

YANG–MILLS EQUATIONS AS INVERSE SCATTERING PROBLEM

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Received 11 May 1977

Non-linear self-duality equations are shown to be conditions of compatibility of two linear equations. All the N -instanton fields are constructed explicitly.

Introduction. Here we propose a method of solving self-duality equations [1]

$$F_{\mu\nu} = \pm F_{\mu\nu}^*, \quad (1)$$

for gauge fields in 4-dimensional Euclidean space. Classical trajectories described by these fields (instantons) correspond to the tunnel transitions between vacua with different topologies as it has been noticed by Gribov in the fall of 1975 (unpublished) and discussed thoroughly in subsequent papers [2–4]. Instantons are conjectured to be responsible for the confinement of quarks [5, 6] and for the resolution of the U(1) problem [2].

The self-dual fields (1) fall into classes defined by a topological charge

$$N = \frac{1}{8\pi^2} \int \text{Sp} F_{\mu\nu} F_{\mu\nu}^*. \quad (2)$$

The one-instanton solutions found in ref. [1] are 8-parametric fields, the parameters being four positions in space, scale and three Euler angles of global isotopic rotations. N -instanton fields should be $8N$ -parametric fields. This was proved in the spring of 1976 by Shwartz (unpublished) using the Atiyah–Singer theorem on indexes and was confirmed afterwards by direct counting the number of zero-energy excitations [7]. Some multi-instanton solutions have been found previously [8], the most general of these being the $5N$ -parametric solution of 't Hooft with fixed relative orientation of isospins.

The method developed allows finding all self-dual

fields ^{†1}. Our basic observation is that the *non-linear* eqs. (1) are just conditions of compatibility for some set of *linear* equations. For example, the equations

$$D_\mu \psi = (\partial_\mu + A_\mu) \psi = 0, \quad (3)$$

are compatible, provided that the operators D_μ commute:

$$[D_\mu, D_\nu] = F_{\mu\nu} = 0. \quad (4)$$

Once eqs. (3) are solved, one could explicitly write down all the fields with $F_{\mu\nu} = 0$: $\partial_\mu \psi + A_\mu \psi = 0$ implies that

$$A_\mu = \psi \partial_\mu \psi^{-1}, \quad (5)$$

i.e., we arrive at purely longitudinal fields, as has been expected.

Self-duality equation as the compatibility condition.

Let us start with Dirac equations for massless two-component fermions:

$$(D_4 + i\mathbf{D} \cdot \boldsymbol{\sigma})\mathbf{u} = 0, \quad (6)$$

$$(D_4 - i\mathbf{D} \cdot \boldsymbol{\sigma})\mathbf{v} = 0. \quad (7)$$

Let us search for solutions of eq. (7) of the following form

$$\mathbf{v} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \psi(\lambda, \mathbf{x}), \quad (8)$$

^{†1} The method to be presented is closely related to the solution of the inverse scattering problem for many-dimensional spaces developed in ref. [9].

for all values of λ . ($\psi(\lambda, x)$ is a 2×2 matrix in the isotopic space.) Then eq. (7) takes the form:

$$L_1 \psi = [\lambda(D_2 - iD_1) + (D_4 + iD_3)] \psi = 0, \quad (9)$$

$$L_2 \psi = [\lambda(D_4 - iD_3) - (D_2 + iD_1)] \psi = 0. \quad (10)$$

These equations are compatible, provided that

$$[L_1, L_2] = 0. \quad (11)$$

This commutator vanishes irrespective of λ only provided that $F_{\mu\nu} = -F_{\mu\nu}^*$. Starting with eq. (6) one arrives in the same way at the equation $F_{\mu\nu} = F_{\mu\nu}^*$. Thus, non-linear self-duality equations are substituted by the linear eqs. (9) and (10).

Convenient and proper variables are complex coordinates and complex fields, introduced as follows:

$$z_1 = \frac{1}{2}(x_2 + ix_1), \quad z_2 = \frac{1}{2}(x_4 + ix_3), \quad (12)$$

$$B_1 = A_2 - iA_1, \quad B_2 = A_4 - iA_3, \quad (13)$$

$$\partial_i = \partial/\partial z_i, \quad \nabla_i = \partial_i + B_i, \quad \bar{\nabla}_i = \bar{\partial}_i - B_i^+. \quad (14)$$

In these variables the equation $F_{\mu\nu} = -F_{\mu\nu}^*$ takes the form:

$$\partial_1 B_2 - \partial_2 B_1 + [B_1, B_2] = 0, \quad (15)$$

$$\partial_i B_i^+ + \bar{\partial}_i B_i + [B_i, B_i^+] = 0. \quad (16)$$

In the linearized form one has

$$(\lambda \nabla_1 + \bar{\nabla}_2) \psi(\lambda, x) = 0, \quad (17)$$

$$(\lambda \nabla_2 - \bar{\nabla}_1) \psi(\lambda, x) = 0. \quad (18)$$

One-instanton solution. Once the ψ -function is found, the fields B_i are given by

$$\lambda B_1 - B_2^+ = \psi(\lambda, x) (\lambda \partial_1 + \bar{\partial}_2) \psi^{-1}(\lambda, x), \quad (19)$$

$$\lambda B_2 + B_1^+ = \psi(\lambda, x) (\lambda \partial_2 - \bar{\partial}_1) \psi^{-1}(\lambda, x). \quad (20)$$

These equations imply that $\tilde{\psi}(\lambda) = (\psi^+)^{-1}(-1/\bar{\lambda})$ satisfies eqs. (17) and (18) if $\psi(\lambda)$ does. Thus, the restriction

$$\psi^+(-1/\bar{\lambda}) = \psi^{-1}(\lambda), \quad (21)$$

will be imposed on the ψ -functions.

Eqs. (19) and (20) yield self-dual fields, provided ψ -functions are found which satisfy eq. (21) and such that the r.h.s. in eqs. (10), (20) have at most a linear

dependence on λ ^{#2}. To proceed further one should specify the analyticity properties of $\psi(\lambda, x)$ as functions of λ . We suppose that the only singularities in the λ -plane are poles whose locations do not depend on x .

Let us start with few simplest cases. Suppose $\psi(\lambda, x)$ be independent of λ . Eq. (21) implies that in this case $\psi^+(x) = \psi^{-1}(x)$, and from eqs. (19), (20) it follows that $B_i = \psi \partial_i \psi^{-1}$. Thus, we obtain all longitudinal fields with stress $F_{\mu\nu} = 0$.

The next step is a two-pole matrix $\psi(x, \lambda)$ ^{#3}; the poles taken to be located at $\lambda = \lambda_0$ and $\lambda = -1/\bar{\lambda}_0$:

$$\psi(x, \lambda) = v \left(uI + \frac{\lambda - \lambda_0}{1 + \lambda \bar{\lambda}_0} fA + \frac{1 + \lambda \bar{\lambda}_0}{\lambda - \lambda_0} \bar{f}A^+ \right), \quad (22)$$

$$\psi^{-1}(x, \lambda) = \left(uI - \frac{\lambda - \lambda_0}{1 + \lambda \bar{\lambda}_0} fA - \frac{1 + \lambda \bar{\lambda}_0}{\lambda - \lambda_0} \bar{f}A^+ \right). \quad (23)$$

Here I, v and A are 2×2 matrices; I is a diagonal unit matrix, $v^+ = v^{-1}$; u and f are some functions, $\bar{u} = u$. From the equation $\psi \psi^{-1} = 1$ it follows that $A^2 = 0$, and that $u^2 = 1 + f\bar{f}\{A, A^+\}$. We impose the normalization $\{A, A^+\} = 1$. After some exercises one could verify that the fields A_μ are nonsingular only provided $\lambda_0 = 0$; thus, we put $\lambda_0 = 0$.

The arbitrary unitary matrix $v(x)$ in expansions (22), (23) corresponds to gauge transformations; to fix a gauge we put $v(x) = 1$. It is convenient to parametrize matrix A as follows:

$$A = \frac{1}{|a|^2 + |b|^2} \begin{bmatrix} ab & a^2 \\ -b^2 & -ab \end{bmatrix}, \quad (24)$$

where $a(x)$ and $b(x)$ are functions to be specified below.

Let us substitute eqs. (22)–(24) into eqs. (19), (20). The residues of the singular λ^{-2} , λ^{-1} , λ^2 and λ^3 terms should vanish identically. The λ^2 and/or λ^3 terms lead to the equations:

$$A \partial_i A = 0, \quad (25)$$

which imply that $a(x)$ and $b(x)$ are analytic functions of two complex variables \bar{z}_i : $a(x) = a(\bar{z}_1, \bar{z}_2)$, $b(x) = b(\bar{z}_1, \bar{z}_2)$. The vanishing of the λ^{-1} and/or λ^2 terms leads to some differential equation, this equation integrates to the algebraic equation:

^{#2} We stress that the fields B_i do not depend on the auxiliary parameter.

^{#3} No single-pole solutions exist due to eq. (21).

$$\frac{\sqrt{1+|f|^2}}{f} = \frac{z_1(a\bar{\partial}_2 b - b\bar{\partial}_2 a) - z_2(a\bar{\partial}_1 b - b\bar{\partial}_1 a) + c(\bar{z}_1, \bar{z}_2)}{|a|^2 + |b|^2} \quad (26)$$

where $c(\bar{z}_1, \bar{z}_2)$ is an integration constant. Thus, we are left with three analytic two-variable functions $a(\bar{z}_i)$, $b(\bar{z}_i)$ and $c(\bar{z}_i)$ to be determined.

A careful study of eq. (26) demonstrates that it has solutions only provided that its r.h.s. is bounded in modulus from below by unity. This condition turns out to be extremely restrictive: only linear functions $a(\bar{z}_i)$ and $b(\bar{z}_i)$ are allowed, and the function $c(\bar{z}_i)$ should be some constant. One of the possible solutions is given by:

$$a = \bar{z}_1, \quad b = \bar{z}_2, \quad c = 2. \quad (27)$$

The eight parameters, the most general one-instanton solution depends on, are: scale c , shifts $a \rightarrow \bar{z}_1 - \bar{z}_1^0$, $b \rightarrow \bar{z}_2 - \bar{z}_2^0$ and three isotopic rotations $a = V_{11}\bar{z}_1 + V_{12}\bar{z}_2$, $b = V_{21}\bar{z}_1 + V_{22}\bar{z}_2$ given by the unitary matrix V , $\det V = 1$. The fields B_i given by solution (27):

$$B_1 = \frac{1}{2(1+z_i\bar{z}_i)} \begin{bmatrix} \bar{z}_1 & 0 \\ -2\bar{z}_2 & -\bar{z}_1 \end{bmatrix}, \quad (28)$$

$$B_2 = -\frac{1}{2(1+z_i\bar{z}_i)} \begin{bmatrix} \bar{z}_2 & 2\bar{z}_1 \\ 0 & -\bar{z}_2 \end{bmatrix},$$

are just the one-instanton solution of ref. [1].

Reproduction of instantons. Suppose N -instanton fields B_i^0 have been found:

$$\lambda B_1^0 - B_2^{0+} = \psi_N(\lambda\partial_1 + \bar{\partial}_2)\psi_N^{-1}, \quad (29)$$

$$\lambda B_2^0 + B_1^{0+} = \psi_N(\lambda\partial_2 - \bar{\partial}_1)\psi_N^{-1}. \quad (30)$$

We look for the $(N+1)$ -instanton ψ -matrix of the form:

$$\psi_{N+1} = \psi_1 \psi_N. \quad (31)$$

The resultant equations for the $(N+1)$ instanton fields B_i are as follows:

$$\lambda B_1 - B_2^+ = \psi_1(\lambda\nabla_1^0 + \bar{\nabla}_2^0)\psi_1^{-1}, \quad (32)$$

$$\lambda B_2 + B_1^+ = \psi_1(\lambda\nabla_2^0 - \bar{\nabla}_1^0)\psi_1^{-1}, \quad (33)$$

where $\nabla_i^0 = \partial_i + B_i^0$.

We use for ψ_1 an expansion reminiscent of eq. (22):

$$\psi_1 = u + \lambda f \tilde{A} + (1/\lambda) \tilde{f} \tilde{A}^+, \quad (34)$$

$$\psi_1^{-1} = u - \lambda f \tilde{A} - (1/\lambda) \tilde{f} \tilde{A}^+. \quad (35)$$

Here $\tilde{A}^2 = 0$, and $u^2 = 1 + |f|^2 \{A, A^+\}$. Eq. (25) is substituted in this case by the equation

$$\tilde{A} \nabla_i^0 \tilde{A} (\partial_1 + B_i^0) \tilde{A} = 0. \quad (36)$$

In order to solve it one should bear in mind that eq. (9) implies that

$$B_i = g \partial_i g^{-1}, \quad (37)$$

where g is the matrix with $\det g = 1$. After the transformation $\tilde{A} = g A g^{-1}$ we get for matrix A eq. (25), their solution being already known:

$$A = \frac{1}{|a|^2 + |b|^2} \begin{bmatrix} ab & a^2 \\ -b^2 & -ab \end{bmatrix}, \quad a = a(\bar{z}_i), \quad b = b(\bar{z}_i). \quad (38)$$

Eq. (26) is substituted by the equation:

$$\frac{(1 + |f|^2 \{AA^+\})^{1/2}}{f} = \frac{z_1(a\bar{\partial}_2 b - b\bar{\partial}_2 a) - z_2(a\bar{\partial}_1 b - b\bar{\partial}_1 a) + c + \phi}{|a|^2 + |b|^2}, \quad (39)$$

where ϕ is constrained by equations

$$A(g^{-1} \bar{\nabla}_2^0 g) A = [\partial_1 \phi / (|a|^2 + |b|^2)] A, \quad (40)$$

$$A(g^{-1} \nabla_1^0 g) A = -[\partial_2 \phi / (|a|^2 + |b|^2)] A, \quad (41)$$

which result from eq. (16). As well as in the one-instanton solution, eq. (39) imposes strict restrictions on the functions $a(\bar{z}_i)$, $b(\bar{z}_i)$ and $c(\bar{z}_i)$: its r.h.s. squared should have a modulus not lower than

$$\{\tilde{A}, \tilde{A}^+\} = A h^{-1} A^+ h + h^{-1} A^+ h A, \quad (42)$$

with $h = g^+ g$. We have not succeeded in finding an explicit form of functions which satisfy this constraint. Nevertheless, one may be convinced that the procedure outlined works indeed step by step. Now we demonstrate how it works for $N=2$.

In this case the fields $B_i^{(0)}$ are those given by eq. (28). The matrix g coupled with these fields according to eq. (37) is given by

$$g = \frac{1}{\sqrt{1+z_i\bar{z}_i}} \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \\ -\bar{z}_2 & z_1 + \bar{z}_1^{-1} \end{bmatrix}. \quad (43)$$

The constraints (40) result in:

$$\phi = - \frac{(z_1 a - z_2 b)^2}{\bar{z}_1^2 (1 + z_i \bar{z}_i)} \tag{44}$$

And, finally, we get:

$$\{A, A^+\} = \left[\frac{|a\bar{z}_1 - bz_2|^2 + |az_2 + b(z_1 + \bar{z}_1^{-1})|^2}{(1 + z_i \bar{z}_i)(|a|^2 + |b|^2)} \right]^2 \tag{45}$$

A general solution of eq. (39) is given by the 8-parametric family of functions a, b and c ; one of these three is

$$a = \bar{z}_1 - \bar{z}_1^{-1}, \quad b = \bar{z}_2, \quad c = 2 + \bar{z}_1^{-2}, \tag{46}$$

which gives a two-instanton solution with $O(3) \times O(2)$ symmetry.

Thus, we have established a recursion procedure of reproduction of instantons.

Conclusions. Despite all its limitations (explicit formulas for all N -instanton fields are not available) our approach looks rather promising. We hope that being developed in a proper way this approach would allow integrating complete Yang–Mills equations, constructing proper action-angle variables and finding a quasiclassical (and hopefully exact) solution of the quantum problem. New conservation laws and hidden symmetry of the Yang–Mills equations would be discovered in this way; this hidden symmetry (is it a two-dimensional infinite-parametric conformal group?) being responsible for the $8N$ -parametric family of instantons with topological charge N . The linear eqs. (15), (16) might turn very useful when fluctuations near instanton fields are analyzed.

A number of assumptions are introduced above which at first sight are far from being self-explanatory. They are justified a posteriori as our solutions saturate an exact bound for the number of arbitrary parameters. Nevertheless, there are many interesting questions to be answered: how the number of poles of $\psi(x, \lambda)$ is connected with the topological charge N , why the locations of poles should be independent of x , what the geometrical meaning is of the constraint given by eq. (39), why there exist single-component fermions in the self-dual gauge fields, and so on.

After this work had been completed and ready for publication, news came (S.I. Gelfand, private communication) about an interesting comment by Atiyah on

the geometrical meaning of the self-duality equations considered on $P^3(c)$ spaces. We present below one of the possible connections of the $P^3(c)$ treatment of the duality equations with our approach.

Let us start with an 8-dimensional space defined by four complex variables $\xi_a = (\xi_1, \xi_2, \xi_3, \xi_4)$. A three-dimensional projective space $p^3(c)$ is obtained after identifying with each other all the points ξ_a which differ by a common complex factor. This space $p^3(c)$ is projected onto the space of our variables z_1 and z_2 as follows:

$$z_1 = \frac{\xi_1 \bar{\xi}_3 + \bar{\xi}_2 \xi_4}{|\xi_3|^2 + |\xi_4|^2}, \quad z_2 = \frac{\xi_2 \bar{\xi}_3 - \bar{\xi}_1 \xi_4}{|\xi_3|^2 + |\xi_4|^2} \tag{A.1}$$

The Yang–Mills fields are lifted onto $p^3(c)$ according to:

$$C_a = \frac{\partial x^\mu}{\partial \xi_a} A_\mu = \frac{\partial z_i}{\partial \xi_a} B_i - \frac{\partial \bar{z}_i}{\partial \xi_a} B_i^\dagger \tag{A.2}$$

The self-duality equation takes on $P^3(c)$ the form:

$$\frac{\partial C_b}{\partial \xi_a} - \frac{\partial C_a}{\partial \xi_b} + [C_a, C_b] = 0, \tag{A.3}$$

so that

$$C_a = \psi \partial \psi^{-1} / \partial \xi_a \tag{A.4}$$

After insertion of formula (A.4) into eq. (A.2) one arrives at eqs. (17), (18) with the parameter λ identified as $\lambda = \xi_3 / \xi_4$.

We express our deep gratitude for useful discussions to V. Dutyshev, S. Manakov, V. Matsaev, N. Nikolaev, S. Novikov, A. Polyakov and L. Faddeev.

Note added (9 August 1977). (i) The functions $a(\bar{z})$ and $b(\bar{z})$ should not vanish identically, otherwise the fields $A_\mu = 0$. (ii) It could be verified that the asymptotical behaviour of the functions $a(\bar{z})$ and $b(\bar{z})$, which define the matrix ψ_1 , coincide with those of the one-instanton solution. As a result, the degree of the mapping at infinity is indeed enlarged by one unit. (iii) Is the iterative procedure presented sufficient to build all $(N+1)$ -instanton solutions or not, remains an open question. Further work in this direction is in progress. We are grateful to the Referee for useful comments and for pointing out few misprints in the original text.

References

- [1] A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. 59B (1975) 85.
- [2] G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8.
- [3] C. Callan, R. Dashen and D. Gross, Phys. Lett. 63B (1976) 334.
- [4] R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172.
- [5] A. Polyakov, Phys. Lett. 59B (1975) 82.
- [6] A. Polyakov, NORDITA preprint (1976).
- [7] R. Jackiw, C. Nohl and C. Rebbi, MIT preprint (1976).
- [8] B. Burlankov and V. Dutyshev, Rept. at the Meeting of the Nuclear Physics Division of the Academy of Sciences (October 1976); Zh. Eksp. Teor. Fiz., to be published; E. Witten, Harvard Univ. preprint (1976).
- [9] V.E. Zakharov and A.B. Shabat, to be published.