# CONSTRUCTION OF INSTANTONS

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A complete construction, involving only linear algebra, is given for all self-dual euclidean Yang–Mills fields.

In ref. [1] it was shown that "instantons", i.e. self-dual solutions of the Yang–Mills equations in the compactified euclidean 4-space $S^4$, corresponded in a precise way to certain real algebraic bundles on the complex projective 3-space $P_3(C)$. Using this correspondence and algebro-geometric results of Horrocks and Barth we shall show that, for the compact classical groups, all instantons have an essentially unique description in terms of linear algebra. For simplicity we will give details only for the case of SU(2), but we shall indicate briefly how to modify this for general $G$.

The construction starts from a complex linear map $A(z)$: $W \to V$, where $\dim W = k$, $\dim V = 2k + 2$, $z$ stands for a complex 4-vector $(z_1, z_2, z_3, z_4)$ and $A(z)$ is itself linear in $z$, i.e. $A(z) = \Sigma_{i=1}^{4} A_i z_i$. We assume $V$ has a fixed bilinear form denoted by $(v_1, v_2)$, which is skew and non-degenerate. For any subspace $U \subset V$ let $U^0$ be its annihilator, i.e. it consists of those $v$ such that $(u, v) = 0$ for all $u \in U$. We now impose the following condition on $A(z)$:

$$\text{for } z \neq 0,\ U_z = A(z)W \subset V \text{ has dimension } k \text{ and is isotropic, i.e. } U_z \subset (U_z)^0. \tag{1}$$

Given condition (1) we take $E_z$ to be the quotient space $(U_z)^0/U_z$. Since $\dim U_z = k$, $\dim (U_z)^0 = k+2$ and $\dim E_z = 2$. Moreover $E_z$ inherits a non-degenerate skew form, and $E_z = E_{\lambda z}$ for all non-zero scalar $\lambda$. Hence the family of $E_z$ defines an algebraic vector bundle over $P_3(C)$ with group $SL(2, C)$. This construction was introduced and studied by Horrocks. The jumping lines of $E$ [2,§2] are lines in $P_3(C)$ joining points $(z)$ and $(\xi)$ such that $(U_z)^0 \cap U_\xi \neq 0$.

To get a bundle corresponding to an SU(2)-instanton we need to impose reality conditions on $A(z)$ as explained in ref. [2]. We fix anti-linear maps $\sigma$ on $W$, $V$, $C^4$ so that $\sigma^2 = 1$ on $W$, $\sigma^2 = -1$ on $C^4$, $V$. On $C^4$ define $\sigma$ explicitly by $\sigma(z_1, z_2, z_3, z_4) = (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)$, and on $V$ we require that $\sigma$ be compatible with the skew form, i.e. $(\sigma v_1, \sigma v_2) = \overline{(v_1, v_2)}$. This gives $V$ a hermitian form $\langle , \rangle$ defined by $\langle v_1, \sigma v_2\rangle = (v_1, v_2)$, and we require this to be *positive definite.* Thus, as in ref. [2], $C^4$ can be identified with the 2-dimensional quarternion space $H^2$, while $V$ can be identified with $H^{k+1}$ (with its standard metric), $\sigma$ being in both cases multiplication by the quaternion $j$.

Our reality condition is now that $A(z)$ be compatible with $\sigma$, i.e.

$$\sigma[A(z)w] = A(\sigma z)\sigma w\ . \tag{2}$$

The definition of the hermitian form shows that the orthogonal space $U^\perp$ of $U \subset V$ is $(\sigma U)^0$. Now condition (2) implies that $\sigma U_z = U_{\sigma z}$ (with $U_z$ as in condition (1)). Hence $(U_z)^0 = (U_{\sigma z})^\perp$ and so, by the positive definiteness of the hermitian form, $(U_z)^0 \cap U_{\sigma z} = 0$. This shows that the "real" lines of $P_3(C)$ (i.e. lines joining $(z)$ to $(\sigma z)$ – corresponding as in ref. [2] to points in the real 4-sphere $S^4$) are never jumping lines. Thus $E$ satisfies the two conditions for an SU(2)-instanton described in ref. [2].

When $A(z)$ satisfies conditions (1) and (2) the resulting SU(2)-instanton, viewed differential-geometrically as a vector bundle with SU(2)-connection, can be constructed as follows. If $x \in S^4$ is represented by the line $(z, \sigma z)$ of $P_3(C)$, take $F_x$ to be the 2-dimensional subspace of $V$ which is orthogonal to both $U_z$ and $U_{\sigma z}$ (note that, by condition (1), $U_z \subset U_z^0 = U_{\sigma z}^{\perp}$). As $x$ varies we obtain a vector bundle $F$ over $S^4$ as sub-bundle of the product $S^4 \times V$. We give $F$ the connection induced, by orthogonal projection, from this product. It follows from the geometrical correspondence of ref. [2] that this connection is self-dual and the instanton number (or Pontrjagin index) is $k$. A direct verification is also not hard.

Our main assertion is that all SU(2)-instantons arise in this way and that gauge equivalence corresponds precisely to the obvious linear equivalence, i.e. replacing $A(z)$ by $PA(z)Q$, where $P$, $Q$ act on $V$, $W$ preserving their structures, i.e. $P \in \mathrm{Sp}(k+1)$, $Q \in \mathrm{GL}(k, R)$. A quick count of parameters (first made by Barth) yields $8k-3$ as the number of effective parameters of our construction.

In view of known results [1,7] this is encouraging but not conclusive evidence that our construction yields all instantons. For this we need a result of Barth [3] characterizing SL(2, C) bundles $E$ on $P_3(C)$ which arise from the Horrocks construction (i.e. from $A(z)$ satisfying condition (1)). The essential condition is the vanishing of the sheaf cohomology group $H^1(P_3, E(-2))$, where $E(-2)$ denotes the sheaf of holomorphic sections of $E \otimes L^{-2}$ and L is the Hopf line-bundle over $P_3$. If $E$ corresponds to a self-dual SU(2)-bundle $F$ on $S^4$, then $H^1(P_3, E(-2))$ can be identified with the space of solutions of the linear differential equation $(\Delta + \frac{1}{6}R)u = 0$, where $u$ is a section of $F$, $\Delta$ is the covariant laplacian of $F$ and $R > 0$ is the scalar curvature of $S^4$. (For the case when $E$ is trivial see Penrose [6] and [2,§4].) Since $\Delta \geqslant 0$ as an operator this equation has no global solutions, other than zero, and so Barth's criterion is fulfilled.

This vanishing theorem is analogous to the one used in refs. [1,7] in computing the dimension of the moduli space. It should be emphasized that vector bundles on $P_3(C)$ exist for which neither of these vanishing theorems hold (Hartshorne, unpublished). Fortunately the extra reality constraints arising from instantons exclude the unpleasant cases and simplify the problem, justifying the speculation in ref. [2,§5].

If $E$ arises by the Horrocks construction from $A(z)$: $W \to V$, then $W$, $V$ can be naturally identified with $H^2(P_3, E(-1))$ and $H^2(P_3, E(-4) \otimes T)$ respectively (where $T$ denotes the tangent bundle), while $A(z)$ is essentially multiplication by $\partial/\partial z$, regarded as a section of $T(-1)$. If $A(z)$ also satisfies condition (2) so that $E$ corresponds to a self-dual SU(2) bundle $F$ over $S^4$, then $W^*$ can be identified with the solutions of the Dirac equation for positive spinors coupled to $F$. The space $V$ consists of sections of $E$ whose covariant derivatives are sums of products of solutions of the above Dirac equation and solutions of the "twistor equation". This shows that $A(z)$ can be reconstructed in an essentially unique way from $F$.

As an example, we may consider the 't Hooft instanton, depending on $k$ real parameters $\lambda_i$ and $k$ points $y_i$ in $R^4$. Writing $z \in C^4$ as a pair of quaternions $(p, q)$, condition (2) implies that $A(z)$ is given by a $(k+1) \times k$ matrix of quaternions, which for the 't Hooft solution is

$$\begin{pmatrix} \lambda_1 p & \cdots & \lambda_k p \\ y_1 p - q & & \\ & \ddots & \\ & & y_k p - q \end{pmatrix},$$

where $y_i$ is represented as a quaternion and $y_i p$ denotes quaternion product.

The potential $A_\mu$ may be obtained from the general $A(z)$ by a simple algebraic process which involves inverting a quaternionic matrix [4]. The diagonal nature of the above matrix leads to the particularly simple form of the 't Hooft potentials.

We now indicate how to modify the construction to deal with other groups than SU(2) = Sp(1). First observe that the construction works for any symplectic group Sp($n$), the only change being that we now take $\dim V = 2k + 2n$. For the orthogonal group O($n$) we take $\dim V = 2k + n$ to have a symmetric rather than a skew form and $W$ becomes symplectic (i.e. $\sigma^2 = -1$). Since any compact Lie group $G$ has a faithful representation as a subgroup of O($n$) we can describe $G$-instantons as O($n$)-instantons with additional structure. For $G = SU(n) \subset O(2n)$ this amounts to imposing compatibility with an orthogonal $J$ with $J^2 = -1$ throughout. Note that the symplectic case can also be treated this way by the embedding $Sp(n) \subset O(4n)$.

In conclusion we should point out that although the construction of $G$-instantons, a problem in non-linear differential equations, has now been reduced to linear algebra, certain points remain to be clarified. In particular, the geometric and topological nature of the space of moduli has still to be extracted from the linear algebra.

More detail on the results described here can be found in refs. [4,5] and will be developed further elsewhere.

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