

Instantons and 2d superconformal field theory

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ABSTRACT: A recently proposed correspondence between 4-dimensional $\mathcal{N} = 2$ SUSY $SU(k)$ gauge theories on $\mathbb{R}^4/\mathbb{Z}_m$ and $SU(k)$ Toda-like theories with Z_m parafermionic symmetry is used to construct four-point $\mathcal{N} = 1$ super Liouville conformal block, which corresponds to the particular case $k = m = 2$.

The construction is based on the conjectural relation between moduli spaces of $SU(2)$ instantons on $\mathbb{R}^4/\mathbb{Z}_2$ and algebras like $\widehat{\mathfrak{gl}}(2)_2 \times \mathcal{NSR}$. This conjecture is confirmed by checking the coincidence of number of fixed points on such instanton moduli space with given instanton number N and dimension of subspace degree N in the representation of such algebra.

KEYWORDS: Conformal and W Symmetry, Supersymmetric gauge theory

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1 Introduction

Alday, Gaiotto and Tachikawa [1] proposed correspondence between Liouville theory and four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. This correspondence has been generalized for the conformal theories with additional symmetries, such as affine Lie algebras, \mathcal{W} algebras, parafermions [2–8]. In particular, it was suggested in [6] that the instanton calculus in the gauge theories on $\mathbb{R}^4/\mathbb{Z}_2$ give rise to the super-Virasoro conformal blocks. The idea to use the \mathbb{Z}_2 symmetric instanton moduli \mathcal{M}_{sym} is based on its conjectural relation to the coset $\widehat{\mathfrak{gl}}(n)_2/\widehat{\mathfrak{gl}}(n-2)_2$ which is isomorphic to $\mathcal{A} = \widehat{\mathfrak{gl}}(2)_2 \times \mathcal{NSR}$ (see [9]–[12] for the relation between instanton moduli spaces and algebras). This relation should mean that the algebra \mathcal{A} acts on the direct sum of certain cohomology of spaces \mathcal{M}_{sym} . In particular the algebra \mathcal{A} has a representation with basis labeled by fixed points for the torus action on the moduli spaces \mathcal{M}_{sym} . In section 3 we show that the number of fixed points actually coincide with the number of states in the certain representation of algebra $\mathcal{A} = \widehat{\mathfrak{gl}}(2)_2 \times \mathcal{NSR}$.

The \mathbb{Z}_2 symmetric instanton moduli \mathcal{M}_{sym} decomposes into several connected components. The only two of these components are used in [6] as the new integration domain. The integral is interpreted the \mathbb{Z}_2 restricted instanton partition function of $\text{SU}(2)$ $\mathcal{N} = 2$ supersymmetric pure gauge theory. This function coincides with so-called Whittaker or Gaiotto [13] limit of the four-point super-Liouville conformal block function for $\mathcal{N} = 1$ super Liouville theory (or equivalently norm of Whittaker vector). In section 3 we consider the number of fixed points on two used components. These numbers coincide with the number of states in the certain representation of algebra $\mathcal{A} = \mathcal{B} \times \mathcal{B} \times \mathcal{F} \times \mathcal{NSR}$ where \mathcal{B} and \mathcal{F} denotes Heisenberg and Clifford algebras respectively.

Then we construct explicit expression of the general four-point super-Liouville conformal block function in terms of \mathbb{Z}_2 restricted instanton partition functions. For considered conformal block the matter fields in the fundamental representation of the gauge group should be included into the instanton calculations. Now, the integral over the moduli

space involves zero modes of the matter fermions. After Nekrasov’s deformation [14] (see also [15, 16]) it takes efficient equivariant form and is handled by means of the localization method. The instanton partition function coincides with the four-point conformal block up to an additional factor related to the algebra $\widehat{\mathfrak{gl}}(2)_2$. We found this factor and checked the new representation up to the level 5/2.

The paper is organized as follows. In section 2 we describe the structure of \mathbb{Z}_2 symmetric moduli spaces \mathcal{M}_{sym} . The section 3 is devoted to combinatorial study of fixed points on such space. The instanton moduli integral evaluation is the subject of section 4. We recall the definition of the four-point conformal block function in super-Liouville theory and give its new expression in terms of colored Young diagrams in section 5.

2 Modified moduli space

We begin by reminding ADHM construction [22] of N -instanton solution in the case of $SU(2)$ gauge group (see also [23–27]). ADHM data consist of complex matrices, two $N \times N$ matrices B_1, B_2 , a $N \times 2$ matrix I and a $2 \times N$ matrix J . The space \mathcal{M}_N which defines all possible N -instanton solutions is given by the following set of conditions:

1. Matrices B_1, B_2, I and J satisfy the following equations

$$[B_1, B_2] + IJ = 0, \tag{2.1}$$

2. The solutions related by $U(N)$ transformations

$$B'_i = gB_i g^{-1}, \quad I' = gI, \quad J' = Jg^{-1}; \quad g \in GL(N) \tag{2.2}$$

are equivalent.

3. Among vectors obtained by the repeated action of B_1 and B_2 on $I_{1,2}$ there exist N linear independent. Here I_1 and I_2 stand for the columns of the matrix I and they are considered as two vectors of N -dimensional vector space V . The vector space V is attached to each point of \mathcal{M}_N . It forms a fiber of the N -dimensional fiber bundle whose base is the moduli space \mathcal{M}_N itself.

The subspace of the Moduli space \mathcal{M}_{sym} is defined by the following additional restriction of \mathbb{Z}_2 symmetry

$$B_1 = -PB_1P^{-1}; B_2 = -PB_2P^{-1}; \quad I = PI; \quad J = JP^{-1}. \tag{2.3}$$

where $P \in GL(N)$ is some gauge transformation. It is clear that $P^2 = 1$. Hence the space V decomposes $V_+ \oplus V_-$ where $Pv = v$ for $v \in V_+$ and $Pv = -v$ for $v \in V_-$.

This new manifold \mathcal{M}_{sym} is a disjoint union of connected components $\mathcal{M}_{\text{sym}}(N_+, N_-)$, where N_+ and N_- denote dimensions of V_+ and V_- correspondingly, $N_+ + N_- = N$. These numbers are fixed inside given connected component of \mathcal{M}_{sym} . Each component is connected and can be considered separately.

The construction of the instanton partition function involves the determinants of the vector field v on \mathcal{M}_N , defined by

$$B_l \rightarrow t_l B_l; \quad I \rightarrow It_v; \quad J \rightarrow t_1 t_2 t_v^{-1} J, \quad (2.4)$$

where parameters $t_l \equiv \exp \epsilon_l \tau$, $l = 1, 2$ and $t_v = \exp a \sigma_3 \tau$.

Fixed points, which are relevant for the determinants evaluation, are found from the conditions:

$$t_l B_l = g^{-1} B_l g; \quad It_v = g^{-1} I; \quad t_1 t_2 t_v^{-1} J = J g. \quad (2.5)$$

The solutions of this system can be parameterized by pairs of Young diagrams $\vec{Y} = (Y_1, Y_2)$ such that the total number of boxes $|Y_1| + |Y_2| = N$. The cells $(i_1, j_1) \in Y_1$ and $(i_2, j_2) \in Y_2$ correspond to the vectors $B_1^{i_1} B_2^{j_1} I_1$ and $B_1^{i_2} B_2^{j_2} I_2$ respectively. It is convenient to use these vectors as a basis in the fiber V attached to some fixed point. Then the explicit form of the ADHM date for the given fixed point is defined straightforwardly

$$\begin{aligned} g_{ss'} &= \delta_{ss'} t_1^{i_s-1} t_2^{j_s-1}, \\ (B_1)_{ss'} &= d_{ss'} \delta_{i_s+1, i_{s'}} \delta_{j_s, j_{s'}}, \\ (B_2)_{ss'} &= d_{ss'} \delta_{i_s, i_{s'}} \delta_{j_s+1, j_{s'}}, \\ (I_\alpha)_s &= \delta_{s, 1_\alpha}, \\ J &= 0, \end{aligned} \quad (2.6)$$

where $s = (i_s, j_s)$, $s' = (i_{s'}, j_{s'})$ denotes the boxes of Young diagrams Y_i , $d_{ss'} = 1$ if s, s' belongs to the same Young diagram and $d_{ss'} = 0$ otherwise, 1_α denotes the corner box $(1, 1)$ of the diagram Y_α .

Coming back to \mathcal{M}_{sym} we note that it contains all fixed points of the vector field (2.4) found above. Eq.(2.3) defines the operator P in the fixed point \vec{Y}

$$P(B_1^{i-1} B_2^{j-1} I_\alpha) = (-1)^{i+j} B_1^{i-1} B_2^{j-1} I_\alpha, \quad (2.7)$$

so that the matrix elements can be found explicitly, $P_{ss'} = (-1)^{i_s+j_s} \delta_{ss'}$. In particular it follows that all fixed points belong to \mathcal{M}_{sym} . The parity characteristic $P(s) = (-1)^{i_s+j_s}$ is assigned to each box in the Young diagrams related to the fixed point. We adopt convenient notation from [6] that a box with coordinates of the same and different parities are respectively white and black. Then $P(s) = 1$ for the white boxes and $P(s) = -1$ for the black ones. Therefore the fixed points can be classified by the numbers of white and black boxes, N_+ and N_- . These numbers are the same as defined above, i.e. they are equal to the dimensions of the subspaces V_+ and V_- of the fibers attached to those points of \mathcal{M}_{sym} which belong to the same component as the fixed point itself.

3 Modified moduli space and $\widehat{\mathfrak{gl}}(2)_2 \times \mathcal{NSR}$ algebra

The norm of the Whittaker vector found in [6] is equal to the sum of contributions of fixed points on connected components $\mathcal{M}_{\text{sym}}(N, N)$ and $\mathcal{M}_{\text{sym}}(N, N-1)$. In this section we

calculate the number of fixed points on such components and discuss the result from the $\widehat{\mathfrak{gl}}(2)_2 \times \text{NSR}$ point of view.

It is convenient to introduce the generating function

$$\chi(q) = \sum_N |\mathcal{M}_{\text{sym}}(N, N)|q^N + \sum_N |\mathcal{M}_{\text{sym}}(N, N-1)|q^{N-1/2}, \tag{3.1}$$

where $|\mathcal{M}_{\text{sym}}(N_+, N_-)|$ is a number of fixed points on $\mathcal{M}_{\text{sym}}(N_+, N_-)$. This number equals to the number of pairs of Young diagrams with N_+ white boxes and N_- black boxes.

Denote by $d(Y) = N_+(Y) - N_-(Y)$ the difference between number of white and black boxes in Young diagram Y . For any integer k we denote by

$$\chi_k(q) = \sum_{d(Y)=k} q^{\frac{|Y|}{2}}, \tag{3.2}$$

the generating function of Young diagrams of given difference $d(Y)$. This function has the form:

$$\chi_k(q) = q^{\frac{2k^2-k}{2}} \prod_{m \geq 0} \frac{1}{(1 - q^{m+1})^2}. \tag{3.3}$$

The formula (3.3) was proved in [17, section 5] for $k = 0$. The factor $q^{\frac{2k^2-k}{2}}$ corresponds to the smallest Young diagram with $d(Y) = k$. For $k > 0$ this diagram consist of $2k - 1$ rows of length $2k - 1, 2k - 2, \dots, 1$. For $k < 0$ this diagram consist of $2|k|$ rows of length $2|k|, 2|k| - 1, \dots, 1$.

The generating function of pairs Young diagrams with $N_+ - N_- = k$ reads

$$\chi_k^{(2)} = \sum_{k_1+k_2=k} \chi_{k_1} \chi_{k_2}, \tag{3.4}$$

Using (3.3) and Jacobi triple product identity

$$\sum_{n \in \mathbb{Z}} (-1)^n t^n q^{n^2} = \prod_{m \geq 0} (1 - q^{2m+2})(1 - q^{2m+1}t)(1 - q^{2m+1}t^{-1}) \tag{3.5}$$

we get

$$\chi(q) = \chi_0^{(2)}(q) + \chi_1^{(2)}(q) = \prod_{m \geq 0} \frac{(1 + q^{m+\frac{1}{2}})^2}{(1 - q^{m+1})^3} = \chi_B(q)^3 \chi_F(q)^2, \tag{3.6}$$

where

$$\chi_B(q) = \prod_{n \in \mathbb{Z}, n > 0} \frac{1}{(1 - q^n)} \tag{3.7}$$

$$\chi_F(q) = \prod_{r \in \mathbb{Z} + \frac{1}{2}, r > 0} (1 + q^r). \tag{3.8}$$

The first terms of the series for $\chi(q)$ looks as follows

$$\chi(q) = 1 + 2q^{1/2} + 4q + 8q^{3/2} + 16q^2 + 28q^{5/2} + \dots \tag{3.9}$$

The formula (3.6) looks very suggestive. The $\chi_B(q)\chi_F(q)$ equals to the character of standard representation of the \mathcal{NSR} algebra with generators L_n, G_r . The remaining part is related to the algebra $\mathcal{B} \times \mathcal{B} \times \mathcal{F}$ where \mathcal{B} is the Heisenberg algebra with generators b_n and relations $[b_n, b_m] = n\delta_{n+m}$ and \mathcal{F} is the Clifford algebra with generators f_r and relations $\{f_r, f_s\} = r\delta_{r+s}$.

Thus equation (3.6) means that the generating function of numbers of fixed points on components $\mathcal{M}_{\text{sym}}(N, N)$ and $\mathcal{M}_{\text{sym}}(N, N - 1)$ equals to the character of representation of the algebra $\mathcal{A} = \mathcal{B} \times \mathcal{B} \times \mathcal{F} \times \mathcal{NSR}$. This representation theory point of view can be exploit similar to [18] (see also [19]).

One can consider the whole space \mathcal{M}_{sym} . The generating function has the form

$$\chi(q) = \sum_N |\mathcal{M}_{\text{sym}}(N)| q^{\frac{N}{2}} = \prod_{n \in \mathbb{Z}, n > 0} \frac{1}{(1 - q^{\frac{n}{2}})^2} \tag{3.10}$$

The result equals to the character of the certain representation of $\widehat{\mathfrak{gl}}(2)_2 \times \mathcal{NSR}$ namely the tensor product of Fock representation of Heisenberg algebra, vacuum representation¹ of $\widehat{\mathfrak{sl}}(2)_2$ and NS representation of \mathcal{NSR} . In other words the generating function of numbers of fixed points on $\mathcal{M}_{\text{sym}}(N)$ equals to the character of representation of the algebra $\mathcal{A} = \widehat{\mathfrak{gl}}(2)_2 \times \mathcal{NSR}$.

Note that appearance of algebras $\mathcal{B} \times \mathcal{B} \times \mathcal{F} \times \mathcal{NSR}$ and $\widehat{\mathfrak{gl}}(2)_2 \times \mathcal{NSR}$ may be related to the fact that $\widehat{\mathfrak{sl}}(2)$ representation of level 2 can be realized by one bosonic and one fermionic field [20].

4 Determinants of the vector field

In [14, 15] the form of $SU(k)$ $\mathcal{N} = 2$ supersymmetric instanton partition function (in what follows we are dealing with $SU(2)$ case) was derived as an integral of the equivariant form. This form is defined in terms of the vector field v acting on the moduli space \mathcal{M}_N .

By means of the localization technique the evaluation of the moduli integral is reduced to the calculation of the determinants [10, 14, 15] of the vector field v in the vicinity of fixed points

$$\mathcal{Z}_N(a, \epsilon_1, \epsilon_2) = \sum_n \frac{1}{\det_n v}. \tag{4.1}$$

Here n numerates fixed points of the vector field.

To evaluate the determinant of the vector field one needs to find all eigenvectors of the vector field on the tangent space passing through the fixed points

$$\begin{aligned} t_i \delta B_i &= \Lambda g \delta B_i g^{-1}, \\ \delta I t &= \Lambda g \delta I, \\ t_1 t_2 t^{-1} \delta J &= \Lambda \delta J g^{-1}. \end{aligned} \tag{4.2}$$

¹More precisely character of vacuum representation in principal grading.

This is equivalent to the following set of equations

$$\begin{aligned}
 \lambda(\delta B_i)_{ss'} &= (\epsilon_i + \phi_{s'} - \phi_s)(\delta B_i)_{ss'}, \\
 \lambda(\delta I)_{sp} &= (a_p - \phi_s)(\delta I)_{sp}, \\
 \lambda(\delta J)_{ps} &= (\epsilon_1 + \epsilon_2 - a_p + \phi_s)(\delta J)_{ps},
 \end{aligned} \tag{4.3}$$

where $\Lambda = \exp \lambda \tau$, $g_{ss} = \exp \phi_s \tau$ and

$$\phi_s = (i_s - 1)\epsilon_1 + (j_s - 1)\epsilon_2 + a_{p(s)}. \tag{4.4}$$

System (4.3) gives all possible eigenvectors of the vector field. We should keep only those which belong to the tangent space. Essentially this means excluding variations breaking ADHM constraints. On the Moduli space

$$[\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J = 0. \tag{4.5}$$

Gauge symmetry can be taken into account in the following manner. We fix a gauge in which $\delta B_{1,2}, \delta I, \delta J$ are orthogonal to any gauge transformation of $B_{1,2}, I, J$. This gives additional constraint

$$[\delta B_l, B_l^\dagger] + \delta I I^\dagger - J^\dagger \delta J = 0. \tag{4.6}$$

The variations in the l.h.s. of (4.5) and (4.6) should be excluded from (4.3). The corresponding eigenvalues are defined from the equations

$$\begin{aligned}
 t_1 t_2 ([\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J) &= \Lambda g \left([\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J \right) g^{-1}, \\
 [\delta B_l, B_l^\dagger] + \delta I I^\dagger - J^\dagger \delta J &= \Lambda g \left([\delta B_l, B_l^\dagger] + \delta I I^\dagger - J^\dagger \delta J \right) g^{-1}.
 \end{aligned} \tag{4.7}$$

One finds the following eigenvalues, which should be excluded from (4.3):

$$\begin{aligned}
 \lambda &= (\epsilon_1 + \epsilon_2 + \phi_s - \phi_{s'}), \\
 \lambda &= (\phi_s - \phi_{s'}).
 \end{aligned} \tag{4.8}$$

Thus, the determinant of the vector field (2.4) is given by

$$\det v = \frac{\prod_{s,s' \in \bar{Y}} (\epsilon_1 + \phi_{s'} - \phi_s)(\epsilon_2 + \phi_{s'} - \phi_s) \prod_{l=1,2; s \in \bar{Y}} (a_l - \phi_s)(\epsilon_1 + \epsilon_2 - a_l + \phi_s)}{\prod_{s,s' \in \bar{Y}} (\phi_{s'} - \phi_s)(\epsilon_1 + \epsilon_2 - \phi_{s'} + \phi_s)} \tag{4.9}$$

Now we consider the action of the vector field (2.4) in \mathcal{M}_{sym} . The tangent space is reduced by the additional requirement (2.3)

$$-\delta B_{1,2} = P \delta B_{1,2} P^{-1}; \quad \delta I = P \delta I; \quad \delta J = \delta J P^{-1}, \tag{4.10}$$

or, on the level of the matrix elements,

$$-(\delta B_{1,2})_{ss'} = P(s)(\delta B_{1,2})_{ss'} P(s'); \quad (\delta I)_{sp} = P(s)(\delta I)_{sp}; \quad (\delta J)_{ps} = (\delta J)_{ps} P(s), \tag{4.11}$$

The first relation in (4.11) means that only eigenvectors $(\delta B_{1,2})_{ss'}$ with the different colors of s and s' belong to Z_{sym} . Similarly, the second one leaves $(\delta J)_{ps}$ only if s is white. The variations, which should be excluded (4.5) and (4.6) belong to \mathcal{M}_{sym} only for the matrix elements between the states of the same color.

Thus, we get the new determinant of the vector field (2.4)

$$\det' v = \frac{\prod_{\substack{s,s' \in \vec{Y} \\ P(s) \neq P(s')}} (\epsilon_1 + \phi_{s'} - \phi_s)(\epsilon_2 + \phi_{s'} - \phi_s) \prod_{\substack{\alpha=1,2; s \in \vec{Y} \\ P(s)=1}} (a_\alpha - \phi_s)(\epsilon_1 + \epsilon_2 - a_\alpha + \phi_s)}{\prod_{\substack{s,s' \in \vec{Y} \\ P(s)=P(s')}} (\phi_{s'} - \phi_s)(\epsilon_1 + \epsilon_2 - \phi_{s'} + \phi_s)} \quad (4.12)$$

Re-expressed in terms of arm-length and leg-length this expression gives

$$\det' v = \prod_{\alpha,\beta=1}^2 \prod_{s \in \diamond Y_\alpha(\beta)} E(a_\alpha - a_\beta, Y_\alpha, Y_\beta | s) (Q - E(a_\alpha - a_\beta, Y_\alpha, Y_\beta | s)), \quad (4.13)$$

here $E(a, Y_1, Y_2 | s)$ are defined as follows

$$E(a, Y_1, Y_2 | s) = a + b(L_{Y_1}(s) + 1) - b^{-1}A_{Y_2}(s), \quad (4.14)$$

where $A_Y(s)$ and $L_Y(s)$ are respectively the arm-length and the leg-length for a cell s in Y . The region $\diamond Y_\alpha(\beta)$ is defined as

$$\diamond Y_\alpha(\beta) = \{(i, j) \in Y_\alpha | P(k'_j(Y_\alpha)) \neq P(k_i(Y_\beta))\}, \quad (4.15)$$

or, in other words, the boxes having different parity of the leg- and arm-factors. Formula (4.13) is similar to the equation (3.7) in [21]. So the contribution of the vector multiplet reads

$$Z_{\text{vec}}^{\text{sym}}(\vec{a}, \vec{Y}) = \frac{1}{\det' v} \quad (4.16)$$

We consider the $\mathcal{N} = 2$ SU(2) theory with 4 fundamental hypermultiplets with masses μ . These hypermultiplets give some additional contribution because of appearance of the N null-modes for each kind of fermions in fundamental representation of the gauge group. The amplitudes ψ of the null-modes can be considered as the fiber V attached to one of the fixed point \vec{Y} . The eigenvalues of the vector field are defined from the equation

$$\lambda \psi_s = (\mu + \phi_s) \psi_s, \quad (4.17)$$

The corresponding contribution of the fundamental hypermultiplets with masses μ_i looks as follows

$$Z_f(\mu_i, \vec{a}, \vec{Y}) = \prod_{i=1}^4 \prod_{\alpha=1}^2 \prod_{s \in Y_\alpha} (\phi(a_\alpha, s) + \mu_i), \quad (4.18)$$

Considering the case of \mathcal{M}_{sym} we impose some restrictions on the set of eigenvectors for the fundamental multiplets. Namely we will assume that $\psi \in V_+$, if N -even and $\psi \in V_-$, if N -odd.

The above consideration suggests the following form of the contributions of the fundamental hyper multiplets

$$\begin{aligned}
 Z_f^{\text{sym}(0)}(\mu_i, \vec{a}, \vec{Y}) &= \prod_{i=1}^4 \prod_{\alpha=1}^2 \prod_{s \in Y_{\alpha, s} \text{-white}} (\phi(a_{\alpha}, s) + \mu_i), \\
 Z_f^{\text{sym}(1)}(\mu_i, \vec{a}, \vec{Y}) &= \prod_{i=1}^4 \prod_{\alpha=1}^2 \prod_{s \in Y_{\alpha, s} \text{-black}} (\phi(a_{\alpha}, s) + \mu_i),
 \end{aligned}
 \tag{4.19}$$

The first of these partition functions correspond to the case with even number of instantons, the second one correspond to the case with odd number of instantons.

5 Four-point Super Liouville conformal block

Two-dimensional super conformal Liouville field theory arises in the context of super string theory in non-critical dimensions of spacetime [28]. The Lagrangian of the theory reads

$$\mathcal{L}_{\text{SLFT}} = \frac{1}{8\pi} (\partial_a \phi)^2 + \frac{1}{2\pi} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) + 2i\mu b^2 \bar{\psi} \psi e^{b\phi} + 2\pi b^2 \mu^2 e^{2b\phi} .
 \tag{5.1}$$

Here μ is the cosmological constant and parameter b is related to the central charge c of the super-Virasoro algebra

$$c = 1 + 2Q^2, \quad Q = b + \frac{1}{b} .
 \tag{5.2}$$

In this paper we are interested in the Neveu-Schwarz sector of the super-Virasoro algebra

$$\begin{aligned}
 &= (n - m)L_{n+m} + \frac{c}{8}(n^3 - n)\delta_{n+m}, \\
 \{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2}c(r^2 - \frac{1}{4})\delta_{r+s}, \\
 [L_n, G_r] &= (\frac{1}{2}n - r)G_{n+r}.
 \end{aligned}
 \tag{5.3}$$

where the subscripts m, n are integer and r, s half-integer. The NS fields are classified in highest weight representations of super-Virasoro algebra. One of the central problems in CFT is the computation of the correlation functions of the primary fields represented by the highest weight vectors. We denote them Φ_{Δ} and Ψ_{Δ} . The highest weight vector Φ_{Δ} is annihilated by all positive-frequency generators and has the conformal dimension Δ defined by $L_0|\Delta\rangle = \Delta|\Delta\rangle$, while $\Psi_{\Delta} \equiv G_{-1/2}\Phi_{\Delta}$. Together fields Φ_{Δ} and Ψ_{Δ} form primary super doublet. In what follows we use the standard parametrization of the conformal dimensions

$$\Delta(\lambda) = \frac{Q^2}{8} - \frac{\lambda^2}{2}.
 \tag{5.4}$$

In [29–33] four-point correlation functions of the primary fields in NS sector were constructed by means of so-called elliptic and c -recursion procedures. By means of the bootstrap technique [34] they are expressed in terms of the basic conformal blocks and structure

constants of the operator algebra [35, 36]. In particular, the 4-point correlation function of bosonic primaries Φ_i with conformal weights Δ_i is given by

$$\begin{aligned} \langle \Phi_1(q)\Phi_2(0)\Phi_3(1)\Phi_4(\infty) \rangle = (q\bar{q})^{\Delta-\Delta_1-\Delta_2} \sum_{\Delta} \left(C_{12}^{\Delta} C_{34}^{\Delta} F_0(\Delta_i|\Delta|q) F_0(\Delta_i|\Delta|\bar{q}) \right. \\ \left. + \tilde{C}_{12}^{\Delta} \tilde{C}_{34}^{\Delta} F_1(\Delta_i|\Delta|q) F_1(\Delta_i|\Delta|\bar{q}) \right). \end{aligned} \quad (5.5)$$

Below we quote the results of [31] for the first few coefficients in the series expansion of the superconformal blocks $F_{0,1}$

$$\begin{aligned} F_0(\Delta_i|\Delta|q) &= \sum_{N=0,1,\dots} q^N F^{(N)}(\Delta_i|\Delta), \\ F_1(\Delta_i|\Delta|q) &= \sum_{N=1/2,3/2,\dots} q^N F^{(N)}(\Delta_i|\Delta), \end{aligned} \quad (5.6)$$

up to $N = 2$:

$$F^{(0)} = 1, \quad (5.7)$$

$$F^{(\frac{1}{2})} = \frac{1}{2\Delta}, \quad (5.8)$$

$$F^{(1)} = \frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta}, \quad (5.9)$$

$$\begin{aligned} F^{(\frac{3}{2})} &= \frac{(1 + 2\Delta + 2\Delta_1 - 2\Delta_2)(1 + 2\Delta + 2\Delta_3 - 2\Delta_4)}{8\Delta(1 + 2\Delta)} \\ &\quad + \frac{4(\Delta_1 - \Delta_2)(\Delta_3 - \Delta_4)}{(1 + 2\Delta)(c + 2(-3 + c)\Delta + 4\Delta^2)}, \end{aligned} \quad (5.10)$$

$$\begin{aligned} F^{(2)} &= \frac{(\Delta + \Delta_1 - \Delta_2)(1 + \Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)(1 + \Delta + \Delta_3 - \Delta_4)}{4\Delta(1 + 2\Delta)} \\ &\quad + \frac{(\Delta^2 - 3(\Delta_1 - \Delta_2)^2 + 2\Delta(\Delta_1 + \Delta_2))(\Delta^2 - 3(\Delta_3 - \Delta_4)^2 + 2\Delta(\Delta_3 + \Delta_4))}{2\Delta(3 + 2\Delta)(-3 + 3c + 16\Delta)} \\ &\quad + \frac{(\Delta_1 - 2(\Delta_1 - \Delta_2)^2 + \Delta_2 + \Delta(-1 + 2\Delta_1 + 2\Delta_2))(\Delta_1, \Delta_2 \rightarrow \Delta_3, \Delta_4)}{(c + 2(-3 + c)\Delta + 4\Delta^2)(3 + 4\Delta(2 + \Delta))}. \end{aligned} \quad (5.11)$$

On the basis of the results of the previous section we suggest the following representation for the NS four-point conformal blocks (5.6):

$$\begin{aligned} \sum_{N=0,1,\dots} q^N \sum_{\vec{Y}, \substack{N_+(\vec{Y})=N \\ N_-(\vec{Y})=N}} Z_{\text{f}}^{\text{sym}(0)}(\mu_i, \vec{a}, \vec{Y}) Z_{\text{vec}}^{\text{sym}}(\vec{a}, \vec{Y}) &= (1 - q)^A F_0(\Delta(\lambda_i)|\Delta(a)|q), \\ \sum_{N=\frac{1}{2}, \frac{3}{2}, \dots} q^N \sum_{\vec{Y}, \substack{N_+(\vec{Y})=N+\frac{1}{2} \\ N_-(\vec{Y})=N-\frac{1}{2}}} Z_{\text{f}}^{\text{sym}(1)}(\mu_i, \vec{a}, \vec{Y}) Z_{\text{vec}}^{\text{sym}}(\vec{a}, \vec{Y}) &= \frac{1}{2}(1 - q)^A F_1(\Delta(\lambda_i)|\Delta(a)|q). \end{aligned} \quad (5.12)$$

The parameters of the conformal block functions are related to those of the instanton partition function in the following manner

$$\begin{aligned}\mu_1 &= \frac{Q}{2} - (\lambda_1 + \lambda_2), & \mu_2 &= \frac{Q}{2} - (\lambda_1 - \lambda_2), \\ \mu_3 &= \frac{Q}{2} - (\lambda_3 + \lambda_4), & \mu_4 &= \frac{Q}{2} - (\lambda_3 - \lambda_4),\end{aligned}\tag{5.13}$$

and

$$A = \left(\frac{Q}{2} - \lambda_1\right) \left(\frac{Q}{2} - \lambda_3\right).\tag{5.14}$$

The formula (5.12) is the main result of this paper. We have checked this formula up to the level 5/2.

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References

- [1] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [[arXiv:0906.3219](#)] [[SPIRES](#)].
- [2] L.F. Alday and Y. Tachikawa, *Affine SL(2) conformal blocks from 4d gauge theories*, *Lett. Math. Phys.* **94** (2010) 87 [[arXiv:1005.4469](#)] [[SPIRES](#)].
- [3] N. Wyllard, *A_{N-1} conformal Toda field theory correlation functions from conformal $N = 2$ SU(N) quiver gauge theories*, *JHEP* **11** (2009) 002 [[arXiv:0907.2189](#)] [[SPIRES](#)].
- [4] A. Mironov and A. Morozov, *On AGT relation in the case of U(3)*, *Nucl. Phys.* **B 825** (2010) 1 [[arXiv:0908.2569](#)] [[SPIRES](#)].
- [5] M. Taki, *On AGT conjecture for pure super Yang-Mills and W-algebra*, *JHEP* **05** (2011) 038 [[arXiv:0912.4789](#)] [[SPIRES](#)].
- [6] V. Belavin and B. Feigin, *Super Liouville conformal blocks from $N = 2$ SU(2) quiver gauge theories*, *JHEP* **07** (2011) 079 [[arXiv:1105.5800](#)] [[SPIRES](#)].
- [7] T. Nishioka and Y. Tachikawa, *Para-Liouville/Toda central charges from M5-branes*, *Phys. Rev.* **D 84** (2011) 046009 [[arXiv:1106.1172](#)] [[SPIRES](#)].
- [8] G. Bonelli, K. Maruyoshi and A. Tanzini, *Instantons on ALE spaces and Super Liouville Conformal Field Theories*, [arXiv:1106.2505](#) [[SPIRES](#)].
- [9] H. Nakajima, *Instantons on ALE spaces, quiver variety, and Kac-Moody algebras*, *Duke Math.* **76** (1994) 365.
- [10] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. I, Invent. Math.* **162** (2005) 313 [[math/0306198](#)].

- [11] M. Varagnolo and E. Vasserot, *On the K-theory of the cyclic quiver variety*, *Internat. Math. Res. Notices* **18** (1999) 1005.
- [12] B. Feigin and A. Tsymbaliuk, *Heisenberg action in the equivariant K-theory of Hilbert schemes via scuffle algebra*, [arXiv:0904.1679](#).
- [13] D. Gaiotto, *Asymptotically free $N = 2$ theories and irregular conformal blocks*, [arXiv:0908.0307](#) [SPIRES].
- [14] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [[hep-th/0206161](#)] [SPIRES].
- [15] R. Flume and R. Poghossian, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, *Int. J. Mod. Phys. A* **18** (2003) 2541 [[hep-th/0208176](#)] [SPIRES].
- [16] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [[hep-th/0211108](#)] [SPIRES].
- [17] A.G. Kuznetsov, *Quiver varieties and Hilbert schemes*, *Mosc. Math. J.* **7** (2007) 673 [[math/0111092](#)].
- [18] V.A. Alba, V.A. Fateev, A.V. Litvinov and G.M. Tarnopolsky, *On combinatorial expansion of the conformal blocks arising from AGT conjecture*, [arXiv:1012.1312](#) [SPIRES].
- [19] A. Belavin and V. Belavin, *AGT conjecture and integrable structure of conformal field theory for $c = 1$* , *Nucl. Phys. B* **850** (2011) 199 [[arXiv:1102.0343](#)] [SPIRES].
- [20] V.A. Fateev and A.B. Zamolodchikov, *Parafermionic currents in the two-dimensional conformal quantum field theory and selfdual critical points in $Z(n)$ invariant statistical systems*, *Sov. Phys. JETP* **62** (1985) 215. [Zh.Eksp.Teor.Fiz.89:380-399,1985], SPHJA,62,215;
- [21] F. Fucito, J.F. Morales and R. Poghossian, *Multi instanton calculus on ALE spaces*, *Nucl. Phys. B* **703** (2004) 518 [[hep-th/0406243](#)] [SPIRES].
- [22] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett. A* **65** (1978) 185 [SPIRES].
- [23] A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, *Phys. Lett. B* **59** (1975) 85 [SPIRES].
- [24] A.A. Belavin and V.E. Zakharov, *Yang-Mills equations as inverse scattering problem*, *Phys. Lett. B* **73** (1978) 53 [SPIRES].
- [25] M.F. Atiyah and R.S. Ward, *Instantons and algebraic geometry*, *Commun. Math. Phys.* **55** (1977) 117 [SPIRES].
- [26] V. Drinfeld, Yu. Manin, *Descriptions of instantons*, *Comm. Math. Phys.* **6** (1978) 177.
- [27] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [[hep-th/0206063](#)] [SPIRES].
- [28] A.M. Polyakov, *Quantum geometry of fermionic strings*, *Phys. Lett. B* **103** (1981) 211 [SPIRES].
- [29] A. Belavin, V. Belavin, A. Neveu and A. Zamolodchikov, *Bootstrap in supersymmetric Liouville field theory. I: NS sector*, *Nucl. Phys. B* **784** (2007) 202 [[hep-th/0703084](#)] [SPIRES].

- [30] L. Hadasz, Z. Jaskólski and P. Suchanek, *Recursion representation of the Neveu-Schwarz superconformal block*, *JHEP* **03** (2007) 032 [[hep-th/0611266](#)] [[SPIRES](#)].
- [31] V.A. Belavin, *$N = 1$ SUSY conformal block recursive relations*, [hep-th/0611295](#) [[SPIRES](#)].
- [32] V.A. Belavin, *On the $N = 1$ super Liouville four-point functions*, *Nucl. Phys. B* **798** (2008) 423 [[arXiv:0705.1983](#)] [[SPIRES](#)].
- [33] L. Hadasz, Z. Jaskolski and P. Suchanek, *Elliptic recurrence representation of the $N = 1$ Neveu-Schwarz blocks*, *Nucl. Phys. B* **798** (2008) 363 [[arXiv:0711.1619](#)] [[SPIRES](#)].
- [34] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys. B* **241** (1984) 333 [[SPIRES](#)].
- [35] R.C. Rashkov and M. Stanishkov, *Three-point correlation functions in $N = 1$ Super Liouville Theory*, *Phys. Lett. B* **380** (1996) 49 [[hep-th/9602148](#)] [[SPIRES](#)].
- [36] R.H. Poghosian, *Structure constants in the $N = 1$ super-Liouville field theory*, *Nucl. Phys. B* **496** (1997) 451 [[hep-th/9607120](#)] [[SPIRES](#)].