Lecture 9.

Renormalizarion Group.

Plan.

1. Wilson's RG transformation by 2 steps.

1.1. Step1. Momenta shell integration.

Let us consider Euclidean ϕ^4 field theory with the cutoff at Λ_0

$$Z_0 = \int_{\Lambda_0} [D\phi_0] \exp\left[-A(\phi_0)\right]$$

$$A(\phi) = \int d^dx[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0\phi^2 + \frac{\lambda_0}{4!}\phi^4] \tag{1}$$

Question: How the path integral depends on Λ_0?

We use the simplest regularization for the calculation of this integral:

$$\phi_0(x) = \int_{|k|<\Lambda_0} \frac{d^dk}{(2\pi)^d}\phi_k \exp\left(\imath kx\right) \tag{2}$$

where only the wavevectors with $|k| < \Lambda_0$ enter the paths integral. It gives the free propagator

$$D(k) = \frac{\Theta(\frac{k^2}{\Lambda_0^2})}{k^2 + m_0^2}$$
$$\Theta(x) = 0, x > 1, \;\; \Theta(x) = 1, x \leq 1 \tag{3}$$

Question: What is the difference between the integral with Λ_0 and $\Lambda_1 < \Lambda_0$?

Let us introduce temporarily the notation

$$\phi_0 = \int_{|k|<\Lambda_0} \frac{d^d k}{(2\pi)^d} \phi_k \exp(\imath kx),$$
$$\phi_1 = \int_{|k|<\Lambda_1} \frac{d^d k}{(2\pi)^d} \phi_k \exp(\imath kx) \tag{4}$$

If $\Lambda_0 > \Lambda_1$ the difference is that the integral over ϕ_0, as compared to the integral over ϕ_1, includes the integration over the models with the momentum

$$\Delta = \{\Lambda_1 < |k| < \Lambda_0\} \tag{5}$$

So one can write

$$\phi_0(x) = \phi_1(x) + \hat{\phi}(x),$$
$$\hat{\phi}(x) = \int_\Delta \frac{d^d k}{(2\pi)^d} \phi_k \exp(\imath kx),$$
$$[D\phi_0] = [D\phi_1][D\hat{\phi}] \tag{6}$$

We will call the field $\phi_1(x)$ the slow mode, while $\hat{\phi}(x)$ will be called the fast mode.

Now the idea is **to integrate first over the fast modes $\hat{\phi}$ and define a new "effective action" for the slow modes ϕ_1:**

$$\int_\Delta [D\hat{\phi}] \exp[-A_0(\phi_1 + \hat{\phi})] = C \exp[-A_1(\phi_1)] \tag{7}$$

The constant C is not important because in correlation functions this constant will enter in nominator and denominator.

1.2. Effective action for ϕ^4.

Let us consider how this procedure works for the case of ϕ^4-model.

$$\begin{aligned}
&\phi_0 = \phi_1 + \hat{\phi}, \\
&Z_0 = \int_{\Lambda_0} [D\phi_0] \exp\left[-A_0(\phi_0)\right] = \\
&\int_{\Lambda_1} [D\phi_1][D\hat{\phi}] \exp\left[-A_0(\phi_1)\right] \int_{\Delta} [D\hat{\phi}] \exp[-\int d^d x [\frac{1}{2}(\partial\hat{\phi})^2 + \frac{1}{2}m_0\hat{\phi}^2 + \frac{\lambda_0}{4!}\hat{\phi}^4]] \\
&\exp[-\frac{\lambda_0}{4!}\int d^d x (4\phi_1^3\hat{\phi} + 6\phi_1^2\hat{\phi}^2 + 4\phi_1\hat{\phi}^3)]
\end{aligned} \tag{8}$$

The terms $\partial\phi_1\partial\hat{\phi}$ and $\phi_1\hat{\phi}$ do not appear because

$$\begin{aligned}
&\int d^d x \phi_1(x)\hat{\phi}(x) = \int_{<\Lambda_1}\int_{\Delta} \frac{d^d k d^d p}{(2\pi)^{2d}} \phi(k)\phi(p) \int d^d x \exp\left(\imath(k+p)x\right) = \\
&\int_{<\Lambda_1} \frac{d^d k}{(2\pi)^d} \int_{\Delta} \frac{d^d p}{(2\pi)^d} \phi(k)\phi(p)\delta(k+p)
\end{aligned} \tag{9}$$

The action

$$\begin{aligned}
A[\phi_1,\hat{\phi}] = \int d^d x [\frac{1}{2}(\partial\hat{\phi})^2 + \frac{1}{2}m_0\hat{\phi}^2 + \frac{\lambda_0}{4!}\hat{\phi}^4 \\
-\frac{\lambda_0}{4!}(4\phi_1^3\hat{\phi} + 6\phi_1^2\hat{\phi}^2 + 4\phi_1\hat{\phi}^3)]
\end{aligned} \tag{10}$$

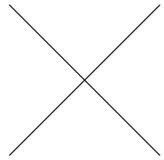
must be viewed in this context as the action for $\hat{\phi}$ with ϕ_1 determining a sorce terms.

Then integration over the fast mode $\hat{\phi}$ can be represented by the diagrams contributions which are generated by the free propagator

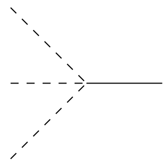
$$< \hat{\phi}(k)\hat{\phi}(-k) >= D_{\Delta}(k) = \frac{\Theta(|k| \in \Delta)}{k^2 + m_0^2} = \underline{\qquad\qquad} \tag{11}$$

and the vertices

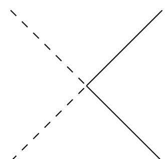
$$-\lambda_0 \int d^d x = \tag{12}$$



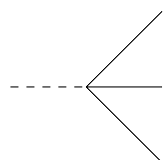
$$-\frac{\lambda_0}{3!} \int d^d x \phi_1^3(x) = \tag{13}$$



$$-\frac{\lambda_0}{2!} \int d^d x \phi_1^2(x) = \tag{14}$$



$$-\lambda_0 \int d^d x \phi_1(x) = \tag{15}$$



The dashed lines represent the field ϕ_1 which carry a momentum $|k| < \Lambda_1$. The solid lines correspond to the field $\hat{\phi}$ with momentum $|k| \in \Delta$.

All solid lines in the diagrams have to be contracted because we integrate over $\hat{\phi}$ leaving ϕ_1 fixed " external legs".

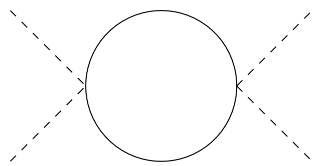
The diagrams exponentiate in terms of connected diagrams so that the action $A_1[\phi_1]$ is given by the sum of connected diagrams:

$$A_1[\phi_1] = -(sum\ of\ all\ connected\ diagrams) \tag{16}$$

The diagrams with no external legs contribute the constant factor above.

1.3. Quasilocal actions.

It is obvious that the action $A_1[\phi_1]$ does not have a form of ϕ^4 action. It containes all powers of ϕ_1 due to diagrams with various numbers of external legs. Moreover the action is no longer local. For example, the diagram (coming from $\phi_1^2(x) < \hat{\phi}^2(x)\hat{\phi}^2(y) >_\Delta \phi_1^2(y)$)



(17)

gives a contribution

$$-\frac{\lambda_0}{8}\int d^dxd^dy\phi_1^2(x)D_\Delta^2(x-y)\phi_1(y) \tag{18}$$

This nonlocality is not so catastrophic. We have violated locality at short distances by introducing cutoff anyway, so this additional nonlocality does not add something new. The **statement is that all nonlocal terms generated this way admit convergent derivative expansions.**

This means the following. Suppose we have evaluated some diagram $D(k_1, ..., k_n)$ with n external legs in the momentum space. One can expand this diagram in power series in all the momenta k_i. This series converge all the way up to $|k_i| < \Lambda_1$ because all internal momentum integration in this diagrams lay above Λ_1 so that dimensinless variables $\frac{k_i}{\Lambda_1}$ are small anough to be in the area of convergency.

This implies that the new action admits meaningful expansion in terms of local contributions

$$A_1[\phi_1] = \int d^dx[\sum_{n=1}^{\infty}\frac{u_{2n}}{2n!}\phi_1^{2n}(x) + \sum_{n=0}^{\infty}\frac{v_{2n}}{2n!}\phi_1^{2n}(\partial\phi_1)^2 + ...] \tag{19}$$

where the dots involve both higher powers of $\partial\phi_1$ and higher derivatives of ϕ_1. The only difference from local action is that the action involve derivatives of arbitrary high order. By this reason the action $A_1[\phi_1]$ is called **quasilocal.**

The conclusion at this point is that by integration out the fast modes $\hat{\phi}$ we can show that **the theory described by** (A_0, Λ_0) **can be equivalently described at low external momenta** $|k| << \Lambda_1$ **by the action** (A_1, Λ_1), where $\Lambda_1 < \Lambda_0$:

$$(A_0, \Lambda_0) \to (A_1, \Lambda_1), \ \Lambda_0 > \Lambda_1 \tag{20}$$

1.4. Step2. Coordinates dilatation and correlation functions relation.

At first glance it may look as a step in a wrong direction because the new action A_1 has complicated quasilocal form.

In fact the integration over the fast mode becomes very powerful move if this integration is combined with the following variable transformation

$$\phi_1(x) = z^{-\frac{1}{2}}(L)\tilde{\phi}_0(\frac{x}{L}) \tag{21}$$

in the functional integral for the correlation functions where

$$L = \frac{\Lambda_0}{\Lambda_1} > 1 \tag{22}$$

and $z^{-\frac{1}{2}}(L)$ is some factor related to the field renormalization (to be discussed later).

To do that let us consider the correlation function

$$\begin{gathered} < \phi_0(x_1)...\phi_0(x_n) >_{(A_0,\Lambda_0)}= \\ \frac{1}{Z_0}\int_{|k|<\Lambda_0}[D\phi_0]\phi_0(x_1)...\phi_0(x_n)\exp\left[-A_0[\phi_0]\right] \end{gathered} \tag{23}$$

Because of we are interested in the low momenta behaviour of this correlation function with the momenta $|k_i| << \Lambda_1 < \Lambda_0$ one can replace

$$\phi_0(x_1)...\phi_0(x_n) \to \phi_1(x_1)...\phi_1(x_n) \tag{24}$$

Then

$$\begin{aligned}
&< \phi_0(x_1)...\phi_0(x_n) >_{(A_0,\Lambda_0)}= \\
&\frac{1}{Z_0}\int_{|k|<\Lambda_1}[D\phi_1]\phi_1(x_1)...\phi_1(x_n)\int_{\Delta}[D\hat{\phi}]\exp\left[-A_0[\phi_1+\hat{\phi}]\right] = \\
&\frac{C}{Z_0}\int_{|k|<\Lambda_1}[D\phi_1]\phi_1(x_1)...\phi_1(x_n)\exp\left[-A_1[\phi_1]\right]
\end{aligned} \tag{25}$$

But

$$Z_0 = C\int_{|k|<\Lambda_1}[D\phi_1]\exp\left[-A_1[\phi_1]\right] = CZ_1 \tag{26}$$

so that we obtain

$$< \phi_0(x_1)...\phi_0(x_n) >_{(A_0,\Lambda_0)}=< \phi_1(x_1)...\phi_1(x_n) >_{(A_1,\Lambda_1)} \tag{27}$$

The variable transformation (21): $\phi_1(x) = z^{-\frac{1}{2}}(L)\tilde{\phi}_0(\frac{x}{L})$ corresponds to the **dilatation of the coordinates with factor $\frac{1}{L}$ and returning the cutoff Λ_1 back to Λ_0**:

$$x \to \frac{x}{L},\ \Lambda_1 \to \Lambda_0 \tag{28}$$

Indeed

$$\begin{aligned}
\phi_1(x) &= \int_{|k|<\Lambda_1}\frac{d^dk}{(2\pi)^d}\phi(k)\exp\left(\imath kx\right) = \\
&= \int_{|k|<\Lambda_1}\frac{d^dk}{(2\pi)^d}\phi(k)\exp\left(\imath Lk\frac{x}{L}\right) = \\
L^{-d}&\int_{|q|<L\Lambda_1=\Lambda_0}\frac{d^dq}{(2\pi)^d}\phi(\frac{q}{L})\exp\left(\imath q\frac{x}{L}\right)
\end{aligned} \tag{29}$$

so that

$$\tilde{\phi}_0(x) = z^{\frac{1}{2}}(L)\phi_1(Lx) = L^{-d} z^{\frac{1}{2}} \int_{|k|<\Lambda_0} \frac{d^d k}{(2\pi)^d} \phi(\frac{k}{L}) \exp{(\imath kx)} \quad (30)$$

contains all the wave vectors $|k| < \Lambda_0$.

We then obtain for correlators

$$\begin{aligned} &< \phi_0(x_1)...\phi_0(x_n) >_{(A_0,\Lambda_0)} = < \phi_1(x_1)...\phi_1(x_n) >_{(A_1,\Lambda_1)} \equiv \\ &\frac{1}{Z_1} \int_{|k|<\Lambda_1} [D\phi_1]\phi_1(x_1)...\phi_1(x_n) \exp{[-A_1[\phi_1]]} = \\ &z^{-\frac{n}{2}} < \tilde{\phi}_0(\frac{x_1}{L})...\tilde{\phi}_0(\frac{x_n}{L}) >_{(\tilde{A}_0,\Lambda_0)} \end{aligned} \quad (31)$$

But now we must identify

$$\phi_0(x) \equiv \tilde{\phi}_0(x) \quad (32)$$

so that the relation (31) can be rewritten as

$$< \phi_0(x_1)...\phi_0(x_n) >_{(A_0,\Lambda_0)} = z^{-\frac{n}{2}}(L) < \phi_0(\frac{x_1}{L})...\phi_0(\frac{x_n}{L}) >_{(\tilde{A}_0,\Lambda_0)} \quad (33)$$

where

$$\tilde{A}_0[\phi_0] \equiv A_1[z^{-\frac{1}{2}}(L)\phi_0(xL)] \quad (34)$$

That is the theories described by A_0 and $\tilde{A}_0$, with the **same cutoff** Λ_0 are different by a scale transformation with the factor L. The transformation

$$A_0 \to \tilde{A}_0 \quad (35)$$

thus represents the scale transformation in the space of effective actions. It is called **the Wilson's RG transformation.**

Let us rewrite the relation (33) in opposit manner and make the Fourier transform of both sides:

$$f.t. < \phi_0(x_1)...\phi_0(x_n) >_{(\tilde{A}_0,\Lambda_0)} = f.t. z^{\frac{n}{2}}(L) < \phi_0(Lx_1)...\phi_0(Lx_n) >_{(A_0,\Lambda_0)} (36)$$

where the comparison is made at low values of momenta $|k_i| << \Lambda_0$. In other words, the theory described by $\tilde{A}_0$ differs from that described by A_0 by a scale transformation of space $x \to \frac{x}{L}$ only.

The Wilson's RG shows what it takes to make such rescaling in field theory. Just rescaling the coordinates in the action is not sufficient. The naive rescaling represented by step 2 above, must be supplemented by the integration over the fast modes in step 1. Roughly speaking, **the rescaling $x \to \frac{x}{L}$ increases the spacial density of the fields degrees of freedom, and one has to dilute them back to the original density by performing step 1**. In this sense **the step 1 can be thought of as transformation of the measure in the path integral**.

2. Geometry of Renormalization flow.

2.1. Wilson's RG semigroup.

We have seen that even the original action A has simple form like ϕ^4, the result of RG transformation is usualy a complicated quasilocal action containing arbitrary high powers and orders of derivatives. On the other hand, if A is generic quasilocal the $\tilde{A}$ will again be of quasilocal form. (It is assumed here that RG transformation is determined by this condition). Under this assumption, **the RG transformation acts in the space of quasilocal actions** which we denote by Σ, so that

$$RG : \Sigma \to \Sigma,\ \tilde{A} = RG_l(A) \tag{37}$$

where $l = \ln(L) \geq 0$ ($L = \frac{\Lambda_0}{\Lambda_1}$).

It is clear that

$$RG_{l_1+l_2}(A) = RG_{l_1}(RG_{l_2}(A)),\ RG_0(A) = A \tag{38}$$

It means that RG is one-parametric **semigroup.** It is not a group because the inverse to RG_l can not be defined. The reason is simple but important- the step 1 leads to a loss of information about the interractions at the cutoff scales.

2.2. Approximate calculation of RG and Wilson's RG flow equation.

In general exact evaluation of the RG transformation for an arbitrary l is at least as difficult as exact evaluation of the functional integral. One can find a good approximation for RG_l with l small enough. Then, according to the composition low (38) the transformation RG_{Nl} can be obtained as N-th iteration of RG_l:

$$RG_{Nl} = RG_l ... RG_l \tag{39}$$

Thus, one can do it step by step, integrating out the shortest scale degrees of freedom.

Hence, it make sense to consider the **infinitesimal** RG transformation

$$\begin{aligned} RG_{\delta l}(A) &= A + B(A)\delta l + ... \\ B(A) &= \frac{d}{dl} RG_l(A)|_{l=0} \end{aligned} \tag{40}$$

In view of (38), the family of actions

$$A_l = RG_l(A_0) \tag{41}$$

where A_0 is an arbitr. initial action from Σ, satisfies the equation

$$\frac{d}{dl} A_l = B(A_l) \tag{42}$$

which is called **Wilson's RG flow equation.** It generates the scale transformations in the space of quasilocal actions Σ. Namely **two actions** A_{l_1}

and A_{l_2} belonging to the same integral curve A_l (RG trajectory) describes the theories which differ by the scale transformation

$$x \to \exp{(l_1 - l_2)}x \tag{43}$$

Thus, the problem of RG transformation splits into 2 steps:

1. Evaluation of $B(A)$.

2. Integration the RG flow equation (42).

2.3. The coordinates in the space of quasilocal actions.

One can introduce some coordinates on Σ to make the RG flow equation more precise. Generic quasilocal action (which is $\phi \to -\phi$ symmetric) can be written as

$$A[\phi_0] = \int d^dx[\sum_{n=1}^{\infty} \frac{u_{2n}}{2n!}\phi_0^{2n}(x) + \sum_{n=0}^{\infty} \frac{v_{2n}}{2n!}\phi_0^{2n}(\partial\phi_0)^2 + \\ \sum_{n=0}^{\infty} \frac{w_{2n+1}}{2n!}\phi_0^{2n+1}(\partial\phi_0)^2\partial^2\phi_0 + ...] \tag{44}$$

Thus the coupling constants u_{2n}, v_{2n}, w_{2n+1},... can be taken as such coordinates. Thus, the space Σ is infinite-dimensional. Some of the coordinates can be eliminated by appropriate renormalization of ϕ_0, one can take for instance $v_0 = 1$.

In terms of these coordinates the RG flow equation can be written in the form

$$\begin{aligned} &\frac{d}{dl}u_{2n}(l) = B_{u_{2n}}(u_{2n}(l), v_{2n}(l), w_{2n+1}(l), ...) \\ &\frac{d}{dl}v_{2n}(l) = B_{v_{2n}}(u_{2n}(l), v_{2n}(l), w_{2n+1}(l), ...) \\ &\frac{d}{dl}w_{2n+1}(l) = B_{w_{2n+1}}(u_{2n}(l), v_{2n}(l), w_{2n+1}(l), ...) \\ &.......... \end{aligned} \tag{45}$$

where $B_{u_{2n}}$, $B_{v_{2n}}$, $B_{w_{2n+1}}$,... are some functions of the coordinates on Σ.

In practical calculations some approximations for these equations are necessary. Most importantly one must find some finite-dimensional approximation for Σ. But before it is usefull to think about what are the properties of the RG flow we must be interested in.

2.4. Geometric properties of RG flow.

Some properties of RG flow can be discussed without refernce to specific form of RG transformation if we made a few general assumptions concerning the actions generated by RG_l transformation. The main assumption is

$$RG_l : \Sigma \to \Sigma \tag{46}$$

which means that RG_l **maps quasilocal action A into quasilocal action A_l and the series representing actions A and A_l converge for the fields ϕ_0 which contain only Fourier components with $|k| \leq \Lambda$.** Unfortunately there is no proof of this statement but in many cases it holds. Hence

$$A_l \in \Sigma, \ for \ l > 0 \ if A_0 \in \Sigma \tag{47}$$

The trajectory A_l gives a set of actions which all **describe essentially the same physical system up to the coordinate rescalings.**

It is usefull to discuss RG action on Σ in terms of physical mass m or in terms of correlation lenght $R_c = m^{-1}$. To define it, consider the 2-point correlation function. Its typical large-distance behaviour is

$$< \phi_0(x)\phi_0(0) > \approx \exp(-\frac{|x|}{R_c}), \ as \ |x| \to \infty \tag{48}$$

Let R_c be the correlation length associated with a system described by the action (A_0, Λ). Of course A_l describes the same system, but in the units in which the cutoff associated with A_l is still Λ, the correlation length will be

$$R_c(l) = \exp(-l) R_c \tag{49}$$

It allows to define various surfaces $\Sigma(R_c = r) \in \Sigma$ of quasilocal actions with fixed correlation lenght $R_c = r$. Of special interest is the **critical surface** Σ_{crit} **with** $R_c = \infty$.

For an arbitrary finite $r \geq 0$ one can consider also the subspaces $\Sigma(R_c \leq r) \subset \Sigma$ of the quasilocal actions such that $R_c \leq r$.

Let us apply RG_l to the subspace $\Sigma(R_c \leq r)$. In view of (49) we find

$$\begin{aligned} &RG_l \Sigma(R_c \leq r) \subset \Sigma(R_c \leq \exp(-l) r) \Rightarrow \\ &RG_l \Sigma \equiv \Sigma(l) \subset \Sigma,\ RG_l \Sigma \neq \Sigma \end{aligned} \tag{50}$$

2.5. RG flow limit and space of local QFT.

Another important subspace is the limiting space

$$RG_\infty \Sigma \equiv \Sigma(\infty) \tag{51}$$

This space can be much smaller then Σ. The basic idea of the Wilson's RG approach is in the hope that $\Sigma(\infty)$ has many less dimensions then Σ. The reason is the following: For large l the Action A_l describes the physical system with actual interactions of much shorter range (the range of interactions is $\approx R_c$) then the range of interractions for the initial action A_0 (in case of a theory on a lattice it means that interaction terms involve only few nearest spins around the given spin). It is unlikely that arbitrary action in Σ has this property. Thus $\Sigma(\infty)$ is expected to be only small subset in Σ.

The RG flow differential equation (42):

$$\frac{d}{dl}A_l = B(A_l) \tag{52}$$

can be integrated backward in l as well as forward. Suppose $A \in \Sigma$ but $A \notin \Sigma(l)$. Then A_{-l} may not be in Σ because A_{-l} can fail to be quasilocal (it may contain long-range interractions) or can have pathological properties like some coefficients which controll convergence of the paths integral can become such that integral with A_{-l} no longer exists.

The subspace $\Sigma(\infty)$ is important exception from this general rule. If one takes $A \in \Sigma(\infty)$ and obtain A_{-l} by integrating backwards we find that

$$A_{-l} \in \Sigma(\infty) \tag{53}$$

To proove this, suppose that $A_{-l} \notin \Sigma(\infty)$ but $A_{-l} \in \Sigma$. Then

$$RG_l(A_{-l}) = A \in \Sigma(\infty) \tag{54}$$

so that A can be reached in finite time l from some point in Σ but this contradicts the definition of $\Sigma(\infty)$. It means that **starting from $\Sigma(\infty)$ one can integrate back and forth, still remaining within $\Sigma(\infty)$.**

For this reason the subspace $\Sigma(\infty)$ is of central importance in local Quantum Field Theory. The problem of QFT is to obtain finite local theory in the limit $\Lambda \to \infty$, with appropriately adjusted bare action A_0. Integrating backward is essentially considering the actions which lead to the same physics as the action A_0 but equipped with larger and larger cutoff Λ mesured in the units of m. It is then clear that **finding renormalized local theories is equivalent to finding actions A from which one can integrate backward whithout limits.** For this reason it is natural **to identify $\Sigma(\infty)$ with the space of local QFT.**

2.6. RG flow fixed points.

The picture of the RG flow at generic point in Σ is rather boring- it just flows. Something interesting happens near singular points of the flow. The most simple but most important points of the RG flow are the **fixed points.**

A fixed point is an action A_* which satisfy the equation

$$B(A_*) = 0 \tag{55}$$

Associated trajectory is trivial

$$A_l = A_* \tag{56}$$

Appendix .

Renormalization in ϕ^4.

1. Regularization and renormalization program.

Recall our **renormalization program** we considered for the ϕ^4 theory.

1.

Start with the action

$$A = \int d^4x (\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4) \tag{57}$$

containing the bare field, bare mass, and bare coupling constant.

2.

Introduce some cutoff with a cutoff momentum Λ (this can be done many ways), to make the integrals over internal momenta finite.

3.

We expect that one can give parameters m_0^2, λ_0, and the field renormalization constant Z certain dependence on Λ:

$$m_0^2 = m_0^2(\Lambda)\ ,\ \lambda_0 = \lambda_0(\Lambda)\ ,\ Z = Z(\Lambda) \tag{58}$$

such that the correlation functions of the renormalized field

$$\phi = Z^{-\frac{1}{2}}(\Lambda)\phi_0,$$
$$< \phi(x_1)...\phi(x_N) >= Z^{-\frac{N}{2}}(\lambda, m, \Lambda) < \phi_0(x_1)...\phi_0(x_N) > \tag{59}$$

have finite $\Lambda \to \infty$ limit.

4.

We rewrite the initial action in terms of renormalized λ, m, $\phi(x)$ introducing counterterms

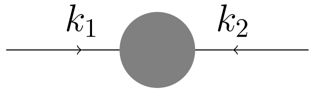
$$A = \int d^4x (\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4) \tag{60}$$

where m is an actual mass and λ is suitably defined finite coupling constant. The identity with the original action implies

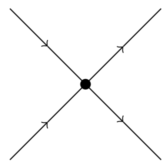
$$1 + \delta Z = Z\ ,\ m^2 + \delta m^2 = Z m_0^2\ ,\ \lambda + \delta\lambda = Z^2\lambda_0 \tag{61}$$

It leads to **the renormalized perturbation theory where the perturbation expansion is going by renormalized coupling constant** λ with the following Faynman rules:

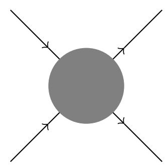
 $$= \frac{1}{k^2 + m^2} \tag{62}$$



$$= (k_1^2 \delta Z - \delta m^2)(2\pi)^4 \delta(k_1 + k_2) \tag{63}$$



$$= \lambda \tag{64}$$



$$= \delta\lambda \tag{65}$$

Therefore we assume the counterterm coefficients themselves depend perturbatively (i.e. as power series) on λ:

$$\begin{aligned} \delta Z &= Z_1 \lambda + Z_2 \lambda^2 + ... \\ \delta m^2 &= b_1 \lambda + b_2 \lambda^2 ... \\ \delta \lambda &= a_1 \lambda + a_2 \lambda^2 + ... \end{aligned} \tag{66}$$

In the cutoff regularization the coefficients Z_i, b_i, a_i depends on the Λ, while in the dimensional regularization the coefficients Z_i, b_i, a_i depends on $\epsilon = 4 - d$.