta,q-1} = \delta_\lambda^\beta$. □

Taking into account $\theta_{\alpha,0} = v_\alpha$ we obtain from Lemma 4.1 that

$$\frac{\partial^2 Z(t)}{\partial t_0^\alpha \partial t_0^1} = v_\alpha(t). \tag{42}$$

Since Z satisfy equations (35) and (40), it is a tau-function of the integrable hierarchy connected with the corresponding Frobenius manifold.

4.4. A_{q-1} *FM and dispersionless limit of Gelfand-Dikij Hierarchy*

The dispersionless limit of the Gelfand-Dikij equations is formulated as follows:

$$\frac{\partial Q}{\partial t_k^\alpha} = [A_{\alpha,k}, Q] = \frac{\partial A_{\alpha,k}}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial A_{\alpha,k}}{\partial y} \frac{\partial Q}{\partial x}, \tag{43}$$

where

$$Q = y^q + u_1(x) y^{q-2} + \cdots + u_{q-1}, \tag{44}$$

and

$$A_{\alpha,k} = \frac{1}{q} c_{\alpha,k} \left(Q^{k+\frac{\alpha}{q}} \right)_+. \tag{45}$$

One can show that these equations are equivalent to the Hamiltonian equation

$$\frac{\partial v^\mu}{\partial t_k^\alpha} = \eta^{\mu\nu} \frac{\partial}{\partial x} \frac{\partial \theta_{\alpha,k+1}}{\partial v^\nu}. \tag{46}$$

[a] To simplify our expressions we write $v(t)$ instead of $v^*(t)$ when it is clear from the context.

4.5. *Formula for tau-function*

As it was derived above the logarithm of the tau-function $Z[\{t_k^\alpha\}]$ is given by

$$Z[\{t_k^\alpha\}] = \frac{1}{2}\int_0^{v^*(t)} C_\alpha^{\beta\gamma}\frac{\partial S}{\partial v^\beta}\frac{\partial S}{\partial v^\gamma}dv^\alpha, \tag{47}$$

where

$$S = \sum_{\alpha=1}^{q-1}\sum_k t_k^\alpha \theta_{\alpha,k} \,. \tag{48}$$

5. Resonance Problem in (p,q) MLG

5.1. *Homogeneity property of string equation. Spectrum for (p,q) case*

Let now only the finite number of the parameters $\{t_k^\alpha\}$ be non-zero. One of them we take equal to one and others be enumerated by two integers (m,n). Here $1 \le m \le q-1$, $1 \le n \le p-1$, where p,q are two coprime integers, $p > q$ and q is a degree of the polynomial Q defined in Eq. (44). Hence, the set of the parameters $\{t_k^\alpha\}$ is replaced by the set $\{t_{mn}\}$. Let us take the action in the form

$$S = \operatorname*{res}_{y=\infty}[Q^{\frac{p+q}{q}} + \sum_{m,n}^{pm-qn>0} t_{mn}Q^{\frac{pm-qn}{q}}], \tag{49}$$

It is easy to check that $Q[y,u_\alpha]$ and $S[u_\alpha,t_{mn}]$ are quasi-homogeneous functions

$$Q[\rho y, \rho^{r_\alpha}u_\alpha] = \rho^q Q[y,u_\alpha], \quad S[\rho^{r_\alpha}u_\alpha, \rho^{\sigma_{mn}}t_{mn}] = \rho^{p+q}S[u_\alpha,t_{mn}]. \tag{50}$$

Here we denote

$$r_\alpha = q-\alpha-1, \quad \sigma_{mn} = p+q-|pm-qn|. \tag{51}$$

We call $\{\sigma_{mn}\}$ the set of the scaling indices of the set $\{t_{mn}\}$. As it was found by Douglas,[3] the numbers $\delta_{mn} = \frac{\sigma_{mn}}{2q}$ coincide with the gravitational dimensions of the physical fields in (p,q) Minimal Liouville gravity.[19]

The function $Z[t_{mn}]$ is a quasi-homogeneous function

$$Z[\rho^{2q\delta_{mn}}t_{mn}] = \rho^{2(p+q)}Z[t_{mn}]. \tag{52}$$

5.2. *The group of the resonance transformations*

Since the scaling indices are integer, the following relation can take place

$$\sigma_{mn} = \sigma_{k_1l_1} + \sigma_{k_2l_2} + \cdots + \sigma_{k_Nl_N}. \tag{53}$$

This is known as a resonance condition. The number of possible resonances grows when p and q increase. A transformation $t_{mn} \to \lambda_{mn}$ of the form

$$\begin{aligned} t_{mn} = \lambda_{mn} &+ \sum_{k_1,l_1,k_2,l_2} A_{mn}^{k_1l_1;k_2,l_2}\lambda_{k_1,l_1}\lambda_{k_2,l_2} \\ &+ \sum_{k_1,l_1,k_2,l_2,k_3,l_3} A_{mn}^{k_1l_1;k_2,l_2;k_3,l_3}\lambda_{k_1,l_1}\lambda_{k_2,l_2}\lambda_{k_3,l_3} + \cdots , \end{aligned} \tag{54}$$

is called resonance transformation if Eq. (53) is satisfied for each term. Besides, by definition, we suggest that the scaling index of λ_{mn} equals to the one of t_{mn}.

It is obvious that

$$t_{mn}(\{\rho^{\sigma_{kl}}\lambda_{kl}\}) = \rho^{\sigma_{mn}} t_{mn}(\{\lambda_{kl}\}), \tag{55}$$

and that the resonance transformation does not change the homogeneity property of the partition function $Z[t_{mn}(\{\lambda_{kl}\})] = \widetilde{Z}[\lambda_{mn}]$

$$\widetilde{Z}[\{\rho^{\sigma_{mn}}\lambda_{mn}\}] = \rho^{p+q}\widetilde{Z}[\{\lambda_{mn}\}]. \tag{56}$$

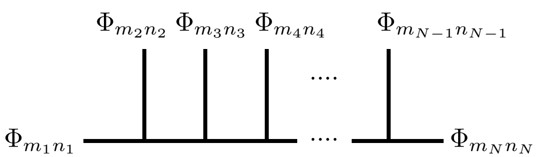
Hence, if we find some solution of the string equation (35) and construct $Z[t_{mn}]$, then we get a family of the solutions $\widetilde{Z}[\{\lambda_{mn}\}] = Z[\{t_{mn}(\{\lambda_{kl})\}\}]$ having the same homogeneity properties with respect to the resonance transformations.

5.3. *The choice of the resonance transformation and of the solution of the string equation*

Now we are in the position to formulate the following problem. We are looking for solutions of the string equation and resonance transformations which gives function $\widetilde{Z}[\{\lambda_{mn}\}]$ satisfying infinite number of constraints known as fusion rules for observables of minimal CFT models $M(p,q)$ and their for the correlators. In what follows we restrict ourself by considering spherical topology. Then these rules can be formulated as follows.

We denote by Φ_{mn}, where $1 \le m \le p$ and $1 \le n \le q$, the primary fields in the minimal model $M(p,q)$ of Conformal field theory. The fields $\Phi_{m,n}$ and $\Phi_{q-m,p-n}$ correspond to the same primary field.

The following graphical representation allows to formulate these restrictions



Here the external lines represent the (arbitrary arranged) primary fields in the correlator $\langle\Phi_{m_1n_1}\Phi_{m_2n_2}\cdots\Phi_{m_Nn_N}\rangle$ (here we assume $N \ge 3$). The fusion rules result to the requirement that the correlation function must be equal to zero if there are no sets of pairs (k_i, l_i) assigned to the internal lines, for which in any vertex of the graph the following condition on the three pairs (m_i, n_i) $(i = 1, 2, 3)$ corresponding to the lines connected to this vertex

$$|m_1 - m_2| + 1 \le m_3 \le \min\{m_1 + m_2 - 1, 2q + 1 - m_1 - m_2\} \quad \text{step 2}, \tag{57}$$

$$|n_1 - n_2| + 1 \le n_3 \le \min\{n_1 + n_2 - 1, 2p + 1 - n_1 - n_2\} \quad \text{step 2}, \tag{58}$$

can not be satisfied any permutation of the pairs.

In addition, from the conformal selection rules for $N = 1$ it follows

$$\langle \Phi_{mn} \rangle = 0, \tag{59}$$

for $(m, n) \neq (1, 1)$ and for $N = 2$

$$\langle \Phi_{m_1 n_1} \Phi_{m_2 n_2} \rangle = 0, \tag{60}$$

for (m_1, n_1) not equal to (m_2, n_2) or $(q - m_2, p - n_2)$.

Now we are going to give a more precise formulation of our main conjecture.

Conjecture 5.1. *There exist a solution of the string equation and a choice of the resonance transformation described above, such that the function*

$$\widetilde{Z}[\{\lambda_{mn}\}] = \langle \exp \sum_{m,n} \lambda_{m,n} O_{m,n} \rangle$$

$$= \sum_{N=0}^{\infty} \sum_{m_i, n_i} \frac{\lambda_{m_1 n_1} \cdots \lambda_{m_N n_N}}{N!} \langle O_{m_1 n_1} \ldots O_{m_N n_N} \rangle, \tag{61}$$

appears the generating function of the correlators in the Minimal Liouville Gravity.

In particular, all correlators $\langle O_{m_1 n_1} \cdots O_{m_N n_N} \rangle$ forbidden by the conformal fusion rules vanish.

6. The Plan of the Solution of the Problem

To solve the formulated above problem we write the action $S(v_\alpha, t_{mn})$ and the generating function $Z[\{t_{mn}\}]$ in terms of new variables $\{\lambda_{mn}\}$ using the resonance change of variables

$$t_{mn} = \lambda_{mn} + A_{mn} \mu^{\delta_{mn}} + \sum_{m_1, n_1}^{\delta_{m_1 n_1} \leq \delta_{mn}} A_{mn}^{m_1 n_1} \mu^{\delta_{mn} - \delta_{m_1 n_1}} \lambda_{m_1 n_1}$$

$$+ \sum_{m_1, n_1, m_2, n_2}^{\delta_{m_1 n_1} + \delta_{m_2 n_2} \leq \delta_{mn}} A_{mn}^{m_1 n_1, m_2 n_2} \mu^{\delta_{mn} - \delta_{m_1 n_1} - \delta_{m_2 n_2}} \lambda_{m_1 n_1} \lambda_{m_2 n_2} + \ldots, \tag{62}$$

where $\mu = \lambda_{11}$ is called the cosmological constant in the continuum approach to MLG.

After performing this transform the action takes the form

$$\widetilde{S}[v_\alpha, \{\lambda_{mn}\}] = S^{(0)}(v_\alpha) + \sum_{m,n} \lambda_{mn} S^{(mn)}(v_\alpha)$$

$$+ \sum_{m_1, n_1, m_2, n_2} \lambda_{m_1 n_1} \lambda_{m_2 n_2} S^{(m_1 n_1, m_2 n_2)}(v_\alpha) + \ldots. \tag{63}$$

The information about the form of the resonance transformation is encoded in the coefficients of $S^{(0)}$, $S^{(mn)}$, etc. From Eqs. (24) and (12.1) we find

$$S^{(0)} = \operatorname*{res}_{y=\infty} \left[Q^{\frac{p+q}{q}} + \sum_{l=1}^{s} A_{1l} \, \mu^{\frac{l+1}{2}} Q^{\frac{p-ql}{q}} \right], \tag{64}$$

where we introduced the new positive integer numbers s and p_0 such that $p = sq+p_0$ and $0 < p_0 < q$.

$$S^{(mn)} = \operatorname*{res}_{y=\infty}\left[Q^{\frac{pm-qn}{q}} + \sum_{l=n+2}^{sm+\lfloor\frac{p_0 m}{q}\rfloor} A^{mn}_{ml}\mu^{\frac{l-n}{2}}\, Q^{\frac{pm-ql}{q}}\right], \tag{65}$$

where A^{mn}_{kl} are the coefficients of the resonance relations and $(l-n)$ is even. The higher coefficients can also be easily written in terms of the coefficients $A^{\{m_i n_i\}}_{kl}$.

The generating function is given by

$$\widetilde{Z}[\{\lambda_{mn}\}] = \frac{1}{2}\int_0^{\mathbf{v}^*} C^{\beta\gamma}_{\alpha}(v)\frac{\partial\widetilde{S}}{\partial v^\beta}\frac{\partial\widetilde{S}}{\partial v^\gamma}dv^\alpha, \tag{66}$$

where $\mathbf{v}^*$ is defined as a function of the parameters $\{\lambda_{mn}\}$ of the Douglas string equation (35).

From now on we will skip the tilde over the functions $\widetilde{S}(\{u_\alpha\},\{\lambda_{mn}\})$ and $\widetilde{Z}(\{\lambda_{mn}\})$.

7. Appropriate Solution

To compute the one-point function which is given by the integral

$$\langle O_{mn}\rangle = \int_0^{v^0_\alpha} C^{\alpha}_{\beta\gamma}\frac{\partial S^{(0)}}{\partial v_\beta}\frac{\partial S^{(mn)}}{\partial v_\gamma}dv_\alpha, \tag{67}$$

we need to know the upper limit in this integral v^0_α which is the solution of the string equation for all couplings (except $\lambda_{11}=\mu$) equal to zero

$$v^0_\alpha = v^*_\alpha(\lambda_{mn})\bigg|_{\lambda_{mn}=0,\lambda_{11}=\mu}. \tag{68}$$

Explicitly, v^0_α satisfies

$$\frac{\partial S^{(0)}}{\partial v_\mu}\bigg|_{v_\alpha=v^0_\alpha} = 0. \tag{69}$$

Using Eqs. (64), (65) and (24), $S^{(0)}$ and $S^{(mn)}$ can be written as

$$S^{(0)} = -\frac{\theta_{p_0,s+1}}{c_{p_0,s+1}} - \sum_{l=1}^{s} A_{1l}\mu^{\frac{l+1}{2}}\frac{\theta_{p_0,s-l}}{c_{p_0,s-l}}, \tag{70}$$

$$S^{(mn)} = -\frac{\theta_{p_0 m,sm-n}}{c_{p_0 m,sm-n}} - \sum_{l=n+2}^{sm+\lfloor\frac{p_0 m}{q}\rfloor} A^{mn}_{ml}\mu^{\frac{l+1}{2}}\frac{\theta_{p_0 m,sm-l}}{c_{p_0 m,sm-l}}. \tag{71}$$

We will use the following proposition from Ref. 7.

Proposition 7.1. *On the line $v_{i>1}=0$,*

$$\begin{cases} k\ \textit{even:}\ \dfrac{\partial\theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,\alpha}\,x_{\lambda,k}\left(-\dfrac{v_1}{q}\right)^{\frac{k}{2}q}, \\ k\ \textit{odd:}\ \frac{\partial\theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,q-\alpha}\,y_{\lambda,k}\left(-\frac{v_1}{q}\right)^{\frac{k-1}{2}q+\lambda}, \end{cases} \tag{72}$$

where

$$x_{\alpha,k} = \frac{\Gamma(\frac{\alpha}{q})}{\Gamma(\frac{\alpha}{q}+\frac{k}{2})(\frac{k}{2})!} \quad \textit{and} \quad y_{\lambda,k} = -\frac{\Gamma(\frac{\alpha}{q})}{\Gamma(\frac{\alpha}{q}+\frac{k+1}{2})(\frac{k-1}{2})!}. \tag{73}$$

Using this statement together with Eq. (70) it is not difficult to see that the string equation (69) has the solutions of the form $v_\alpha^0 = 0$ for $\alpha \neq 1$ and the coordinate v_1^0 is a root of the equation

$$\frac{\partial S^{(0)}}{\partial v_{p_0}} = 0, \quad \text{if } s - \text{odd}, \tag{74}$$

or

$$\frac{\partial S^{(0)}}{\partial v_{q-p_0}} = 0, \quad \text{if } s - \text{even}. \tag{75}$$

Here we assume that after taking derivative we set all v_α for $\alpha \neq 1$ to zero. More explicitly these equations can be written as

$$\sum_{k=-1:2:s} B^{\text{odd}}_{p_0,k}\left(-\frac{v_1}{q}\right)^{\frac{s-k}{2}q} = 0, \quad \text{if } s - \text{odd}, \tag{76}$$

or

$$\sum_{k=-1:2:s} B^{\text{odd}}_{p_0,k}\left(-\frac{v_1}{q}\right)^{\frac{s-k-1}{2}q} = 0, \quad \text{if } s - \text{even}, \tag{77}$$

where

$$B^{\text{odd}}_{p_0,k} = \frac{x_{p_0,s-k}}{c_{p_0,s-k}} A_{1,k}\mu^{\frac{k+1}{2}}, \tag{78}$$

and

$$B^{\text{even}}_{p_0,k} = \frac{y_{p_0,s-k}}{c_{p_0,s-k}} A_{1,k}\mu^{\frac{k+1}{2}}, \tag{79}$$

where $A_{1,-1} = 1$.

8. One-Point Functions

As it was shown in Ref. 7, the structure constant in the flat coordinates on the line $v_{\alpha>0} = 0$, for $\alpha \geq \beta \geq \gamma$

$$C_{\alpha\beta\gamma} = \left(-\frac{v_1}{q}\right)^{\frac{\alpha+\beta+\gamma-q-1}{2}} \quad \text{if } \frac{\alpha+\beta+\gamma-q-1}{2} \in \mathbb{N}_0 \quad \text{and} \\ \alpha+\beta-\gamma \in [1, q-1], \quad \text{otherwise } 0, \tag{80}$$

where $\mathbb{N}_0$ is the set of non-negative integers. Using Eq. (72) we find for s odd and $(sm-n)$ even

$$\langle O_{mn} \rangle = \int_0^{v_1^0} C_{q-1,p_0,p_0 m} \frac{\partial S^{(0)}}{\partial v_{p_0}} \frac{\partial S^{(mn)}}{\partial v_{p_0 m}} dv_1. \tag{81}$$

Taking into account Eq. (80) we conclude that the correlation function is zero for $m \neq 1$. Hence, in this case from the selection rules we obtain

$$\langle O_{1n} \rangle = \int_0^{v_1^0} \Big(- \frac{v_1}{q} \Big)^{p_0 - 1} \frac{\partial S^{(0)}}{\partial v_{p_0}} \frac{\partial S^{(1n)}}{\partial v_{p_0}} dv_1 = 0. \tag{82}$$

For s odd and $(sm-n)$ odd,

$$\langle O_{mn} \rangle = \int_0^{v_1^0} C_{q-1, p_0, q-p_0 m} \frac{\partial S^{(0)}}{\partial v_{p_0}} \frac{\partial S^{(mn)}}{\partial v_{q-p_0 m}} dv_1, \tag{83}$$

and the structure constant here is not equal to zero only if $q - p_0 m = p_0$ as it follows from Eq. (80). Therefore the gravitational dimension

$$[\langle O_{mn} \rangle] = \frac{p+q}{q} - \delta_{mn} = \frac{sm-n}{2} + \frac{s+1}{2} + \frac{p_0 m + p_0}{2q}, \tag{84}$$

is integer, the correlation function is analytic and we shell not consider it.[6]

Similarly, for s even and $(sm-n)$ even, we obtain the following consequence of the selection rules

$$\langle O_{1n} \rangle = \int_0^{v_1^0} \Big(- \frac{v_1}{q} \Big)^{q - p_0 - 1} \frac{\partial S^{(0)}}{\partial v_{q-p_0}} \frac{\partial S^{(1n)}}{\partial v_{q-p_0}} dv_1 = 0. \tag{85}$$

Finally, if s even and $(sm-n)$ odd, we find again that the expressions for the one point correlation functions are analytic.

A simple analysis shows that the number of these equations is equal to the number of the coefficients arising in the first order in the resonance relation. Hence the requirement of absence of the one point functions fixes uniquely unknown coefficients $B_{p_0,k}$ in the expressions (76) and (77).

Thus we arrive to the conclusion that the special solution of the string equation considered above ensure the requirements of the selection rules in agreement with the general prescription described in the previous section.

We note also that the variety of (p,q) models of minimal Liouville Gravity is split in two subclasses according to the condition that $\lfloor p/q \rfloor$ be either even or odd. In each case we find distinct sets of requirements formulated above leading to zero valued one point functions.

9. Two-Point Functions

We are now going to consider the two-point function. From Eq. (66) we find

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle = \sum_{\gamma=1}^{q-1} \int_0^{v_1^0} dv_1 \left(- \frac{v_1}{q} \right)^{\gamma - 1} \frac{\partial S^{(m_1 n_1)}}{\partial v_\gamma} \frac{\partial S^{(m_2 n_2)}}{\partial v_\gamma}. \tag{86}$$

It follows from Eq. (72) that $\frac{\partial S^{(mn)}}{\partial v^\gamma} \neq 0$ if one of the following two conditions is satisfied

$$\begin{aligned} &1) \quad \gamma = m p_0 \bmod q \quad \text{and} \quad (sm-n) - \text{even}, \\ &2) \quad \gamma = q - m p_0 \bmod q \quad \text{and} \quad (sm-n) - \text{odd}. \end{aligned} \tag{87}$$

Similarly to the consideration in the previous section we find four cases where the two point function can be non-zero. In two cases: where the first pair (m_1, n_1) satisfies first condition while the second pair (m_2, n_2) is subject of the second condition and vice versa, we find the regular expression for the two point function. Thus, we are left with the two options where both pairs satisfy either the first or the second condition in Eq. (87).

Explicitly, in the case when both $(sm - n_1)$ and $(sm - n_2)$ are even we get the following requirement

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_0^{v_1^0} dv_1 \left(-\frac{v_1}{q} \right)^{mp_0 - 1} \frac{\partial S^{(mn_1)}}{\partial v_{mp_0}} \frac{\partial S^{(mn_2)}}{\partial v_{mp_0}} = 0 \quad \text{if } n_1 \neq n_2. \tag{88}$$

Making the substitution

$$t = 2\left(\frac{v_1}{v_1^0} \right)^q - 1, \tag{89}$$

and denoting

$$\frac{\partial S^{(mn)}}{\partial v_{mp_0}} = L_{\frac{sm-n}{2}}(t), \tag{90}$$

we find the following consequence of the diagonality condition

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_{-1}^{1} dt \, (1+t)^{\frac{mp_0 - q}{q}} L_{\frac{sm-n_1}{2}}(t) L_{\frac{sm-n_2}{2}}(t) = 0 \quad \text{if } n_1 \neq n_2. \tag{91}$$

Hence, the selection rules for the two-point correlation numbers requires that the polynomials $L_{\frac{sm-n}{2}}$ form an orthogonal set of Jacobi polynomials

$$\frac{\partial S^{(mn)}}{\partial v_{mp_0}} = \frac{pm - qn}{q} P^{(0, \frac{mp_0 - q}{q})}_{\frac{sm-n}{2}}(t), \quad \text{for } (sm-n) - \text{even}. \tag{92}$$

In the second case, where both $(sm - n_1)$ and $(sm - n_2)$ are odd, we have

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_0^{v_1^0} dv_1 \left(-\frac{v_1}{q} \right)^{q - mp_0 - 1} \frac{\partial S^{(mn_1)}}{\partial v_{q-mp_0}} \frac{\partial S^{(mn_2)}}{\partial v_{q-mp_0}} = 0 \quad \text{if } n_1 \neq n_2. \tag{93}$$

Denoting

$$\frac{\partial S^{(mn)}}{\partial v_{q-mp_0}} = (1+t)^{\frac{mp_0}{q}} L_{\frac{sm-n-1}{2}}(t), \tag{94}$$

we find the following consequence of the diagonality condition for the two-point correlation function in this case

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_{-1}^{1} dt \, (1+t)^{\frac{mp_0}{q}} L_{\frac{sm-n_1-1}{2}}(t) L_{\frac{sm-n_2-1}{2}}(t) = 0 \quad \text{if} \quad n_1 \neq n_2. \tag{95}$$

It means that

$$\frac{\partial S^{(mn)}}{\partial v_{q-mp_0}} = \frac{pm - qn}{q} (1+t)^{\frac{mp_0}{q}} P^{(0, \frac{mp_0}{q})}_{\frac{sm-n-1}{2}}(t) \quad \text{for } (sm-n) - \text{odd}. \tag{96}$$

At last, inserting these explicit expressions for the derivatives of $S^{(mn)}$ to the equations (82) and (85) we arrive to the condition

$$\langle O_{1n}\rangle = \int_{-1}^{1}(1+t)^{\frac{p_0-q}{q}}L_{\frac{s+1}{2}}(t)P^{(0,\frac{p_0-q}{q})}_{\frac{s-n}{2}}(t)dt = 0, \tag{97}$$

in the case where s is odd and n is odd and greater than 1. And

$$\langle O_{1n}\rangle = \int_{-1}^{1}(1+t)^{\frac{p_0}{q}}L_{\frac{s}{2}}(t)P^{(0,\frac{p_0}{q})}_{\frac{s-n-1}{2}}(t)dt = 0, \tag{98}$$

in case where s is even and n is odd and greater than 1. Here we introduced the polynomial $L_n(t)$

$$\frac{\partial S^{(0)}}{\partial v_{p_0}}(t) = L_{\frac{s+1}{2}}(t), \tag{99}$$

for s odd,

$$\frac{\partial S^{(0)}}{\partial v_{q-p_0}}(t) = (1+t)^{\frac{p_0}{q}}L_{\frac{s}{2}}(t). \tag{100}$$

for s even.

Taking into account these equations, the order of the polynomials $\frac{\partial S^{(0)}}{\partial v_{p_0}}$ and $\frac{\partial S^{(0)}}{\partial v_{q-p_0}}$ and the string equations (74), (75) we obtain the following explicit expressions

$$\frac{\partial S^{(0)}}{\partial v_{p_0}} = \frac{p+q}{q}\left(P^{(0,\frac{p_0-q}{q})}_{\frac{s+1}{2}}(t) - P^{(0,\frac{p_0-q}{q})}_{\frac{s-1}{2}}(t)\right), \tag{101}$$

if s is odd and

$$\frac{\partial S^{(0)}}{\partial v_{q-p_0}} = \frac{p+q}{q}(1+t)^{\frac{p_0}{q}}\left(P^{(0,\frac{p_0}{q})}_{\frac{s}{2}}(t) - P^{(0,\frac{p_0}{q})}_{\frac{s-2}{2}}(t)\right), \tag{102}$$

if s is even.

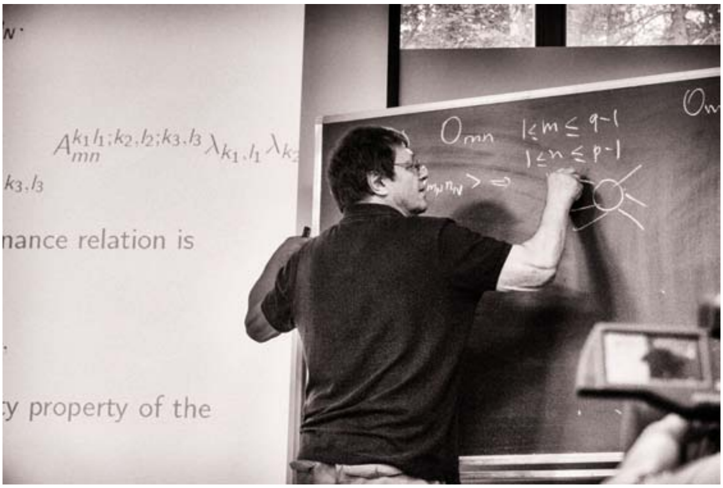
10. Conclusions

In the present talk we have described the relation between the approach to (p,q) models of Minimal Liouville gravity based on the Douglas string equation, on one hand, and the Frobenius manifolds of A_{q-1} type on the other. As a result of this relation the generating function of correlation numbers in MLG is represented by the logarithm of the tau-function of the corresponding integrable hierarchy. All necessary information is encoded in the solution of the Douglas string equation and in the resonance relations between the parameters of the integrable hierarchy and the coupling constants of MLG. Using this relation and some special properties of the flat coordinates on the Frobenius manifold, we have found the appropriate solution of the Douglas string equation. This result generalizes analogues result found recently for Unitary models of Minimal Liouville gravity.[7] We have shown that the appropriate solution is consistent with the basic requirements of the conformal selection rules arising on the levels of one- and two-point correlation functions. Namely, the number of the parameters of the resonance transformations is exactly the number of the constraints following from the selection rules. Resolving these constraints we have found explicit form of the resonance transformations in terms of Jacoby polynomials. It would be interesting to investigate if this matching persists for multi-point correlation functions when the fusion rules of the underlying minimal models of CFT should be taken into account. This analysis requires also knowing the explicit form of the structure constants of the Frobenius algebra in the flat coordinates. We plan to study these questions in the near future.

References

1. A.A. Belavin and V.A. Belavin, *Frobenius Manifolds, Integrable Hierarchies and Minimal Liouville Gravity*, *JHEP* **09**, 151 (2014).
2. A.M. Polyakov, *Quantum Geometry of Bosonic Strings*, *Phys. Lett.* **B103**, 207–210 (1981).
3. M.R. Douglas, *Strings in less than one-dimension and the generalized KdV hierarchies*, *Phys. Lett.* **B238**, 176 (1990).
4. G.W. Moore, N. Seiberg and M. Staudacher, *From loops to states in 2-D quantum gravity*, *Nucl. Phys.* **B362**, 665–709 (1991).
5. A. Belavin and A. Zamolodchikov, *On Correlation Numbers in 2D Minimal Gravity and Matrix Models*, *J. Phys.* **A42**, 304004 (2009), arXiv:0811.0450 [hep-th].
6. A. Belavin, B. Dubrovin and B. Mukhametzhanov, *Minimal Liouville Gravity correlation numbers from Douglas string equation*, *JHEP* **1401**, 156 (2014), arXiv:1310.5659 [hep-th].
7. V. Belavin, *Unitary Minimal Liouville Gravity and Frobenius Manifolds*, arXiv:1405.4468 [hep-th].
8. A. Belavin, A.M. Polyakov and A. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl. Phys.* **B241**, 333–380 (1984).
9. V. Kazakov, A.A. Migdal and I. Kostov, *Critical Properties of Randomly Triangulated Planar Random Surfaces*, *Phys. Lett.* **B157**, 295–300 (1985).
10. V. Kazakov, *Ising model on a dynamical planar random lattice: Exact solution*, *Phys. Lett.* **A119**, 140–144 (1986).
11. V. Kazakov, *The Appearance of Matter Fields from Quantum Fluctuations of 2D Gravity*, *Mod. Phys. Lett.* **A4**, 2125 (1989).
12. M. Staudacher, *The Yang-Lee edge singularity on a dynamical planar random surface*, *Nucl. Phys.* **B336**, 349 (1990).
13. E. Brezin and V. Kazakov, *Exactly solvable field theories of closed strings*, *Phys. Lett.* **B236**, 144–150 (1990).
14. M.R. Douglas and S.H. Shenker, *Strings in Less Than One-Dimension*, *Nucl. Phys.* **B335**, 635 (1990).
15. D.J. Gross and A.A. Migdal, *Nonperturbative Two-Dimensional Quantum Gravity*, *Phys. Rev. Lett.* **64**, 127 (1990).
16. I. Krichever, *The Dispersionless Lax equations and topological minimal models*, *Commun. Math. Phys.* **143**, 415–429 (1992).
17. B. Dubrovin, *Integrable systems in topological field theory*, *Nucl. Phys.* **B379**, 627–689 (1992).

18. R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *Topological strings in d less than 1*, *Nucl. Phys.* **B352**, 59–86 (1991).
19. V. Knizhnik, A.M. Polyakov and A. Zamolodchikov, *Fractal Structure of 2D Quantum Gravity*, *Mod. Phys. Lett.* **A3**, 819 (1988).
20. P.H. Ginsparg, M. Goulian, M. Plesser and J. Zinn-Justin, *(p, q) String actions*, *Nucl. Phys.* **B342**, 539–563 (1990).
21. P. Di Francesco and D. Kutasov, *Unitary Minimal Models Coupled To 2-d Quantum Gravity*, *Nucl. Phys.* **B342**, 589–624 (1990).



A.A. Belavin, Chernogolovka, 2015.