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On correlation numbers in 2D minimal gravity and matrix models*

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Abstract

We test recent results for the four-point correlation numbers in minimal Liouville gravity against calculations in the one-matrix models, and find full agreement. In the process, we construct the resonance transformation which relates coupling parameters λ_k of the Liouville gravity to the couplings t_k of the matrix models, up to the terms of the order 4. We also conjecture the general form of this transformation.

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1. Introduction

At present, there are two relatively independent approaches to 2D quantum gravity. One is the continuous approach, in which the theory is defined through the functional integral over the Riemannian³ metric $g_{\mu\nu}(X)$, with appropriate gauge fixing. The choice of the conformal gauge leads to quantum Liouville theory [1] (coupled to matter fields), and for that reason this approach is often called the Liouville gravity. The other is the discrete approach, based on the idea of approximating the fluctuating 2D geometry by an ensemble of planar graphs, so that the continuous theory is recovered in the scaling limit where the planar graphs of very large size dominate. The discrete approach is usually referred to as the matrix models, since technically the ensemble of the graphs is usually generated by the perturbative expansion of the integral over $N \times N$ matrices, with N sent to infinity to guarantee the planarity of the graphs (see, e.g., [2] and references therein). Since these two approaches stem from the same idea of fluctuating geometry, they are expected to produce identical results for the physical quantities. Indeed, this expectation was confirmed by explicit calculations of some quantities in a number of particular models [3–5]. However, the techniques involved in calculations

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³ Here we always have in mind the Euclidean version of the theory.

by the two approaches are very different, and satisfactory conceptual understanding of the relation between those techniques is still lacking. In the absence of that, more checks of the agreement of the results seem to be desirable.

The most easily defined objects in the quantum gravity are the n -point ‘correlation numbers’—the integrated correlation functions

$$C_{k_1 \dots k_n} = \langle O_{k_1} \dots O_{k_n} \rangle, \quad O_k = \int_{\mathbb{M}} \mathcal{O}_k(X). \tag{1.1}$$

Here $\mathcal{O}(X)$ are some local densities (two forms) on the manifold \mathcal{M} which may involve both the ‘matter’ and the metric degrees of freedom localized at $X \in \mathbb{M}$, and the expectation value is taken over the fluctuations of both the matter and the geometry. The generating function,

$$Z(\{\lambda_k\}) = Z_0 \left\langle \exp \left\{ \sum_k \lambda_k O_k \right\} \right\rangle, \tag{1.2}$$

may be regarded as the vacuum partition function of the original theory perturbed by adding the fields $\mathcal{O}_k(X)$ to the action density, with the coupling constants $-\lambda_k$. Throughout this paper we limit our attention to the case when \mathbb{M} is topologically a sphere. Even in this case, while it is relatively easy to evaluate the n -point correlation numbers in solvable matrix models, the Liouville gravity approach to these quantities involves technically complicated integration over the $(n - 3)$ -dimensional moduli space of a sphere with n punctures. For this reason, most previous comparisons of the matrix models and the Liouville gravity results were limited to one-, two- and three-point correlation numbers (notable exceptions are the analysis in [5]). A few years ago, a new technique for handling the moduli integrals was developed by Alexei Zamolodchikov and one of the authors [6]. It applies to the so-called ‘minimal gravity’ models (minimal CFT coupled to the Liouville mode); by using the higher Liouville equations of motion [7] it allows one to reduce the moduli integrals to the boundary terms. In this way, the (partial) result for the four-point correlation numbers in the minimal gravity was obtained in [6].

In this paper we test the result of [6] against the corresponding correlation numbers from the matrix models. There is an important subtlety which makes this comparison less than straightforward. In a local field theory, the integrated correlation functions of type (1.1) suffer from intrinsic ambiguity generally referred to as the ‘contact terms’. Since (1.1) is the integral

$$\int_{X_1, \dots, X_n} \langle \mathcal{O}_{k_1}(X_1) \dots \mathcal{O}_{k_n}(X_n) \rangle, \tag{1.3}$$

the correlation numbers may pick up contributions from delta-like terms in the integrand, when two or more points X_i collide. Although most of the contact terms are not determined by the field theory itself, it is well known that any change of contact terms is equivalent to a certain analytic change of the coupling parameters in (1.2):

$$\lambda_k \rightarrow \tilde{\lambda}_k = \lambda_k + \sum_{k_1 k_2} C_k^{k_1 k_2} \lambda_{k_1} \lambda_{k_2} + \dots \tag{1.4}$$

Of course, this relation is at the heart of the renormalization theory, and indeed the ambiguity of the contact terms just reflects the freedom of making finite renormalizations. Usually, the ambiguity is resolved by appealing to the dimensional analysis. The local fields $\mathcal{O}_k(X)$ can be chosen in such a way that they (and hence the associated coupling parameters in (1.2)) have definite mass dimensions, so that

$$O_k \sim [\text{mass}^2]^{\delta_k}, \quad \lambda_k \sim [\text{mass}^2]^{-\delta_k}. \tag{1.5}$$

Then, if one does not allow for any ‘auxiliary’-dimensional parameters, only transformations (1.4) which respect the balance of the dimensions are admitted, i.e., the nonlinear (say, n th order) terms in (1.4) are admissible only if they satisfy the ‘resonance conditions’⁴

$$\delta_k = \delta_{k_1} + \dots + \delta_{k_n}. \tag{1.6}$$

We generally refer to the nonlinear terms in (1.4) as the ‘resonance terms’.

In the absence of the resonances, no nonlinear finite renormalizations (1.4) consistent with the dimensional counting are allowed, and thus the above ambiguity is completely fixed. However, when the minimal CFT is coupled to the gravity, appearance of the resonances turns out to be very common. Therefore, this ambiguity needs to be resolved before a meaningful comparison between the correlation numbers obtained by the two different methods can be made. The problem was first outlined by Moore, Seiberg and Staudacher [8], who also found the solution in a number of special cases. In particular, addressing the case of the p -critical point of the one-matrix model and the corresponding minimal gravity $\mathcal{MG}_{2/2p+1}$ (the minimal CFT $\mathcal{M}_{2/2p+1}$ coupled to the Liouville gravity), they partly determined the resonance terms in the relation between the coupling parameters, which allowed them to establish equivalence up to the level of two-point correlation numbers.

In what follows, we extend this analysis to three- and four-point correlation numbers and find perfect agreement between the matrix models and minimal gravity results. In the process, we determine the higher-order resonance terms by demanding that the higher-order correlation numbers satisfy the fusion rules inherent to the minimal gravity. At the end, we conjecture the full resonance transformation which relates coupling parameters in the p -critical one-matrix models and minimal gravity $\mathcal{MG}_{2/2p+1}$.

2. Minimal gravity $\mathcal{MG}_{2/2p+1}$

Specific models of the Liouville gravity are defined by the content of the ‘matter’ field theory which is placed on the 2D manifold with the fluctuating metric. Perhaps the simplest models are defined by choosing the minimal CFT $\mathcal{M}_{q'/q}$ as the ‘matter’ theory. As in [9], we use the term ‘minimal gravity’ $\mathcal{MG}_{q'/q}$ for such models. In this work we restrict our attention to the models with $q' = 2$ and $q = 2p + 1$, $p = 1, 2, 3, \dots$. The minimal gravity $\mathcal{MG}_{2/2p+1}$ is very likely to correspond to the p -critical point in the general one-matrix model [10].

2.1. Minimal model $\mathcal{M}_{2/2p+1}$

The Kac table of the minimal CFT $\mathcal{M}_{2/2p+1}$ is a single row of length $2p$, the entries being the degenerate primary fields $\Phi_{(1,n)}$, $n = 1, 2, \dots, 2p$. Due to the mandatory identification $\Phi_{(1,n)} = \Phi_{(1,2p+1-n)}$, the model actually has only p -independent primary fields. We will use the abbreviated notation

$$\Phi_k = \Phi_{(1,k+1)} \tag{2.1}$$

for them, so that Φ_0 is the identity operator. All the independent primaries are listed by letting k run through the range $k = 0, 1, 2, \dots, p - 1$ (and in what follows, unless stated otherwise, it is assumed that k lies in this range), but it is often convenient to extend the range to the full Kac table, $k = 1, 2, \dots, 2p$, with the above identification,

$$\Phi_{2p-k-1} = \Phi_k, \tag{2.2}$$

⁴ The term is taken from the vocabulary of the renormalization group, where precisely these conditions characterize the nonlinear terms in the beta-functions which cannot be eliminated by analytic transformations of the coupling parameters.

kept in mind. In particular, with this convention the fusion rules of $\mathcal{M}_{2/2p+1}$ take the simple form

$$[\Phi_{k_1}][\Phi_{k_2}] = \sum_{k=|k_1-k_2|:2}^{k_1+k_2} [\Phi_k], \tag{2.3}$$

where, as usual, $[\Phi_k]$ stands for the irreducible Virasoro representation associated with the primary field Φ_k . Here and below, the symbol $\sum_{k=m:2}^{m'}$ denotes the sum with the step 2, in which k runs over the values $m, m+2, m+4, \dots, \leq m'$. While in (2.3), it is assumed that k_1, k_2 lie within the domain $[0, 1, \dots, p-1]$; the summation index k is allowed to run outside it, where the terms $[\Phi_k]$ are understood as $[\Phi_{2p-k-1}]$.

Note that the identification (2.2) breaks the naive parity symmetry $\Phi_k \rightarrow (-)^k \Phi_k$ of (2.3), so that $[\Phi_k]$ with odd (even) k may appear in the rhs of (2.3) with even (odd) k_1+k_2 . In particular, the correlation functions,

$$\langle \Phi_{k_1}(X_1) \Phi_{k_2}(X_2) \dots \Phi_{k_n}(X_n) \rangle, \tag{2.4}$$

do not necessarily vanish when $k_1+k_2+\dots+k_n$ is odd.

On the other hand, conformal invariance and the fusion rules (2.3) do force many correlation functions to vanish. Thus, the conformal invariance demands vanishing of all the one-point correlation functions except for $\langle \Phi_0(X) \rangle$, as well as the diagonal form of the two-point functions, i.e.,

$$\langle \Phi_k(X) \rangle = 0 \quad \text{unless} \quad k = 0, \tag{2.5}$$

$$\langle \Phi_{k_1}(X_1) \Phi_{k_2}(X_2) \rangle = 0 \quad \text{unless} \quad k_1 = k_2, \tag{2.6}$$

while the fusion rules (2.3) impose restrictions on the multi-point correlation functions (2.4) with $n \geq 3$. Those restrictions can be written in a compact form:

$$(2.4) = 0 \quad \text{if} \quad \begin{cases} k_1 + \dots + k_{n-1} < k_n & \text{for } k_1 + \dots + k_n \text{ even} \\ k_1 + \dots + k_n < 2p - 1 & \text{for } k_1 + \dots + k_n \text{ odd,} \end{cases} \tag{2.7}$$

if one assumes that all k_i are in the range $[0, 1, \dots, p-1]$, and that k_n is the largest of them, i.e. $k_i \leq k_n$. Below, we generally refer to the case of even or odd $k_1 + \dots + k_n$ as the even and odd sectors.

2.2. Coupling to Liouville gravity

According to [1, 11], coupling a conformally invariant ‘matter’ field theory to the fluctuating metric $g_{\mu\nu}(x)$ leads, in the conformal gauge, to the famous Liouville action,

$$\mathcal{A}_L = \frac{1}{4\pi} \int_{\mathbb{M}} \sqrt{\hat{g}} [\hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + Q \hat{R} \varphi + 4\pi \mu e^{2b\varphi}] d^2x, \tag{2.8}$$

for the field φ related to the conformal factor in $g_{\mu\nu} = e^{2b\varphi} \hat{g}_{\mu\nu}$, where $\hat{g}_{\mu\nu}$ is an arbitrary fixed ‘background’ metric on \mathbb{M} . The parameters b and Q are determined by the central charge c_M of the conformal ‘matter’:

$$26 - c_M = 1 + 6Q^2 \quad Q = 1/b + b. \tag{2.9}$$

In our case, the ‘matter’ is the minimal CFT $\mathcal{M}_{2/2p+1}$ with $c_M = -6p + 10 - \frac{12}{2p+1}$, i.e.,

$$b = \sqrt{\frac{2}{2p+1}}. \tag{2.10}$$

The parameter μ in (2.8) is dimensional:

$$\mu \sim [\text{mass}]^2, \tag{2.11}$$

and interpreted as the cosmological constant.

The operators $O_k = \int_{\mathbb{M}} \mathcal{O}_k(x)$ are constructed by ‘gravitational dressing’ of the primary fields Φ_k :

$$\mathcal{O}_k(x) = \Phi_k(x) e^{2a_k \varphi(x)} \sqrt{\hat{g}} d^2x, \tag{2.12}$$

where $a_k = b \frac{k+2}{2}$. This choice of the parameters a_k ensures that the integrand in (2.12) is indeed a density. These parameters also determine the mass dimensions of the operators O_k, δ_k in (1.5) [3, 11],

$$\delta_k = -\frac{a_k}{b} = -\frac{k+2}{2}. \tag{2.13}$$

Proper gauge fixing leads to the following expression for the correlation numbers (1.1) with $n \geq 3$:

$$\langle O_{k_1} \dots O_{k_n} \rangle = \int_{x_1, \dots, x_{n-3}} \langle O_{k_1}(x_1) \dots O_{k_{n-3}}(x_{n-3}) \tilde{\mathcal{O}}_{k_{n-2}}(x_{n-2}) \tilde{\mathcal{O}}_{k_{n-1}}(x_{n-1}) \tilde{\mathcal{O}}_{k_n}(x_n) \rangle_{\text{Matter Liouville}}, \tag{2.14}$$

where $\tilde{\mathcal{O}}_k = C \bar{C} e^{2a_k \varphi} \Phi_k$ are scalars (zero forms) associated with the densities \mathcal{O}_k . Three points, x_{n-2}, x_{n-1}, x_n , can be chosen at will, and the integration is performed over the remaining $n-3$ points x_1, \dots, x_{n-3} (interpreted as the moduli of the n -punctured sphere). The expectation value at the right-hand side of (2.14) involves both the ‘matter’ and the Liouville sectors. Due to the factorized form (2.12), the integrand in (2.14) is a product of the correlation functions (2.4) and the correlation functions of the Liouville exponentials:

$$\langle e^{2a_{k_1} \varphi(x_1)} \dots e^{2a_{k_n} \varphi(x_n)} \rangle_{\text{Liouville}}. \tag{2.15}$$

For $n = 3$ no integration is necessary, and the three-point correlation numbers are obtained by multiplying the three-point functions (2.4) by the Liouville three-point function (2.15). The result takes a very simple form [9]:

$$\langle O_{k_1} O_{k_2} O_{k_3} \rangle = -\mu^{\delta_{k_1} + \delta_{k_2} + \delta_{k_3}} N_{k_1 k_2 k_3} \mathcal{N}_p \prod_{i=1}^3 \text{Leg}_L(k_i). \tag{2.16}$$

Here

$$\mathcal{N}_p = (2p - 1)(2p + 1)(2p + 3), \tag{2.17}$$

and the ‘Leg factors’

$$\text{Leg}_L(k) = \frac{(-)^{\frac{k}{2}}}{2\pi^{\frac{k}{2}}} \left[\gamma \left(\frac{2}{2p+1} \right) \right]^{-\frac{k+1}{2}} \left[\gamma \left(\frac{2(k+1)}{2p+1} \right) \right]^{\frac{1}{2}} \frac{\Gamma(p - 1/2)}{\Gamma(p - k - 1/2)}. \tag{2.18}$$

($\gamma(t) = \Gamma(t)/\Gamma(1-t)$) will play a very small role in our discussion⁵. One can always get rid of them by multiplicative renormalization of the operators O_k and the coupling parameters λ_k in (1.2):

$$O_k \rightarrow \frac{1}{\text{Leg}_L(k)} O_k, \quad \lambda_k \rightarrow \text{Leg}_L(k) \lambda_k. \tag{2.19}$$

⁵ Note that $\text{Leg}_L(0) = 1/2$. It is also interesting to note that for odd k the factor (2.18) returns pure imaginary values. This reflects the known property of the odd- k operators Φ_k in the minimal CFT $\mathcal{M}_{2/2p+1}$. For example, $\mathcal{M}_{2/5}$ is known to describe the Yang–Lee edge criticality [12]; in this identification the operator Φ_1 corresponds to the Ising-order parameter, which couples to the pure imaginary magnetic field.

The most important entry in (2.16) is the ‘fusion coefficients’ $N_{k_1 k_2 k_3}$ which take the value 1 if three-point fusion rules are satisfied, and turn to zero otherwise. Explicitly,

$$N_{k_1 k_2 k_3} = \begin{cases} 1 & \text{if } k_1 + k_2 + k_3 \geq 2p - 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k_1 + k_2 + k_3 \text{ odd} \quad (2.20)$$

and

$$N_{k_1 k_2 k_3} = \begin{cases} 1 & \text{if } k_1 + k_2 \geq k_3 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k_1 + k_2 + k_3 \text{ even,} \quad (2.21)$$

where in writing (2.21) we have assumed that k_3 is the maximal of the numbers k_i , i.e. $k_1, k_2 \leq k_3$. This factor, of course, is inherited from the ‘matter’ three-point functions $\langle \Phi_{k_1} \Phi_{k_2} \Phi_{k_3} \rangle$.

Since Φ_0 is the identity operator in $\mathcal{M}_{2/2p+1}$, insertion of O_0 in (1.1) is equivalent to taking the derivative with respect to μ , more precisely

$$\langle O_0 O_{k_1} \dots O_{k_n} \rangle = -Z_0^{-1} \frac{\partial}{\partial \mu} (Z_0 \langle O_{k_1} \dots O_{k_n} \rangle), \quad (2.22)$$

where

$$Z_0 \sim \mu^{\frac{2p+3}{2}} \quad (2.23)$$

is the Liouville partition function of a sphere. Therefore, the two- and one-point correlation numbers are easily deduced from (2.16):

$$\langle O_k O_{k'} \rangle = \delta_{k,k'} \frac{\mathcal{N}_p \mu^{2\delta_k}}{2p - 2k - 1} \text{Leg}_L^2(k), \quad \langle O_k \rangle = -\delta_{k,0} (p + 3/2) \mu^{-1}. \quad (2.24)$$

Again, the diagonal form of the two-point numbers and vanishing of all but one of the one-point numbers can be traced to the corresponding properties of the ‘matter’ correlation functions, equations (2.6) and (2.5).

In [6], the integration over the moduli in (2.14) is performed in the case $n = 4$. The result, specialized for $\mathcal{MG}_{2/2p+1}$, can be written as

$$\langle O_{k_1} O_{k_2} O_{k_3} O_{k_4} \rangle = \mathcal{N}_p \mu^{\sum \delta_{k_i}} C_{k_1 k_2 k_3 k_4} \prod_{i=1}^4 \text{Leg}_L(k_i), \quad (2.25)$$

where, if one chooses k_1 to be the smallest of the numbers k_i (i.e., $k_1 \leq k_2, k_3, k_4$), the factor $C_{k_1 k_2 k_3 k_4}$ has the form

$$C_{k_1 k_2 k_3 k_4} = (k_1 + 1)(p + k_1 + 3/2) - \sum_{i=2}^4 \sum_{s=-k_1:2}^{k_1} \left| p - k_i - s - \frac{1}{2} \right|. \quad (2.26)$$

An important qualification applies to this result. Equation (2.26) was obtained under the assumption that the number of conformal blocks in the decomposition of the $\mathcal{M}_{2/2p+1}$ correlation function $\langle \Phi_{k_1} \Phi_{k_2} \Phi_{k_3} \Phi_{k_4} \rangle$ is exactly k_1 . To put it in an explicit form, assume that k_i are arranged in non-increasing order,

$$k_1 \leq k_2 \leq k_3 \leq k_4 \leq p - 1. \quad (2.27)$$

Then, in the even sector, this assumption is fulfilled iff

$$k_1 + k_4 \leq k_2 + k_3 \quad (2.28)$$

(which incidentally guarantees that the even-sector fusion rules are satisfied). In the odd sector, this condition also requires that the fusion rules are satisfied with sufficient redundancy:

$$-k_1 + k_2 + k_3 + k_4 \geq 2p - 1, \quad (2.29)$$

which, in turn, demands the validity of (2.28) (since $k_4 \leq p - 1$). Thus, in both even and odd sectors, when (2.28) breaks down, the validity of (2.26) is questionable [6]. Indeed, we will see in section 4.2 that at $k_1 + k_4 > k_2 + k_3$ (2.26) deviates from the matrix model result.

As was mentioned in the introduction, the minimal gravity $\mathcal{MG}_{2/2p+1}$ is likely to be the world-sheet theory of the p -critical point in the one-matrix model [10]. This identification was confirmed by explicit comparison of the one- and two-point correlation numbers [8]. The aim of this paper is to extend the analysis to the higher-order correlation numbers, and in particular to test the new result (2.26) against the matrix model's calculations.

3. One-matrix model

The matrix models technique was extensively studied in the literature. Most of what we will need here can be found in [2] and references therein.

3.1. p -critical point

The one-matrix model exhibits an infinite set of multi-critical points, labelled by the integer $p = 1, 2, 3, \dots$. In the scaling limit near the p -critical point, the partition function of the sphere is expressed through the solution of the ‘string equation’

$$\mathcal{P}(u) = 0, \tag{3.1}$$

where $\mathcal{P}(u)$ is the $(p + 1)$ -degree polynomial:

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}, \tag{3.2}$$

with the parameters t_k describing the relevant deviations from the p -critical point⁶. The singular part of the matrix model partition function $Z(t_0, t_1, \dots, t_{p-1})$ is expressed through (3.2) as follows⁷:

$$Z = \frac{1}{2} \int_0^{u_*} \mathcal{P}^2(u) \, du, \tag{3.3}$$

where $u_* = u_*(t_0, t_1, \dots, t_{p-1})$ is the suitably chosen root of the polynomial (3.2), i.e., $\mathcal{P}(u_*) = 0$. It is important to remember that (3.3) really gives only the singular part of the matrix model partition function. The actual matrix integral has also a regular part, analytic in all the parameters t_k at $\{t_k\} = 0$; in the spirit of the scaling theory of criticality, the regular part is disregarded as non-universal. For this reason the choice of 0 as the lower limit in the integral (3.3) is largely arbitrary. One can replace it by any constant, or indeed any regular function of t_k , at the cost of adding the regular terms to Z .

There is strong evidence [8, 10] that (3.3) (with u_* taken to be the maximal real root of $\mathcal{P}(u)$) provides the matrix model description of the minimal gravity $\mathcal{MG}_{2/2p+1}$, perturbed by the operators (2.12). In this identification $-t_0$ is interpreted as the cosmological constant μ in (2.8). Obvious re-scaling symmetry $u \rightarrow au, t_k \rightarrow a^{k+2} t_k$ of equation (3.1) allows one to ascribe ‘mass dimensions’ to the parameters t_k ,

$$t_k \sim [\text{mass}^2]^{\frac{k+2}{2}}, \tag{3.4}$$

and then $Z \sim [\text{mass}^2]^{\frac{2p+3}{2}}$, in agreement with (2.13) and (2.23).

⁶ Note that our labelling of the parameters t_k is different from that in [8]; our t_k are t_{p-k-1} in [8].

⁷ In the matrix models technique it actually emerges through the equation $\partial^2 Z / \partial t_{p-1}^2 = u_*(t_0, \dots, t_{p-1})$. Equation (3.3) gives ‘physical’ solution of this equation.

Comparison at the level of the correlation numbers involves two subtleties noted long ago in [8]. First, it is obvious from (2.13) that at sufficiently large p there are many resonances (1.6) between the dimensions δ_k . Therefore, the relation between the parameters t_k in (3.1) and the minimal gravity couplings λ_k may involve the resonance terms

$$t_k = C_k \lambda_k + \sum_{n=2}^{\lfloor \frac{k+2}{2} \rfloor} \sum_{k_1, \dots, k_n=0}^{p-1} C_k^{k_1 \dots k_n} \lambda_{k_1} \dots \lambda_{k_n} \tag{3.5}$$

with the coefficients constrained by the condition

$$C_k^{k_1 \dots k_n} = 0 \quad \text{unless} \quad \sum_{i=1}^n k_i = k + 2 - 2n. \tag{3.6}$$

The sum over $k_i = 0, 1, \dots, p-1$, in (3.5) includes $k_i = 0$ to take into account the possibility that the integer powers of the cosmological constant μ appear in the right-hand side; by definition

$$\lambda_0 = -\mu. \tag{3.7}$$

The coefficients C_k have very little significance. They can be removed by trivial renormalizations of the parameters λ_k (or t_k) similar to (2.19). Physical equivalence between the p -critical matrix model and the minimal gravity $\mathcal{MG}_{2/2p+1}$ would imply that by the special choice of the coefficients in $C_k^{k_1 \dots k_n}$ in (3.5) the partition function (3.3), expressed through $\{\lambda_k\}$, can be made identical to the generating function (1.2) of the minimal gravity, up to regular terms. This was the idea put forward by Moore *et al* [8], who have verified the identity up to the two-point correlation numbers. In the following subsection, we warm up by re-deriving their result in somewhat different language, and then proceed to the analysis of the higher orders.

The quantities to be compared with the correlation numbers of $\mathcal{MG}_{2/2p+1}$ are the coefficients of the expansion

$$Z = Z_0 + \sum_{k=1}^{p-1} \lambda_k Z_k + \sum_{k_1 k_2=1}^{p-1} \frac{\lambda_{k_1} \lambda_{k_2}}{2} Z_{k_1 k_2} + \dots + \sum_{k_1, \dots, k_n=1}^{p-1} \frac{\lambda_{k_1} \dots \lambda_{k_n}}{n!} Z_{k_1 \dots k_n} + \dots \tag{3.8}$$

of the matrix model partition function (3.3) in the powers of $\lambda_1, \dots, \lambda_{p-1}$, with $\lambda_0 = -\mu$ kept fixed. By dimensional analysis

$$Z_{k_1 \dots k_n} = z_{k_1 \dots k_n} \mu^{\frac{2p+3-2n-\sum k_i}{2}}, \tag{3.9}$$

with $z_{k_1 \dots k_n}$ being just numbers. The coefficients with even $\sum_{i=1}^n k_i$ (even sector) are half-integer powers of μ , and thus definitely belong to the singular part of the partition function. However, $Z_{k_1 \dots k_n}$ with odd $\sum_{i=1}^n k_i$ (odd sector) involve integer powers of μ . When also

$$\sum_{i=1}^n k_i \leq 2p + 3 - 2n, \tag{3.10}$$

the odd-sector coefficients are non-negative powers of μ , and thus belong to the regular part of the partition function. As such, they can be adjusted at will, and will be of no interest in our analysis⁸. Note that this inequality is always satisfied for $n = 1, 2$, but at $n \geq 3$

⁸ However, note that these regular terms are ‘special’; in that they fully agree with the scaling (the coefficients $z_{k_1 \dots k_n}$ in (3.9) are dimensionless). Whereas generic regular terms of the full microscopic partition function violate the scaling, and thus are definitely beyond control of continuous field theory, there is much reason to think that in quantum gravity the special regular terms can be attributed some universal meaning (see [13]). In this work, we do not address this interesting line of questioning.

negative powers of μ appear. Therefore, it is meaningful to compare the odd-sector correlation numbers with $\sum k_i > 2p + 3 - 2n$ with the results in $\mathcal{MG}_{2/2p+1}$. We will return to this point in sections 4 and 5.

3.2. One- and two-point correlation numbers

When one plugs (3.5) into (3.2), the polynomial takes the form

$$\mathcal{P}(u) = \mathcal{P}_0(u) + \sum_{k=1}^{p-1} \lambda_k \mathcal{P}_k(u) + \dots + \sum_{k_i=1}^{p-1} \frac{\lambda_{k_1} \dots \lambda_{k_n}}{n!} \mathcal{P}_{k_1 \dots k_n}(u) + \dots, \quad (3.11)$$

where $\mathcal{P}_0(u)$ and $\mathcal{P}_{k_1 \dots k_n}(u)$ are the polynomials whose coefficients involve non-negative powers of μ . By dimensional analysis,

$$\begin{aligned} \mathcal{P}_0(u) &= u^{p+1} + C'_0 \mu u^{p-1} + C''_0 \mu^2 u^{p-3} + \dots \\ \mathcal{P}_k(u) &= C_k u^{p-k-1} + C'_k \mu u^{p-k-3} + C''_k \mu^2 u^{p-k-5} + \dots \\ &\dots \end{aligned} \quad (3.12)$$

(here C_k are the same as in (3.5), and C'_k, C''_k, \dots , are dimensionless constants related to the higher-order coefficients in (3.5)), and in general $\mathcal{P}_{k_1 \dots k_n}(u)$ are polynomials of the degree,

$$p + 1 - 2n - \sum k_i, \quad (3.13)$$

of similar structure. Of course, only polynomials of non-negative degree appear, so that the sum in (3.11) is finite.

It is essential to note that all the polynomials are either even or odd:

$$\mathcal{P}_{k_1 \dots k_n}(-u) = (-1)^{p+1-\sum k_i} \mathcal{P}_{k_1 \dots k_n}(u), \quad (3.14)$$

because only integer powers of μ can appear in (3.12). We use this symmetry to rewrite equation (3.3) in a somewhat different form, more convenient for our analysis below. We split the integration domain into two pieces:

$$\int_0^{u_*} = \int_{u_0}^{u_*} + \int_0^{u_0}, \quad (3.15)$$

where u_0 stands for the root of $\mathcal{P}_0(u)$ associated with u_* , i.e. u_* at $\lambda_1, \dots, \lambda_{p-1} = 0$. Note that

$$u_0 = a_0 \mu^{\frac{1}{2}}, \quad (3.16)$$

with some constant a_0 which plays no significant role in our discussion below. The integrand in (3.3) involves the products

$$\mathcal{P}_{k_1 \dots k_m}(u) \mathcal{P}_{k_{m+1} \dots k_n}(u)$$

which are even or odd in u depending on whether $k_1 + \dots + k_n$ is even or odd. For the even terms we can extend the integration in the second piece in (3.15) to the domain $[-u_0, u_0]$:

$$\int_0^{u_0} \rightarrow \frac{1}{2} \int_{-u_0}^{u_0}. \quad (3.17)$$

On the other hand, it is not difficult to see that the contributions of the odd terms in that piece involve only non-negative integer powers of μ , and thus belong to the regular part of the partition function. Thus, up to regular terms

$$Z = \frac{1}{2} \int_{u_0}^{u_*} \mathcal{P}^2(u) du + \frac{1}{4} \int_{-u_0}^{u_0} \mathcal{P}^2(u) du, \quad (3.18)$$

and since the regular terms are of no interest to us here, in what follows we study the partition function (3.18).

The first order of business is to determine $\mathcal{P}_0(u)$ and $\mathcal{P}_k(u)$. Since $u_* = u_0 + O(\lambda_k)$, the first term in (3.18) does not contribute to the first three orders in the expansion (3.8), and one finds

$$Z_0 = \frac{1}{4} \int_{-u_0}^{u_0} \mathcal{P}_0^2(u) du, \tag{3.19}$$

$$Z_k = \frac{1}{2} \int_{-u_0}^{u_0} \mathcal{P}_0(u) \mathcal{P}_k(u) du, \tag{3.20}$$

$$Z_{k_1 k_2} = \frac{1}{2} \int_{-u_0}^{u_0} [\mathcal{P}_{k_1}(u) \mathcal{P}_{k_2}(u) + \mathcal{P}_0(u) \mathcal{P}_{k_1 k_2}(u)] du. \tag{3.21}$$

Agreement with (2.24) requires that all Z_k vanish. Then (3.20) suggests that all the polynomials $\mathcal{P}_k(u)$ must be orthogonal to $\mathcal{P}_0(u)$ on the interval $[-u_0, u_0]$, with the measure 1. Since the degrees of all $\mathcal{P}_{k_1 k_2}(u)$ are smaller than $p - 2$ (indeed, smaller than $p - 4$), the second term in (3.21) may be disregarded, and then agreement with the diagonal form of the two-point correlation numbers in (2.24) requires that $\mathcal{P}_k(u)$ themselves form an orthogonal set of polynomials on that interval. One concludes that $\mathcal{P}_k(u)$, up to normalization, are the Legendre polynomials:

$$\mathcal{P}_k(u) = C_k g_k u_0^{p-k-1} P_{p-k-1}(u/u_0). \tag{3.22}$$

Here C_k are the same as in (3.12), and

$$g_k = \frac{(p-k-1)!}{(2p-2k-3)!}. \tag{3.23}$$

We review some trivia about the Legendre polynomials in appendix A (see, e.g., [14] for systematic display). Furthermore, since $\mathcal{P}_0(u)$ is $(p+1)$ -degree polynomial, with no u^p term, and vanishing at u_0 , one finds

$$\mathcal{P}_0(u) = g u_0^{p+1} [P_{p+1}(u/u_0) - P_{p-1}(u/u_0)], \tag{3.24}$$

where the normalization constant,

$$g = \frac{(p+1)!}{(2p+1)!}, \tag{3.25}$$

is inserted to guarantee that the highest-order term in (3.24) is just u^{p+1} , as in (3.2). Incidentally, the coefficient in front of u^{p-1} in (3.24) determines the relation between t_0 in (3.2) and the cosmological constant

$$t_0 = -\frac{1}{2} \frac{p(p+1)}{2p-1} u_0^2. \tag{3.26}$$

Equation (3.19) yields

$$Z_0 = u_0^{2p+3} \frac{g^2(2p+1)}{(2p+3)(2p-1)}, \tag{3.27}$$

and then from (3.21)

$$\frac{Z_{kk'}}{Z_0} = \delta_{k,k'} \frac{\mathcal{N}_p \mu^{-k-2}}{2p-2k-1} \text{Leg}_M^2(k), \tag{3.28}$$

where \mathcal{N}_p is the same as in (2.17), and

$$\text{Leg}_M(k) = \frac{g_k C_k}{(2p+1) g a_0^{k+2}}. \tag{3.29}$$

Equation (3.28) reproduces the structure of the two-point numbers in (2.24), and the identity of (3.29) with the Leg factors (2.18) fixes the normalization constants C_k . Equations (3.22), (3.24) are equivalent to equations (4.24) and (4.28) in [8]⁹.

4. Three- and four-point correlation numbers

Before proceeding to the higher-order correlation numbers, it is useful to get rid of annoying factors in (3.22) and (3.24). We trade λ_k for the dimensionless couplings

$$s_k = \frac{g_k u_0^{-k-2}}{g(2p+1)} \lambda_k, \tag{4.1}$$

and write the polynomial (3.2) as

$$\mathcal{P}(u) = g(2p+1)u_0^{p+1} Q(u/u_0), \tag{4.2}$$

where $Q(x)$ is the polynomial of degree $p+1$, as in (3.11), we will think of it as the power series in s_k :

$$Q(x) = Q_0(x) + \sum_{k=1}^{p-1} s_k Q_k(x) + \sum_{k_1 k_2} \frac{s_{k_1} s_{k_2}}{2} Q_{k_1 k_2}(x) + \dots \tag{4.3}$$

Equations (3.24) and (3.22) then tell us that

$$Q_0(x) = \frac{P_{p+1}(x) - P_{p-1}(x)}{2p+1} = \int P_p(x) dx \tag{4.4}$$

and

$$Q_k(x) = P_{p-k-1}(x). \tag{4.5}$$

It is convenient also to trade the partition function (3.3) for the dimensionless quantity $\mathcal{Z} = \mathcal{Z}(s_1, \dots, s_{p-1})$,

$$Z = g^2(2p+1)^2 u_0^{2p+3} \mathcal{Z}, \tag{4.6}$$

given by

$$\mathcal{Z} = \frac{1}{2} \int_1^{x_*} Q^2(x) dx + \frac{1}{4} \int_{-1}^1 Q^2(x) dx, \tag{4.7}$$

where $x_* = x_*(s_1, \dots, s_{p-1})$ is the largest real root of $Q(x)$. Note that $x_*(0, 0, \dots, 0) = 1$, and

$$Q'_0(1) = 1, \quad Q_k(1) = 1. \tag{4.8}$$

Up to the Leg factors, the correlation numbers are the ratios

$$\mu^{\sum \delta_{k_i}} \mathcal{Z}_{k_1 k_2 \dots k_n} / \mathcal{Z}_0 \tag{4.9}$$

of the coefficients of the expansion

$$\mathcal{Z} = \mathcal{Z}_0 + \sum_{k=1}^{p-1} s_k \mathcal{Z}_k + \dots + \sum_{k_1, \dots, k_n=1}^{p-1} \frac{s_{k_1} \dots s_{k_n}}{n!} \mathcal{Z}_{k_1 \dots k_n} + \dots \tag{4.10}$$

Note that with (4.4), equation (4.7) yields

$$\mathcal{Z}_0 = \frac{1}{4} \int_{-1}^1 Q_0^2(x) dx = \mathcal{N}_p^{-1}, \tag{4.11}$$

where \mathcal{N}_p is precisely the factor (2.17).

⁹ The relation

$$\int_1^\infty P_n(x) e^{-lx} = \sqrt{\frac{2}{\pi l}} K_{n+1/2}(l)$$

between the Legendre polynomials and the Macdonald functions of half-integer order makes the identity evident.

4.1. Three-point numbers

Evaluation of the coefficients $\mathcal{Z}_{k_1 k_2 k_3}$ is straightforward; we defer the calculations to appendix B. The result is

$$\mathcal{Z}_{k_1 k_2 k_3} = -1 + \frac{1}{2} \int_{-1}^1 [\mathcal{Q}_{k_1 k_2}(x) \mathcal{Q}_{k_3}(x) + \mathcal{Q}_{k_1 k_3}(x) \mathcal{Q}_{k_2}(x) + \mathcal{Q}_{k_2 k_3}(x) \mathcal{Q}_{k_1}(x)] dx. \quad (4.12)$$

The first term -1 reproduces (2.16), except for the fusion rule factor $N_{k_1 k_2 k_3}$, equations (2.20), (2.21). The role of the second term is to fix that discrepancy. Note that in this case we need not worry about the odd sector. Recall from (3.10) that when $k_1 + k_2 + k_3$ is odd and $< 2p - 1$, i.e. when the odd-sector fusion rules (2.20) are violated, the terms with $\mathcal{Z}_{k_1 k_2 k_3}$ belong to the regular part of the partition function. Therefore, we only need to look at the case when $k_1 + k_2 + k_3$ is even, where (2.21) demands that the second term in (4.12) turns to 1 at all configurations of k_1, k_2, k_3 such that $k_1 + k_2 > k_3$ (as in (2.21) we assume that $k_1, k_2 \leq k_3$), to cancel the first term in (4.12). To reproduce the fusion rule factor $N_{k_1 k_2 k_3}$ we need to have

$$\frac{1}{2} \int_{-1}^1 \mathcal{Q}_{k_3}(x) \mathcal{Q}_{k_1 k_2}(x) dx = \begin{cases} 1 & \text{if } k_1 + k_2 < k_3 \\ 0 & \text{if } k_1 + k_2 \geq k_3. \end{cases} \quad (4.13)$$

Since $\mathcal{Q}_k(x) = P_{p-k-1}(x)$, this is achieved by taking (see equation (A.8) in appendix A)

$$\mathcal{Q}_{k_1 k_2}(x) = P'_{p-k_1-k_2-2}(x), \quad (4.14)$$

where the prime denotes the derivative of the Legendre polynomial with respect to x . Note that now for some admissible values of k_1, k_2 the index in (4.14) can take negative values; throughout this paper we adopt the convention that $P_n(x)$ with negative n are identically zero. With (4.14), we have

$$\mathcal{Z}_{k_1 k_2 k_3} / \mathcal{Z}_0 = -N_{k_1 k_2 k_3} \mathcal{N}_p, \quad (4.15)$$

in exact agreement with (2.16).

4.2. Four-point numbers

Direct calculation (appendix B) yields

$$\mathcal{Z}_{k_1 k_2 k_3 k_4} = \mathcal{Z}_{k_1 k_2 k_3 k_4}^{(0)} + \mathcal{Z}_{k_1 k_2 k_3 k_4}^{(1)}, \quad (4.16)$$

where

$$\mathcal{Z}_{k_1 k_2 k_3 k_4}^{(0)} = -F(-2) + \sum_{i=1}^4 F(k_i - 1) - F(k_{(12|34)}) - F(k_{(13|24)}) - F(k_{(14|23)}), \quad (4.17)$$

and

$$\mathcal{Z}_{k_1 k_2 k_3 k_4}^{(1)} = \frac{1}{2} \int_{-1}^1 [\mathcal{Q}_{k_1 k_2 k_3} \mathcal{Q}_{k_4} + \mathcal{Q}_{k_4 k_1 k_2} \mathcal{Q}_{k_3} + \mathcal{Q}_{k_3 k_4 k_1} \mathcal{Q}_{k_2} + \mathcal{Q}_{k_2 k_3 k_4} \mathcal{Q}_{k_1}] dx. \quad (4.18)$$

In (4.17)

$$F(k) = P'_{p-k-2}(1) = \frac{1}{2}(p-k-1)(p-k-2)\Theta_{p-2,k}, \quad (4.19)$$

with $\Theta_{k,k'}$ being the step function,

$$\Theta_{k,k'} = \begin{cases} 1 & \text{for } k \geq k' \\ 0 & \text{for } k < k', \end{cases} \quad (4.20)$$

and we use the notation

$$k_{(ij|lm)} = \min(k_i + k_j, k_l + k_m). \quad (4.21)$$

Like in (4.12), the role of the term (4.18) is to enforce the fusion rules, and the polynomials $Q_{k_1 k_2 k_3}(x)$ are to be determined from this requirement.

The analysis is simpler in the even sector, so let us start with this case. Assume again that the numbers k_1, k_2, k_3, k_4 are arranged as in (2.27), so that in (4.17) we always have

$$k_{(12|34)} = k_1 + k_2, \quad k_{(13|24)} = k_1 + k_3. \tag{4.22}$$

Recalling (4.5), and counting the degrees of the polynomials $Q_{k_1 k_2 k_3}(x)$, one observes that (4.18) vanishes when the even-sector fusion rules (2.7) are satisfied. On the other hand, when the fusion rules are violated, the inequality (2.28) is violated as well, and we have

$$k_{(14|23)} = k_2 + k_3 < p - 1, \tag{4.23}$$

where the last inequality follows from $k_4 \leq p - 1$. With (4.22) and (4.23), expression (4.17) evaluates to

$$-\frac{1}{2}(k_4 - k_1 - k_2 - k_3 - 2)(2p - 3 - k_1 - k_2 - k_3 - k_4). \tag{4.24}$$

Thus, for (4.16) to satisfy the even-sector fusion rules (2.7) the integral,

$$\int_{-1}^1 Q_{k_1 k_2 k_3}(x) Q_{k_4}(x) dx, \tag{4.25}$$

has to return (4.24) with the opposite sign. This uniquely determines the polynomials $Q_{k_1 k_2 k_3}$:

$$Q_{k_1 k_2 k_3}(x) = P''_{p - \sum k_i - 3}(x) \tag{4.26}$$

(see equation (A.9) in appendix A).

Now, in the odd sector, as was explained in section 2.2, the coefficients (4.16) with $\sum k_i \leq 2p - 5$ correspond to regular terms in the partition function Z , and thus can be disregarded. For that reason, in particular, we can pay no attention to the second term, equation (4.18). The only singular terms potentially violating the odd-sector fusion rules (2.7) are those with

$$\sum k_i = 2p - 3. \tag{4.27}$$

But it is easy to check that these terms actually vanish. Indeed, if again k_i are arranged as in (2.27), and $k_1 + k_4 \leq k_2 + k_3$ (in which case it follows from (4.27) that also $k_1 + k_4 \leq p - 1$), (4.17) is given by (4.24) which vanishes at (4.27). If instead $k_2 + k_3 \leq k_1 + k_4$ (and hence $k_2 + k_3 \leq p - 1$), expression (4.17) reduces to

$$(k_1 + 1)(2p - 3 - k_1 - k_2 - k_3 - k_4), \tag{4.28}$$

again vanishing at (4.27).

Once the fusion rules are taken care of, we can compare (4.16) with the $\mathcal{MG}_{2/2p+1}$ four-point correlation number. It is not difficult to check that when (2.28) is satisfied, (4.17) reproduces (2.26) perfectly.

However, when instead $k_2 + k_3 < k_1 + k_4$, (4.17) no longer agrees with (2.26). This is expected, since the Liouville gravity calculations in [6] were made under a certain assumption which is violated in this case. The matrix model result (4.16), which is valid at all configurations of k_i , may provide a hint on how to modify the Liouville gravity analysis when the number of conformal blocks in the $\mathcal{M}_{2/2p+1}$ four-point function is smaller than k_1 .

5. Multi-point correlation numbers

In principle, one can extend the analysis of the previous section to n -point correlation numbers with $n > 4$. Although at the moment no specific results for these multi-point numbers in the

minimal gravity are available, one thing is known. Since the correlation functions (2.4) of $\mathcal{M}_{2/2p+1}$ vanish when the fusion rules (2.7) are violated, the correlation numbers (2.14) then vanish as well. This requirement for the n -point numbers imposes strong conditions on the form of the polynomials $Q_{k_1 \dots k_{n-1}}(x)$, which fix them uniquely, and thus one must be able to determine the complete form of the polynomial $Q(x)$, equation (4.3), step by step in n .

In fact, the problem seems over-determined. Indeed, suppose we have already constructed the expansion (4.3) up to the order $n-1$, and thus $Q_0, Q_k, \dots, Q_{k_1 \dots k_{n-2}}$ are already determined. Then $Q_{k_1 \dots k_{n-1}}$ enters the expression for the n th order coefficient $Z_{k_1 \dots k_n}$ only through the ‘counterterm’

$$\frac{1}{2} \int_{-1}^1 Q_{k_1 \dots k_{n-1}}(x) Q_{k_n}(x) dx, \tag{5.1}$$

and n similar terms which differ from this by cyclic permutations of k_1, \dots, k_n . The polynomials $Q_{k_1 \dots k_{n-1}}$ must be chosen in such a way that these terms cancel all other contributions to $Z_{k_1 \dots k_n}$ when the even-sector fusion rules (2.7) are violated, i.e. when $k_1 + \dots + k_{n-1} > k_n$. But since the degree of the polynomial $Q_{k_1 \dots k_{n-1}}(x)$ is $p+3-2n-(k_1 + \dots + k_{n-1})$ (see equation (3.13)), the integral (5.1) actually vanishes at $k_1 + \dots + k_{n-1} > k_n + 4 - 2n$. For $n \geq 4$ a window,

$$k_n > \sum_{i=1}^{n-1} k_i > k_n + 4 - 2n, \tag{5.2}$$

opens in the configurations of k_i violating the even-sector fusion rules, where the term (5.1) seems to be incapable of doing its job of fixing the fusion rules. A similar problem exists in the odd sector. For $n \geq 4$ there is a window,

$$2p-1 > \sum_{i=1}^n k_i > 2p+3-2n, \tag{5.3}$$

in the configurations of k_i , where the odd-sector fusion rules (2.7) are violated, but corresponding coefficients $Z_{k_1 \dots k_n}$ are singular (involve negative integer powers of μ ; see section 3.1).

We have seen in section 4.2 that at $n=4$ the problem takes care of itself, in both even and odd sectors. The first factor in equation (4.24) forces the coefficients $Z_{k_1 k_2 k_3 k_4}$ to vanish within the window (5.2), and the last factor in (4.24) and (4.28) makes sure that they vanish within the window (5.3) as well.

The same phenomenon should persist for higher $n > 4$, for otherwise we would face incurable disagreement between the correlation numbers in $\mathcal{MG}_{2/2p+1}$ and the matrix model. We have calculated the five-point correlation numbers $C_{k_1 k_2 k_3 k_4 k_5}$, and indeed they automatically vanish within both even- and odd-sector windows, equations (5.2) and (5.3). We do not present this calculation here, for the result is somewhat cumbersome, and anyway at the moment there is nothing on the minimal gravity side to compare it with. But as a byproduct of this calculation we have determined the four-index polynomials $Q_{k_1 \dots k_4}$,

$$Q_{k_1 k_2 k_3 k_4}(x) = P_{p-\sum k-4}'''(x), \tag{5.4}$$

where $\sum k = k_1 + k_2 + k_3 + k_4$.

6. Discussion

Identification of $\mathcal{MG}_{2/2p+1}$ as the world-sheet theory of the p -critical one-matrix model suggests that, by choosing suitable resonance terms in the relation between the couplings

t_k in (3.2) and λ_k , the matrix model correlation numbers can be put in agreement with the fusion rules of $\mathcal{MG}_{2/2p+1}$. Technically, this is done by constructing the polynomial $Q(x)$, equation (4.3), order by order in s_k . In sections 4 and 5, we have executed this program up to the fourth order.

For higher n direct calculations become rather involved. But a quick glance at (4.5), (4.14), (4.26) and (5.4) immediately suggests the general form,

$$Q_{k_1 \dots k_n}(x) = \left(\frac{d}{dx}\right)^{n-1} P_{p-\sum k-n}(x), \tag{6.1}$$

where again $\sum k = k_1 + \dots + k_n$. Then, using (A.6), the full polynomial $Q(x)$ can be neatly written as the integral

$$Q(x) = - \oint_0 \frac{(1 - 2xz + z^2 - \sum_{k=1}^{p-1} 2s_k z^{k+2})^{1/2} dz}{z^{p+2}} \frac{dz}{2\pi i}, \tag{6.2}$$

where the integration is over a small circle around 0. So far, we did not find a proof that all correlation numbers computed from (4.7) with $Q(x)$ given by (6.2) obey all fusion rules (2.7). Thus, (6.2) remains a conjecture.

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Appendix A. Legendre polynomials

The Legendre polynomials $P_n(x)$ are n th order polynomials which form an orthogonal system on the interval $[-1, 1]$ with the weight 1,

$$\int_{-1}^1 P_n(x) P_{n'}(x) dx = \frac{2\delta_{n,n'}}{2n+1}. \tag{A.1}$$

The standard normalization is such that

$$P_n(1) = 1. \tag{A.2}$$

Explicitly,

$$P_n(x) = \frac{2^{-n}}{n!} \frac{d^n}{dx^n} [x^2 - 1]^n = 2^{-n} \sum_{l=0}^{\lfloor n/2 \rfloor} (-1)^l \frac{(2n-2l)!}{l!(n-l)!(n-2l)!} x^{n-2l}. \tag{A.3}$$

Another closed expression is given in terms of the hypergeometric series,

$$P_n(x) = {}_2F_1 \left(-n, n+1, 1; \frac{1-x}{2} \right), \tag{A.4}$$

i.e.

$$P'_n(1) = \frac{n(n+1)}{2}, \quad P''_n(1) = \frac{(n-1)n(n+1)(n+2)}{8}, \quad \text{etc.} \tag{A.5}$$

Yet another closed form is in terms of the contour integral

$$P_n(x) = \oint_0 \frac{(1 - 2xz + z^2)^{-1/2} dz}{z^{n+1}} \frac{dz}{2\pi i}. \tag{A.6}$$

The following relations are useful in our analysis:

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n + 1)P_n(x), \tag{A.7}$$

they are valid for all $n = 0, 1, 2, 3, \dots$, if one assumes that $P_{-1}(x) = 0$.

Besides the orthogonality condition (A.1), we need integrals involving the derivatives of the Legendre polynomials:

$$\frac{1}{2} \int_{-1}^1 P'_n(x) P_m(x) dx = E_{n+m-1} \Theta_{n,m+1}, \tag{A.8}$$

$$\frac{1}{2} \int_{-1}^1 P''_n(x) P_m(x) dx = E_{n+m} \Theta_{n,m+2} \frac{(n+m+1)(n-m)}{2}, \tag{A.9}$$

and in general

$$\begin{aligned} \frac{1}{2} \int_{-1}^1 P_n^{(l)}(x) P_m(x) dx &= E_{n+m-l} \Theta_{n,m+l} \\ &\times \frac{2^{-l+1}}{(l-1)!} \prod_{s=0}^{l-2} (n+m+l-1-2s)(n-m+l-2-2s), \end{aligned} \tag{A.10}$$

where $P^{(l)}(x)$ stands for the l th derivative. Here $\Theta_{n,m} = P_{n-m}(1)$ is the step function (4.20), and

$$E_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{A.11}$$

Integrating (A.9) by parts, we have

$$\frac{1}{2} \int_{-1}^1 P'_n(x) P'_m(x) dx = E_{n+m} \left[\Theta_{m,n} \frac{n(n+1)}{2} + \Theta_{n,m} \frac{m(m+1)}{2} \right]. \tag{A.12}$$

Appendix B. Evaluation of $\mathcal{Z}_{k_1 \dots k_n}$

Generally, the coefficient $\mathcal{Z}_{k_1 \dots k_n}$ is computed by taking the n th order derivative of (4.7) with respect to the parameters s_{k_1}, \dots, s_{k_n} . The result naturally splits into two pieces,

$$\mathcal{Z}_{k_1 \dots k_n} = \mathcal{Z}_{k_1 \dots k_n}^{(*)} + \mathcal{Z}_{k_1 \dots k_n}^{(\text{int})}, \tag{B.1}$$

corresponding to two terms (4.7). Since $x_* = 1 + O(s_k)$, the term $\mathcal{Z}_{k_1 \dots k_n}^{(*)}$ appears only starting from $n = 3$, and in general it has the ‘local’ form, i.e., it is built from the polynomials $Q_0(x), Q_k(x), \dots$ and their x -derivatives, taken at $x = 1$. On the other hand, the term $\mathcal{Z}_{k_1 \dots k_n}^{(\text{int})}$ represents explicit dependence of the integrand in the second term in (4.7) on the parameters s_k ; starting from $n = 3$, it exists entirely due to the presence of nonlinear terms in the transformation (3.5). Generally, it has the form

$$\mathcal{Z}_{k_1 \dots k_n}^{(\text{int})} = \sum_{r=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2} \int_{-1}^1 [Q_{l_1 \dots l_r}(x) Q_{l_{r+1} \dots l_n}(x)] dx, \tag{B.2}$$

where (l_1, \dots, l_n) are permutations of the numbers (k_1, \dots, k_n) , and the symbol $[\dots]$ signifies the sum of all *distinct* terms generated by the permutations.

The calculation of the ‘local’ term $\mathcal{Z}_{k_1 \dots k_n}^{(*)}$ is more tedious since it involves taking derivatives of x_* , but it can be streamlined by reducing it to the residue calculus. Write

$$\int_1^{x_*} Q^2(x) dx = \oint_{1, x_*} Q^2(x) \log \frac{Q(x)}{Q_0(x)} \frac{dx}{2\pi i}, \tag{B.3}$$

where the integration contour encircles the points 1 and x_* , but leaves outside all other roots of $Q(x)$ and $Q_0(x)$. For the sake of the expansion in s_k , we may write $Q(x)$ as

$$Q(x) = Q_0(x) + \tilde{Q}(x), \tag{B.4}$$

where $\tilde{Q}(x)$ represents all the s_k -dependent terms in expansion (4.3), so that (B.3) takes the form

$$\frac{1}{2} \oint_{1, x_*} Q_0^2(x) R(\tilde{Q}(x)/Q_0(x)) \frac{dx}{2\pi i} \tag{B.5}$$

with $R(t) = (1+t)^2 \log(1+t)$. When $R(t)$ is expanded as the power series

$$R(t) = t + \frac{3}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{30}t^5 + \dots, \tag{B.6}$$

the l th term produces l th order pole at $x = 1$ in the integrand in (B.5). Then the local terms are computed by evaluating the residues.

B.1. Results

For $n = 3$, one finds

$$\mathcal{Z}_{k_1 k_2 k_3} = -\frac{Q_{k_1}(1)Q_{k_2}(1)Q_{k_3}(1)}{Q'_0(1)} + \frac{1}{2} \int_{-1}^1 [Q_{l_1 l_2}(x)Q_{l_3}(x)] dx, \tag{B.7}$$

which reduces to (4.12) in view of (4.8).

For $n = 4$, we have

$$\mathcal{Z}_{k_1 k_2 k_3 k_4}^{(*)} = -\frac{[Q_{l_1} Q_{l_2} Q_{l_3} Q_{l_4}] Q''}{(Q'_0)^3} + \frac{[Q'_{l_1} Q_{l_2} Q_{l_3} Q_{l_4}]}{(Q'_0)^2} - \frac{[Q_{l_1 l_2} Q_{l_3} Q_{l_4}]}{Q'_0}, \tag{B.8}$$

where all the Q 's are taken at $x = 1$, and the brackets [...] have the same meaning as in (B.2); for instance, the last term is in fact the sum of six terms.

The integral term in this case has two parts:

$$\mathcal{Z}_{k_1 k_2 k_3 k_4}^{(\text{int})} = \frac{1}{2} \int_{-1}^1 [Q_{l_1 l_2}(x)Q_{l_3 l_4}(x)] + \frac{1}{2} \int [Q_{l_1 l_2 l_3}(x)Q_{l_4}(x)] dx. \tag{B.9}$$

The first term here involves only the polynomials $Q_{kk'}(x)$ given by (4.14); it is evaluated using (A.12) and when combined with (B.8), yields (4.17). The second part in (B.9) provides (4.18).

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