Lecture 11.

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1. RG critical fixed points.

We are going to continue the discussion of general properties of RG flow.

At generic point in the space of quasi-local actions Σ the pattern of RG flow is rather boring- it just flows. Something interesting is expected to happen near singular points of the flow. The simplest and probably the most important ones are the **fixed points**.

1.1. Critical fixed points.

By definition, the fixed point is an action A_* which satisfy the equation

$$B(A_*) = 0 \tag{1}$$

The corresponding RG trajectory is trivial:

$$A_l = A_*, \;\; for\ all\ l. \tag{2}$$

As the correlation length measured in the units Λ is

$$R_c(l) = \exp(-l) R_c \tag{3}$$

for A_l, this means that for A_* we must have

$$either\ R_c = 0,\ or\ R_c = \infty. \tag{4}$$

By the definition, A_* with $R_c = \infty$ belongs to the **critical surface:**

$$A_* \in \Sigma_{crit},\ if\ R_c = \infty. \tag{5}$$

We call the fixed points A_* with $R_c = \infty$ the critical fixed points.

1.2. RG flow topology around two critical points.

To illustrate the topological ideas related to the Wilson's RG, we consider the simplest possible form of topology. In the analysis of the topology we will assume continuity of the RG flow, i.e. the functions B defining the RG flow at some point

$$\frac{d}{dl} A_l = B(A_l) \tag{6}$$

are continuous functions of their arguments. The general proof of this statement is not avalable, but various examples support this assumption.

Assume there are only two fixed points, one with $R_c = 0$ and another one with $R_c = \infty$. Denote them P_0 and P_∞.

As $R_c(l) = \exp(-l) R_c$, any trajectory which starts at finite R_c will approach the point P_0 where $R_c = 0$. On the other hand trajectories which start on critical surface Σ_{crit} will approach P_∞. The only point where some trajectory can leave Σ_{crit} is P_∞. So there must be a trajectory (let us call it U) going from P_∞ to P_0.

Consider also a trajectory G which starts close to Σ_{crit}. This one first approaches P_∞ while staying close to Σ_{crit}, then spends very long time l in the vicinity of P_∞ (by continuity $B(A)$ is small when A is close to P_∞) and then departs along some way close to U, finally going towards P_0.

Now let us look for the space $\Sigma(\infty)$ in this picture. **It is clear that $\Sigma(\infty)$ is U with the points P_∞, P_0 included**

$$\Sigma(\infty) = U \tag{7}$$

Starting with any action $A \in U$ one can go back along the trajectory arbitrary amount of time l without leaving the space Σ (actually without leaving U). The reason is that $R_c = \infty$ on U only at the point P_∞. To reach P_∞ from any point on U with finite R_c takes infinite time l (when moving backwards). Starting from other point in Σ and integrating backwards, one generally leaves Σ in sufficient amount of time.

We see in this example the role played by critical points in the formation of $\Sigma(\infty)$. **Therefore finding critical fixed points is of central importance in the RG approach**.

1.3. Vicinity of critical fixed point in 2d approximation.

Suppose A_* is the critical fixed point:

$$B(A_*) = 0 \tag{8}$$

Then near this point

$$A = A_* + \delta A \tag{9}$$

where δA is infinitesimal. It must be thought as a vector in the tangent space $T\Sigma$ taken at the point A_*

$$\delta A \in T\Sigma|_{A_*} \tag{10}$$

The RG transformations can be linearized near the point A_*:

$$B(A_* + \delta A) = K(\delta A) + O(\delta A^2) \tag{11}$$

where K is a linear operator acting in this tangent space,

$$K : T\Sigma|_{A_*} \to T\Sigma|_{A_*} \tag{12}$$

As it is conventional in the theory of linear operators, we will write

$$K(\delta A) = K\delta A \tag{13}$$

The RG flow in this close vicinity of A_* is described by the linear differential equation

$$\frac{d}{dl}(\delta A) = K\delta A. \tag{14}$$

This equation can be easily solved provided one can find the eigenvectors Ψ_α and associated eigenvalues of K:

$$K\Psi_\alpha = k_\alpha \Psi_\alpha. \tag{15}$$

The solution then takes the form

$$\delta A(l) = \sum_{\alpha} C_{\alpha} \exp{(k_{\alpha} l)} \Psi_{\alpha}, \tag{16}$$

where C_{α} are determined by the initial vector $\delta A(0) = \delta A_0$.

It is clear that the pattern of the RG flow near the A_* is directly related to the eigenvalues k_{α}.

To illustrate this let us assume that there are only two eigenvalues (2 dim. approximation of Σ), one positive and one negative

$$k_+ > 0, \;\; k_- < 0. \tag{17}$$

There is a natural coordinate system on $T\Sigma|_{A_*}$ associated with the eigenvectors Ψ_+ and Ψ_-. All trajectories starting at the Ψ_- axis never leave it; they flow towards the fixed point A_*. As the correlation lenght can only decrease with l, the entire axis Ψ_- must belong to Σ_{crit} (where by definition $R_c = \infty$). The trajectories starting at the Ψ_+ also remain at this axis, but flow away from the fixed point. For sufficiently large l these trajectories leave the domain of validity the linear approximation (14) so that (16) says nothing about their ultimate destination. The same is true for any generic trajectory G starting at neither axis: at sufficiently large l it leaves the vicinity of the fixed point and its fate cannot be predicted within the linear approximation.

However there is an important difference between the generic trajectories like G and any trajectory U that lies on the axis Ψ_+. Starting at any point on the axis Ψ_+ one can integrate backward for indefinite negative l without leaving the vicinity of the critical fixed point A_*; in fact corresponding $A(-l)$ approaches A_* as $l \to \infty$. This is in contrast with the negative time behaviour of generic trajectories. Associated $A(-l)$ with sufficiently large l leave the vicinity of A_* and the question if they remain in Σ for $l \to \infty$ cannot be answered within the linear approximation. On the other hand the linear approximation is good enough to recognize in this example at least one-dimensional component of the subspace $\Sigma(\infty)$ associated with the point A_*. The axis Ψ_+ is the linear approximation to this component. Precise statement is

$$\Psi_+ \in T\Sigma(\infty)|_{A_*}. \tag{18}$$

1.4. Vicinity of critical fixed point in general case, unstable manifold.

It is easy to generalize the conclusions obtained in the 2d example. Let us define two special subspaces of the tangent $T\Sigma|_{A_*}$: let F_+ be the linear subspace spanned by the eigenvectors of K with positive eigenvalues k:

$$F_+ = \oplus_{\alpha, k_\alpha > 0} \mathbb{R}\Psi_\alpha. \tag{19}$$

Similarly

$$F_- = \oplus_{\alpha, k_\alpha < 0} \mathbb{R}\Psi_\alpha. \tag{20}$$

This separation leaves aside very interesting possibility of having zero eigenvalues we will discuss later. Also it is assumed here that all k_α are real. At this point this assumption looks arbitrary. We will see later that complex k_α contradict quantum mechanical interpretation of field theory.

Simple analysis of (16) shows that

1. The fixed point $A_* \in \Sigma_{crit}$ and

$$F_- = T\Sigma_{crit}|_{A_*}. \tag{21}$$

2. The fixed point A_* also belongs to some component of Sgm_∞ and

$$F_- \subset T\Sigma_\infty|_{A_*}. \tag{22}$$

Let us define the subspace $U(A_*) \subset \Sigma(\infty)$ by the condition that all points of $U(A_*)$ lie on RG trajectories which, when integrated backward, converge to the fixed point A_*, i.e.

$$A_0 \in U(A_*) \Rightarrow A_l \to A_* \; as \; l \to -\infty. \tag{23}$$

The subspace $U(A_*)$ **is called the unstable manifold of the fixed point** A_*. Obviously $A_* \in U(A_*)$. If we ignore the possibility of zero eigenvalues then the tangent space to $U(A_*)$ taken at the fixed point A_* itself coincides with the space F_+:

$$F_+ = TU(A_*)|_{A_*}. \tag{24}$$

The reason why A_* can belong to the space $\Sigma(\infty)$ of dimensionality **greater** than that of $U(A_*)$ is illustrated by the following simple example.

Suppose we have two fixed points, the A_* and A'_*, such that $A_* \in U(A'_*)$. Then there must be at least one RG trajectory in $U(A'_*)$ which flows to A_* (this is by definition of unstable manifold U). This trajectory has $R_c = \infty$ and therefore it simultaneously belongs to $\Sigma(\infty)$ and Σ_{crit}. It follows that the unstable manifold $U(A_*)$ has at least one dimension less then $U(A'_*)$.

Understanding the global topological properties of the RG flow, like the one illustrated in the above example are beyond the capabilities of linear analysis. However, as we have seen, it is within the power of linear analysis to prove that **each critical fixed point** A_* **generates a manifold** $U(A_*)$ **which belongs to** $\Sigma(\infty)$ and

$$dim U(A_*) = dim F_+ \tag{25}$$

In other words, **the critical fixed point** A_* **generates** $dimF_+$**-parameter family of finite local QFT**.

2. RG transformation of composite fields.

2.1. Composite fields as a tangent space of the quasilocal actions.

It is important to understand the meaning of the above geometric notions in terms of local field degrees of freedom.

Recall the generic quasi local action can be written as integral

$$A = \int d^d x \mathcal{L},$$
$$\mathcal{L} = \sum_{k=1} \frac{u_{2k}}{2k!}\phi_0^{2k} + \sum_{k=0}^{\infty} \frac{v_{2k}}{2n!}\phi_0^{2k}(\partial\phi_0)^2 + ... \tag{26}$$

Then

$$\delta A = \int d^d x \mathcal{L},$$
$$\delta\mathcal{L} = \sum_{k=1} \frac{\delta u_{2k}}{2k!}\phi_0^{2k} + \sum_{k=0}^{\infty} \frac{\delta v_{2k}}{2n!}\phi_0^{2k}(\partial\phi_0)^2 + ... \tag{27}$$

In other words δA is an integral of linear combination of local composite fields:

$$\delta A = \int d^d x \sum_{\alpha} \delta\lambda_\alpha O_\alpha(x), \ O_\alpha = \phi_0^{2n}, \phi_0^{2n}(\partial\phi_0)^2, ... \tag{28}$$

and $\delta\lambda_\alpha$ are the variations of the coupling constants. Hence,

$$F \equiv T\Sigma|_{A_*} \tag{29}$$

can be identified with the space of scalar composite fields spanned by O_α.

2.2. RG transformation of composite fields.

How the RG transformation acts on these composite fields?

Consider a correlation function involving one of more of $O_\alpha(x)$

$$< O_\alpha(x)... >= \frac{1}{Z}\int [D\phi_0](O_\alpha(x)...) \exp\left[-A[\phi_0]\right]. \tag{30}$$

where the dots represent other local insertions. To perform the step 1 of the Wilson's RG transformation we must split

$$\phi_0(x) = \phi_1(x) + \hat{\phi}(x) \tag{31}$$

and then integrate the fast mode $\hat{\phi}(x)$ out. Since, $O_\alpha(x)$ in general is a polynomial function of $\phi_0(x)$ and its derivatives at the point x. Therefore

$$O_\alpha(\phi_1(x) + \hat{\phi}(x), der.) = \sum_\beta Y_\alpha^\beta(\hat{\phi}(x), der.)O_\beta(\phi_1(x), der.), \tag{32}$$

where Y_α^β are local functions of the fast mode and its derivatives. For example

$$(\phi_1(x) + \hat{\phi}(x))^n = \sum_{k=0}^{n} Y_n^k(\hat{\phi})\phi_1^{n-k}(x),\ Y_n^k = C_n^k\hat{\phi}^k(x). \tag{33}$$

Therefore the integral over $\hat{\phi}(x)$ can be written as

$$\frac{1}{Z}\int[D\hat{\phi}](O_\alpha(\phi_1(x)+\hat{\phi}(x),der.)...)\exp\left[-A(\phi_1+\hat{\phi})\right] = \frac{1}{Z_1}(\sum_\beta y_\alpha^\beta(L)O_\beta(\phi_1(x),der.)...)\exp\left[-A_1(\phi_1)\right], \tag{34}$$

where A_1 is the action appearing in step 1 of the Wilson's RG transformation with the parameter L and the coefficients y_α^β are the expectation values of Y_α^β over the ensemble of the fast mode fluctuations:

$$y_\alpha^\beta(L) =< Y_\alpha^\beta(\hat{\phi}(x),der.) >_{\hat{\phi}} . \tag{35}$$

In writing this equation we have assumed that the other insertions represented by the dots are placed at the distances $>> \frac{1}{\Lambda}$ from x and therefore all diagrams with the fast mode propagators connecting these insertions can be ignored (claster property). In general, the coefficients $y_\alpha^\beta(L)$ are complicated sums of Feynmann diagrams. **But in any case (35) shows that in the step 1 any composite field O_α gets replaced by a linear combination of composite fields with L-dependent coefficients**.

This statement remains true after we perform the step 2, which amounts to the change of variables

$$\phi_1(x) = z^{-\frac{1}{2}}(L)\phi_0(\frac{x}{L}) \tag{36}$$

As O_β in (35) are typically polynomials of ϕ_1 and its derivatives taken at x, we have

$$\sum_\beta y_\alpha^\beta(L)O_\beta(z^{-\frac{1}{2}}(L)\phi_0(\frac{x}{L}), z^{-\frac{1}{2}}\partial\phi_0(\frac{x}{L}),...) = \sum_\beta z_\alpha^\beta(L)O_\beta(\phi_0(\frac{x}{L}), \frac{\partial}{\partial(\frac{x^i}{L})}\phi_0(\frac{x}{L}),...), \tag{37}$$

i.e. **under the full RG transformation the composite field O_α gets replaced by the linear combination of composite fields with L-dependent coefficients**. The exact statement is

$$< O_\alpha(x)... > |_A =< \sum_\beta z_\alpha^\beta(L) O_\beta(\frac{x}{L})... > |_{A'}, \tag{38}$$

where

$$A' = RG_l(A), \ \ l = \log(L) \tag{39}$$

is the RG transformed action. Thus, we can write

$$RG_{l,A} O_\alpha = \sum_\beta z_\alpha^\beta(L) O_\beta \tag{40}$$

stressing the fact that RG transformation acts on the fields linearly. Additional index A is written down to remind that this transformation depends on the action A.

2.3. RG transformation of composite fields at the fixed-point.

Let A be a fixed point A_*. Then $A' = A$ and (40) yelds linear relation for the correlation functions of the composite fields. Recall that by the composition low of the RG transformations,

$$RG_{l+l'} = RG_l RG_{l'} \tag{41}$$

in this case the coefficients z_α^β must obey the equation

$$z_\alpha^\beta(l + l') = \sum_\gamma z_\alpha^\gamma(l') z_\gamma^\beta(l) \tag{42}$$

It means that $z_\alpha^\beta(l)$ must satisfy linear differential equation with l-dependent coefficients:

$$\frac{d}{dl} z_\alpha^\beta(l) = -\sum_\gamma D_\alpha^\gamma z_\gamma^\beta(l),\ \ D_\alpha^\beta = -\frac{d}{dl} z_\alpha^\beta(l)|_{l=0}. \tag{43}$$

These coefficients D_α^β are nothing else then the matrix elements of the linear operator D corresponding to the infinitesimal transformation (40)

$$RG_{\delta l, A_*} = I - \delta l D + O(\delta l^2). \tag{44}$$

Let us denote $\Phi_\alpha(x)$ the eigenvectors of the operator D, i.e. special linear combinations of the composite fields $O_\alpha(x)$ which satisfy the equations

$$D\Phi_\alpha \equiv D_\alpha^\beta \Phi_\beta = D_\alpha \Phi_\alpha, \tag{45}$$

where the notation D_α has been used for the corresponding eigenvalues of D. Then

$$RG_{l,A_*}\Phi_\alpha = \exp(-lD_\alpha)\Phi_\alpha, \tag{46}$$

and for the fixed-point theory the equations (38) reads

$$< \Phi_{\alpha_1}(x_1)...\Phi_{\alpha_n}(x_n) > |_{A_*} = L^{-D_{\alpha_1}}...L^{-D_{\alpha_n}} < \Phi_{\alpha_1}(\frac{x_1}{L})...\Phi_{\alpha_n}(\frac{x_n}{L}) > |_{A_*}. \tag{47}$$

For the two-point correlation functions, which depend only on the distance between the two points, it follows from (47)

$$< \Phi_\alpha(x)\Phi_\beta(y) > |_{A_*} = |x-y|^{-D_\alpha - D_\beta} G_{\alpha\beta} \tag{48}$$

where $G_{\alpha\beta}$ are the constants. In general, the equation expresses the **scale invariance** of the theory described by the fixed-point action A_*. The eigenvalues D_α are called the **scale dimensions (or anomalous scale dimensions)** of the fields Φ_α. If A_* is not a free-field theory, there is no reason to expect them to coincide with canonical dimensions.