Lecture 8

Standard model and Georgi-Glashow Grand Unification model.

Plan.

1. Standard Model.

1.1. Gauge group and bosonic sector.

The gauge group of the Standard Model is

$$SU(3) \times SU(2) \times U(1) \tag{1}$$

where $SU(3)$ is responsible for the strong interactions of quarks and hence, we have to add strong interaction coupling constant g_s to the constants g, $\acute{g}$ of the electro-weak interaction. Thus, the Standard Model have additional

$SU(3)$-gauge symmetry and $SU(3)$ gauge fields transforming by the rule

$$G^A_\mu(x)t^A \to U(x)G^A_\mu(x)t^A U^{-1}(x) + \frac{\imath}{g_s}U(x)\partial_\mu U^{-1}(x),$$
$$U(x) = \exp\left(\imath\alpha^A(x)t^A\right) \in SU(3) \quad (2)$$

where t^A, $A = 1, ..., 8$ are generators of $su(3)$-Lie algebra

$$[t^A, t^B] = \imath f^{ABC} t^C \quad (3)$$

So we add to the electro-weak Lagrangian the $SU(3)$ gauge fields contribution

$$L_G = -\frac{1}{4}F^A_{\mu\nu}(F^A)^{\mu\nu},$$
$$F^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu + g_s f^{ABC} G^B_\mu G^C_\nu \quad (4)$$

The Higgs bosons are $SU(3)$-singlets so they do not interract to the $SU(3)$ gauge fields.

1.2. Quarks multiplets.

The quarks of all generations sit in the fundamental $SU(3)$-representation so that they are 3-components complex vectors regardless of chirality. In this representation the $su(3)$-generators are given by Gell-Mann matrices (see Appendix).

Hence, all the covariant derivatives from the electro-weak theory (see Appendix) have to be extended by $SU(3)$-gauge fields:

$$Q^i_L \to Q^{i\alpha}_L,\ u_R \to u^\alpha_R,\ d_R \to d^\alpha_R,$$
$$D_\nu Q^{i\alpha}_L = \partial_\nu Q^{i\alpha}_L - \imath g_s G^A_\nu (t^A)^{\alpha\beta} Q^{i\beta}_L - \frac{\imath g}{2} A^a_\nu (\sigma^a)^i_j Q^{j\alpha}_L - \frac{\imath \acute{g}}{2} B_\nu (Y_L)^i_j Q^{j\alpha}_L,$$
$$D_\nu u^\alpha_R = \partial_\nu u^\alpha_R - \imath g_s G^A_\nu (t^A)^{\alpha\beta} u^\beta_R - \frac{\imath \acute{g}}{2} B_\nu Y_{Ru} u^\alpha_R,$$
$$D_\nu d^\alpha_R = \partial_\nu d^\alpha_R - \imath g_s G^A_\nu (t^A)^{\alpha\beta} d^\beta_R - \frac{\imath \acute{g}}{2} B_\nu Y_{Rd} d^\alpha_R \quad (5)$$

where $\alpha = 1, ..., 3$ labels the elements of $SU(3)$-multiplet. The quarks Lagrangian now takes the form

$$L_q = \bar{Q_L}^{i\alpha}(\imath\gamma^\nu D_\nu)^{\alpha\beta}Q_L^{i\beta} + \bar{u_R}^\alpha(\imath\gamma^\nu D_\nu)^{\alpha\beta}u_R^\beta + \bar{d_R}^\alpha(\imath\gamma^\nu D_\nu)^{\alpha\beta}d_R^\beta - \lambda_d\bar{Q}_L^{i\alpha}\Phi^i d_R^\alpha - \lambda_d\bar{d}_R^\alpha(\Phi^i)^\dagger Q_L^{i\alpha} - \lambda_u\epsilon^{ij}\bar{Q}_L^{i\alpha}\Phi^j u_R^\alpha - \lambda_u\epsilon^{ij}\bar{u}_R^\alpha(\Phi^j)^\dagger Q_L^{i\alpha}. \quad (6)$$

Due to Higgs effect Yukawa interaction terms gives the standard mass terms for quarks

$$-\frac{1}{\sqrt{2}}\lambda_d v(\bar{d}_R d_L + \bar{d}_L d_R) - \frac{1}{\sqrt{2}}\lambda_u v(\bar{u}_R u_L + \bar{u}_L u_R). \quad (7)$$

1.3. Leptons multiplets.

The leptons of all generations sit in the $SU(3)$-singlets so they do not interact with $G_\mu^A(x)$ gauge fields.

1.4. J_μ^5 *current and anomalies.*

There is an additional symmetry in the theory since the chiral fermions are present in the theory

$$\psi \to \exp{(\imath\alpha\gamma^5)}\psi \quad (8)$$

since the chiral fermions are present. This symmetry leads at the classical level to the axial current conservation low

$$\partial^\mu J_\mu^5 = 0,\ \ J_\mu^5 = \bar{\psi}\gamma^5\gamma_\mu\psi \quad (9)$$

But due to quantum anomaly this current does not conserv.

Since in electro-weak sector vector bosons interact with vector and axial currents this anomaly breaks the gauge symmetry of the theory. It may lead to nonrenormalizable theory. **But the anomaly in the theory is canceled.**

2. Georgi-Glashow model of Grand Unification.

2.1. Standard model coupling constants evolution and Grand Unification idea.

The Standard Model with gauge group $SU(3) \times SU(2) \times U(1)$ has 3 independent coupling constants g_3, g_2, g_1 such that

$$g_3 > g_2 > g_1 \tag{10}$$

at the energy level $<< m_W \approx 100GeV$. It follows from Renormalization group equations that g_3 and g_2 decrease as the energy scale grows:

$$\begin{aligned} \frac{1}{g_3^2(E)} &= \frac{1}{g_3^2(M)} + \frac{1}{16\pi^2}(11 - \frac{4n}{3}) \log{(\frac{E}{M})^2}, \\ \frac{1}{g_2^2(E)} &= \frac{1}{g_2^2(M)} + \frac{1}{16\pi^2}(\frac{22}{3} - \frac{4n}{3} - \frac{1}{6}) \log{(\frac{E}{M})^2}, \end{aligned} \tag{11}$$

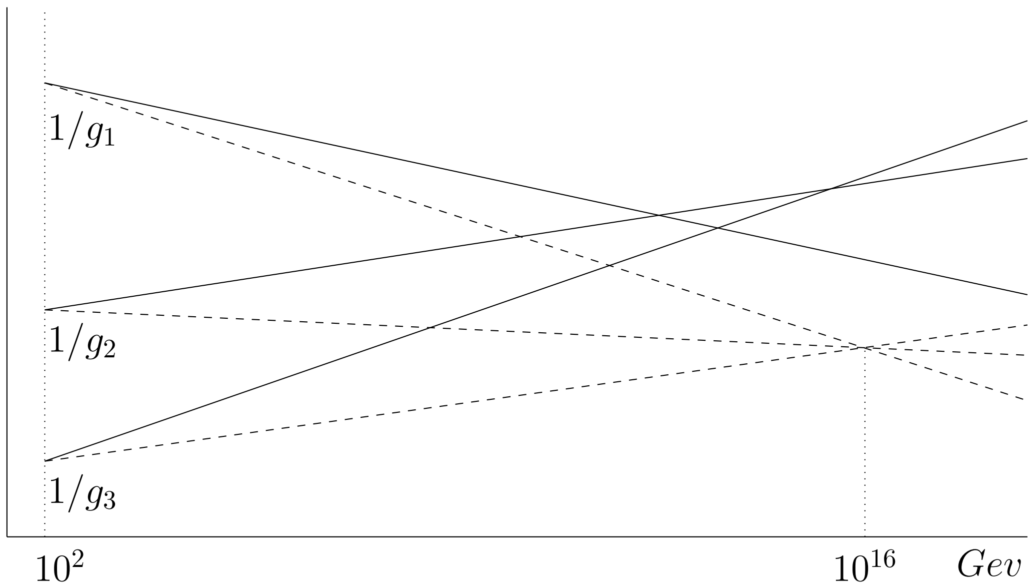
while the $U(1)$ gauge coupling goes like

$$\frac{1}{g_1^2(E)} = \frac{1}{g_1^2(M)} - \frac{1}{16\pi^2}(\frac{4n}{3} + \frac{1}{10}) \log{(\frac{E}{M})^2} \tag{12}$$

where n is the number of generations. When $n = 3$ we find that

$$\begin{aligned} \frac{1}{g_3^2(E)} &= \frac{1}{g_3^2(M)} + \frac{7}{16\pi^2} \log{(\frac{E}{M})^2} \\ \frac{1}{g_2^2(E)} &= \frac{1}{g_2^2(M)} + \frac{19}{6}\frac{1}{16\pi^2} \log{(\frac{E}{M})^2} \\ \frac{1}{g_1^2(E)} &= \frac{1}{g_1^2(M)} - \frac{41}{10}\frac{1}{16\pi^2} \log{(\frac{E}{M})^2} \end{aligned} \tag{13}$$

As the energy grows the coupling constants $g_i(E)$ approach to each other maximally close at $E \approx 10^{16}$ GeV. (though they do not intersect at one point):



(14)

One can explain this behaviour at high energies if we assume that at some energy scale around $E \approx 10^{16} GeV$ they coinside. Then at lower energies, $g_{3,2}$ will encrease because of renormalization group equations (asymptotic freedom), while g_1 will decrease, which is realy observed at low energies.

An even more interesting hypothesis whould be that **all the Standard Model gauge symmetries are the subgroups of some larger gauge symmetry group which is sponaneously broken** at large energies.

The most simple choise for such group is $SU(5)$. Then the $SU(3) \times SU(2) \times U(1)$ coupling constants ere related to the $SU(5)$ coupling constant as

$$g_5 = g_3 = g_2 = \sqrt{\frac{5}{3}} g_1 \tag{15}$$

The idea that the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ is emebeded into a large simple gauge group is called Grand

Unification. The particular $SU(5)$ choice has been proposed by Georgi and Glashow.

In fact one can modify the renormalization group equation in such a way to make the gauge couplings intersecting at some point. It can be achieved in $N=1$ **supersymmetric gauge theories.**

2.2. $SU(5)$ multiplets of Standard Model particles.

Because of the rank of the Standard Model group $SU(3)\times SU(2)\times U(1)$ is 4 the minimal Grand unification group must have rank 4. The Georgi and Glashow poposed to take the $SU(5)$ as a Grand Unification gauge group. It is $5^2-1=24$-dimensional group.

It is obvious that

$$SU(3)\times SU(2)\times U(1)\subset SU(5) \tag{16}$$

The embeding (16) is fixed by the Higgs field vacuum expectation value $<\Omega|\Phi|\Omega>$. The Higgs field is in adjoint representation of the $SU(5)$:

$$\Phi(x)=\Phi^a(x)T^a \tag{17}$$

where T^a , $a=1,...,24$ are the fundamental 5-representation matrices of the Lie agebra of the group (see Appendix 3).

The Higgs field vaccum expectation value which is invariant w.r.t. the $SU(3)\times SU(2)\times U(1)$ can be taken to be proportional to the weak hypercharge generator Y

$$< \Omega|\Phi|\Omega >= \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} v = \frac{Y}{2} v \tag{18}$$

(where we put $Y = \sqrt{\frac{5}{3}} T^{24}$

$$T^{24} = \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{19}$$

$Tr(T^{24})^2 = \frac{1}{2}$).

Let us try to sort out the fermions of Standard model in the $SU(5)$ irreducible representations.

We have the following particles

$$\begin{pmatrix} u_1 & u_2 & u_3 & \nu_e \\ d_1 & d_2 & d_3 & e \\ c_1 & c_2 & c_3 & \nu_\mu \\ s_1 & s_2 & s_3 & \mu^- \\ t_1 & t_2 & t_3 & \nu_\tau \\ b_1 & b_2 & b_3 & \tau^- \end{pmatrix} \tag{20}$$

The first generation of particles occupies the first and second rows. The second generation consists of the third and forth rows. The third generation consists of the last two rows.

Looking at the generators T^a (see Appendix 2.) it becomes clear that fundamental $SU(5)$ representation 5 is decomposed w.r.t $SU(3) \times SU(2) \times U(1)$ as

$$5 = (3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2}) \tag{21}$$

The conjugated representation decomposes similarly

$$5^* = (3^*, 1, \frac{1}{3}) \oplus (1, 2, -\frac{1}{2}) \tag{22}$$

so we see that 5^* representation of $SU(5)$ can be realized for $(\bar{d}_1, \bar{d}_2, \bar{d}_3, e, \nu_e)$.

To get the representation like $(3, 2, \frac{1}{6})$ we decompose the tensor product 5×5 into the symmetric and anti-symmetric parts:

$$\begin{aligned} 5 \times 5 = ((3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2})) \times ((3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2})) = \\ (6, 1, -\frac{2}{3})_s \oplus (3^*, 1, -\frac{2}{3})_a \oplus (3, 2, \frac{1}{6})_s \oplus (3, 2, \frac{1}{6})_a \oplus \\ \oplus (1, 3, 1)_s \oplus (1, 1, 1)_a = \\ 15_s \oplus 10_a \end{aligned} \tag{23}$$

Hence the representations we need are in 10_a representation

$$\begin{aligned} 10_a = (3^*, 1, -\frac{2}{3})_a \oplus (3, 2, \frac{1}{6})_a \oplus (1, 1, 1)_a = \\ \begin{pmatrix} 0 & \bar{u}_3 & -\bar{u}_2 & u_1 & d_1 \\ -\bar{u}_3 & 0 & \bar{u}_1 & u_2 & d_2 \\ \bar{u}_3 & \bar{u}_2 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & \bar{e} \\ -d_1 & -d_2 & -d_3 & -\bar{e} & 0 \end{pmatrix} \end{aligned} \tag{24}$$

which is the second fundamental $SU(5)$ representation. The rest particles from the first generation can be placed into

$$5^* = (\bar{d}_1, \bar{d}_2, \bar{d}_3, e, \nu_e) \tag{25}$$

Thus we see that the one generation of leptons and quarks fits into $5^* \oplus 10$**!** Other generations can be placed similarly.

We therefore define left-handed Weyl fermions in 5^* representation $\Psi^i(x)$ and a left-handed Weyl fermions in 10 representation $\Psi_{ij}(x) = -\Psi_{ji}(x)$. The covariant derivatives of these fields are given by

$$\begin{aligned} D_\mu \Psi^i = \partial_\mu \Psi^i - \imath g_5 A^a_\mu (\bar{T}^a)^i_k \Psi^k = \\ \partial_\mu \Psi^i + \imath g_5 A^a_\mu (T^a)^i_k \Psi^k, \\ D_\mu \Psi_{ij} = \partial_\mu \Psi_{ij} - \imath g_5 A^a_\mu (T^a_{10})^{kn}_{ij} \Psi_{kn} = \\ \partial_\mu \Psi_{ij} - \imath g_5 A^a_\mu ((T^a)^k_i \Psi_{kj} + (T^a)^n_j \Psi_{in}) \end{aligned} \tag{26}$$

so that the kinetic terms are

$$\imath \Psi^\dagger_i \bar{\sigma}^\mu D_\mu \Psi^i + \frac{\imath}{2} \Psi^{\dagger ij} \bar{\sigma}^\mu D_\mu \Psi_{ij}. \tag{27}$$

2.3. Quark's electric charges.

Placing the Standard model particles into $SU(5)$ multiplets one can explain the values of electric charges of quarks. Indeed, electric charge operator Q is an element of Lie algebra $su(5)$. Acting by this operator on the vectors from 5^* we find

$$Q\Psi^i = q_i \Psi^i \tag{28}$$

From the other hand we have by definition

$$0 = TrQ = q_{\bar{d}_1} + q_{\bar{d}_2} + q_{\bar{d}_3} + q_e + q_{\nu_e} \Leftrightarrow$$
$$q_{d_1} = q_{d_2} = q_{d_3} = \frac{1}{3} q_e \tag{29}$$

It explains why the charges of quarks are rational, **but does not explain why the charge of electron is quantized.**

2.4. Higgs potential for spontaneous $SU(5)$ symmetry breaking.

The most general ($\Phi \leftrightarrow -\Phi$ symmetric) renormalizable Higgs field potential which breaks $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$ takes the form

$$V(\Phi) = -\frac{\mu^2}{2} Tr(\Phi^2) + \frac{\lambda_1}{4} (Tr(\Phi)^2)^2 + \frac{\lambda_2}{4} Tr(\Phi^4) \tag{30}$$

The vacuum average (18) breaking the $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$ gives the following value of V

$$V(v) = -\frac{\mu^2}{2} \frac{5}{6} v^2 + \frac{1}{4} (\frac{25}{36} \lambda_1 + \frac{35}{216} \lambda_2) v^4 \tag{31}$$

Nonzero value of v minimazing the potential is given by

$$v^2 = \frac{36\mu^2}{30\lambda_1 + 7\lambda_2} \tag{32}$$

and the Higgs boson aqquires the mass

$$m_h^2 = \frac{5\mu^2}{3} \tag{33}$$

According to Higgs mechanism, Goldstones modes make the masses of gauge fields along the broken symmetries to be non-zero. The matrix of $SU(5)$ gauge fields is given by

$$\begin{pmatrix} G_1^1 - \frac{cB}{3} & G_1^2 & G_1^3 & \frac{X_1^1}{\sqrt{2}} & \frac{X_1^2}{\sqrt{2}} \\ G_2^1 & G_2^2 - \frac{cB}{3} & G_2^3 & \frac{X_2^1}{\sqrt{2}} & \frac{X_2^2}{\sqrt{2}} \\ G_3^1 & G_3^2 & G_3^3 - \frac{cB}{3} & \frac{X_3^1}{\sqrt{2}} & \frac{X_3^2}{\sqrt{2}} \\ \frac{X_1^{\dagger 1}}{\sqrt{2}} & \frac{X_1^{\dagger 2}}{\sqrt{2}} & \frac{X_1^{\dagger 3}}{\sqrt{2}} & \frac{W^3+cB}{2} & \frac{W^+}{\sqrt{2}} \\ \frac{X_2^{\dagger 1}}{\sqrt{2}} & \frac{X_2^{\dagger 2}}{\sqrt{2}} & \frac{X_2^{\dagger 3}}{\sqrt{2}} & \frac{W^-}{\sqrt{2}} & \frac{-W^3+cB}{2} \end{pmatrix} \tag{34}$$

where $G_1^1+G_2^2+G_3^3=0$, B is hypercharge gauge field, $\sqrt{2}W^{\pm}=W^1 \pm \imath W^2$, and W^3 are the $SU(2)$ gauge fields. Thus the fields X_α^i from $(3,2,-\frac{5}{6})$, where the index α is $SU(3)$ index while i is $SU(2)$ index, become massive with

$$M_X \approx g_5 v \approx 10^{16} Gev \tag{35}$$

The $SU(3) \times SU(2) \times U(1)$ gauge fields interact to the quarks and leptons with

$$g_3 = g_2 = \frac{\sqrt{3}}{\sqrt{5}} g_1 = g_5 \tag{36}$$

Thus instead of 3 arbitrary gauge coupling constants, we had in the Standard model, we have only one gauge coupling constant as we expected from Standard Model coupling constants evolution.

In order for fermions become massive one needs also to add Yukawa interaction terms.

2.5. Quark-lepton interaction and proton decay.

One can also find from the Lagrangian the couplings of X with quarks and leptons (from the first generation for example):

$$
\begin{aligned}
-g_5[X^{\dagger a}_{1\mu}(d^{\dagger}_a\bar{\sigma}^{\mu}e - \bar{e}^{\dagger}\bar{\sigma}^{\mu}d_a + u^{\dagger b}\bar{\sigma}^{\mu}u^c\epsilon_{abc}) + \\
X^{\dagger a}_{2\mu}(-d^{\dagger}_a\bar{\sigma}^{\mu}\nu + \bar{e}^{\dagger}\bar{\sigma}^{\mu}u_a + d^{\dagger b}\bar{\sigma}^{\mu}u^c\epsilon_{abc})] + h.c. = \\
-g_5X^{\dagger a}_{i\mu}(\epsilon^{ij}d^{\dagger}_a\bar{\sigma}^{\mu}l_j - \epsilon^{ij}\bar{e}^{\dagger}\bar{\sigma}^{\mu}q_{ja} + q^{\dagger bj}\bar{\sigma}^{\mu}u^c\epsilon_{abc}) + h.c. = \\
-g_5X^{\dagger a}_{i\mu}J^{\mu i}_a + h.c.
\end{aligned}
\tag{37}
$$

Because of quark-lepton terms in this expression the exchange of an X boson can violate baryon and lepton conservation low and can lead to a **proton decay**:

$$
\begin{aligned}
ud \xrightarrow{X} e^+\bar{u} \\
p = uud \xrightarrow{X} u\bar{u}e^+ = \pi^0e^+
\end{aligned}
\tag{38}
$$

Proton decay has not been observed. The limit on rate $\frac{1}{\tau}$ for $p \to \pi^0e^+$ decay is $\tau > 10^{33}$ years. It gives the estimation

$$
M_X > 3 \times 10^{15}Gev
\tag{39}
$$

Appendix 1. Electro-weak Theory (GWS).

2.1. Gauge group, bosonic sector and Higgs effect (reminder).

It is given by YM field theory with gauge group $SU(2)\times U(1)$ interracting with a dublet of complex scalar fields $\Phi(x) = (\phi^1(x), \phi^2(x))$ with the following rule of gauge transformations

$$\Phi(x) \rightarrow \exp\left(\imath\alpha^a(x)\frac{\sigma^a}{2} + \imath\frac{\beta(x)}{2}\right)\Phi(x),$$
$$A_\nu(x) \equiv A^a_\nu(x)\frac{\sigma^a}{2} \rightarrow U(x)A_\nu(x)U^{-1}(x) + \frac{\imath}{g}U(x)\partial_\nu U^{-1}(x),$$
$$B_\nu(x) \rightarrow B_\nu(x) + \frac{\imath}{2\acute{g}}\partial\beta(x) \tag{40}$$

where $U(x) = \exp\left(\imath\alpha^a(x)\frac{\sigma^a}{2}\right)$.

The corresponding part of the Lagrangian is given by

$$L(A, B, \Phi) = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}|D_\mu\Phi|^2,$$
$$D_\mu\Phi = \left(\partial_\mu - \imath g A^a_\mu\frac{\sigma^a}{2} - \imath\frac{\acute{g}}{2}B_\mu\right)\Phi \tag{41}$$

It is supposed that Φ acquires the vacuum expectation value

$$\Phi_0 = \frac{1}{\sqrt{2}}(0, v) \tag{42}$$

due to the self-interaction

$$V(\Phi) = -\mu^2\Phi^\dagger\Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 \tag{43}$$

so that the subgroup of matrices leaves the vacuum vector fixed:

$$\exp\left(\imath\beta(x)\left(\frac{\sigma^3+1}{2}\right)\right)\Phi_0 = \Phi_0,$$
$$\exp\left(\imath\beta(x)\left(\frac{\sigma^3+1}{2}\right)\right) \in U(1)_{em} \subset U(1)\times U(1) \subset SU(2)\times U(1) \tag{44}$$

isomorphic to $U(1)$ group. Therefore we have a massless gauge boson while 3 other bosons becomes massive. Indeed

$$D_\mu(\Phi_0 + \phi(x)) = \left(\partial_\mu - \imath g A^a_\mu\frac{\sigma^a}{2} - \imath\frac{\acute{g}}{2}B_\mu\right)(\Phi_0 + \phi(x)) =$$
$$D_\mu\phi(x) - \left(\imath g A^a_\mu\frac{\sigma^a}{2} + \imath\frac{\acute{g}}{2}B_\mu\right)\Phi_0,$$
$$\Phi^\dagger_0\left(gA^a_\mu\frac{\sigma^a}{2} + \frac{\acute{g}}{2}B_\mu\right)\left(gA^{a\mu}\frac{\sigma^a}{2} + \frac{\acute{g}}{2}B^\mu\right)\Phi_0 =$$
$$\frac{1}{2}\frac{v^2}{4}\left(g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + \acute{g}B_\mu)^2\right) \tag{45}$$

Hence, it make sense to introduce the following combinations of gauge bosons

$$
\begin{aligned}
W^{\pm}_{\mu} &= \frac{1}{\sqrt{2}}(A^1 \mp \imath A^2)_{\mu},\ m_W = \frac{gv}{2}, \\
Z_{\mu} &= \frac{1}{\sqrt{g^2+\acute{g}^2}}(gA^3 - \acute{g}B)_{\mu},\ m_Z = \sqrt{g^2+\acute{g}^2}\frac{v}{2} \\
A_{\mu} &= \frac{1}{\sqrt{g^2+\acute{g}^2}}(\acute{g}A^3 + gB)_{\mu},\ m_A = 0
\end{aligned} \tag{46}
$$

For the case of general representation of the gauge group $SU(2)\times U(1)$

$$
\begin{aligned}
&D_{\mu} = \partial_{\mu} - \imath g A^a_{\mu}T^a - \imath\acute{g}YB_{\mu} = \\
&\partial_{\mu} - \imath\frac{g}{\sqrt{2}}(W^+T^+ + W^-T^-)_{\mu} - \frac{\imath}{\sqrt{g^2+\acute{g}^2}}Z_{\mu}(g^2T^3 - \acute{g}^2Y) \\
&-\frac{\imath g\acute{g}}{\sqrt{g^2+\acute{g}^2}}A_{\mu}(T^3+Y)
\end{aligned} \tag{47}
$$

where $T^{\pm} = T^1 \pm \imath T^2$.

It is natural to identify the EM gauge potential coupling to the charge of electron

$$
e = \frac{g\acute{g}}{\sqrt{g^2+\acute{g}^2}} \tag{48}
$$

and determine the electric charge operator as

$$
Q_{em} = T^3 + Y \tag{49}
$$

It is also convenient to introduce the mixing angle Θ_W by the relation

$$
\begin{aligned}
&\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} cos\Theta_W & -sin\Theta_W \\ sin\Theta_W & cos\Theta_W \end{pmatrix}\begin{pmatrix} A^3 \\ B \end{pmatrix} \Leftrightarrow \\
&cos\Theta_W = \frac{g}{\sqrt{g^2+\acute{g}^2}},\ sin\Theta_W = \frac{\acute{g}}{\sqrt{g^2+\acute{g}^2}}
\end{aligned} \tag{50}
$$

Then we will have

$$D_\mu =$$
$$\partial_\mu - \imath\frac{g}{\sqrt{2}}(W^+T^+ + W^-T^-)_\mu - \imath\frac{g}{cos\Theta_W}Z_\mu(T^3 - sin^2\Theta_W Q_{em}) - \imath e A_\mu Q_{em},$$
$$g = \frac{e}{sin\Theta_W} \qquad (51)$$

and $m_W = m_Z cos\Theta_W$. Experimental data: $m_W = 80Gev$, $m_Z = 91Gev$, $m_H = 126Gev$ (2012).

2.2. Fermionic sector, leptons multiplets.

The leptons (which are fermions) interract to W-bosons only by the left-handed components while the right-handed components do not interract to W. Thus, the left-handed components sit at $SU(2)$ dublets:

$$\begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L, \begin{pmatrix} \nu_\mu(x) \\ \mu^-(x) \end{pmatrix}_L, \begin{pmatrix} \nu_\tau(x) \\ \tau^-(x) \end{pmatrix}_L \qquad (52)$$

Each component of the each dublet is a left-handed Weyl spinor w.r.t. Lorentz group:

$$\gamma^5 \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L = - \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L \qquad (53)$$

The upper components describe the 3 kinds of neutrino, while the bottom components describe the electron, muon and τ-lepton.

The right-handed components of leptons sit at $SU(2)$ singlets:

$$e^-_R(x),\ \mu^-_R(x),\ \tau^-_R(x) \qquad (54)$$

They are right-handed Weyl spinors:

$$\gamma^5 e^-_R(x) = e^-_R(x) \qquad (55)$$

Each left-handed dublet together with the corresponding right-handed singlet form a generation of leptons.

In what follows we concentrate on the first generation and introduce the notation

$$\begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L \equiv \begin{pmatrix} E_L^1(x) \\ E_L^2(x) \end{pmatrix}, \; E_R(x) \equiv e_R^-(x) \tag{56}$$

Then the corresponding part of the Lagrangian is given by

$$\begin{aligned} L_{lept} &= \bar{E}_L{}^i(\imath\gamma^\nu D_\nu)E_L^i + \bar{E}_R(\imath\gamma^\nu D_\nu)E_R - \lambda_e \bar{E}_L^i \Phi^i E_R - \lambda_e \bar{E}_R(\Phi^i)^\dagger E_L^i, \\ D_\nu E_L^i &= \partial_\nu E_L^i - \frac{\imath g}{2} A_\nu^a (\sigma^a)_j^i E_L^j - \frac{\imath \acute{g}}{2} B_\nu (Y_L)_j^i E_L^j, \\ D_\nu E_R &= \partial_\nu E_R - \frac{\imath \acute{g}}{2} B_\nu Y_R E_R \end{aligned} \tag{57}$$

where $i, j = 1, 2$ and λ_e is a coupling constant leading to the masses of leptons (λ_e is renormalizable constant so that it is a parameter of the model). Due to the vacuum average (42) the leptons get masses

$$\begin{aligned} \Delta L_{lept} &= -\lambda_e \bar{E}_L^i \Phi^i E_R - \lambda_e \bar{E}_R (\Phi^i)^\dagger E_L^i = -\frac{\lambda_e v}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L) ... \Rightarrow \\ m_e &= \frac{\lambda_e v}{\sqrt{2}} \end{aligned} \tag{58}$$

Notice that we get the standard mass term for fermions $-\frac{\lambda_e v}{\sqrt{2}}\bar{e}e$ **so that** $e_L \to e_R$ **transition is recovered!**

One could add the standard mass term for leptons directly but it would destroy the gauge invariance:

$$\begin{aligned} E_L(x) &\to \exp\left(\imath\alpha^a(x)\frac{\sigma^a}{2} + \imath\beta(x)Y_L\right)E_L(x), \\ E_R(x) &\to \exp\left(\imath\beta(x)Y_R\right)E_R(x) \end{aligned} \tag{59}$$

Thus, **the Dirac's electron is a superposition of** e_L **and** e_R **which are completely different particles from the point of view of weak forces!**

To define Y_L and Y_R we use the relation (49). In the fundamental $SU(2)$-representation, which is used for the left-handed leptons

$$T^3 = \frac{1}{2}\sigma^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{60}$$

Hence,

$$Y_L = Q_{em} - T^3 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{61}$$

Right-handed leptons are singlets, that is $T^3 = 0$ but e_R^-, μ_R^-, τ_R^- have $Q_{em} = -1$ and hence

$$Y_R = -1 \tag{62}$$

2.3. Quarks multiplets and Lagrangian.

Quarks are included into GWS model similarly to the leptons:

$$Q_1 = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}_L, \; Q_2 = \begin{pmatrix} c(x) \\ s(x) \end{pmatrix}_L, \; Q_3 = \begin{pmatrix} t(x) \\ b(x) \end{pmatrix}_L \tag{63}$$

Each component of the each dublet is a left-handed Weyl spinor w.r.t. Lorentz group.

$$\gamma^5 \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}_L = - \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}_L \tag{64}$$

The right-handed components of quarks sit at $SU(2)$ singlets:

$$u_R(x),\; d_R(x),\; c_R(x),\; s_R(x),\; t_R(x),\; b_R(x) \tag{65}$$

The quadruples (Q_1, u_R, d_R)**,...,**(Q_3, t_R, b_R) **form a generations of quarks.**

The Lagrangian is given similar to (57). For the first quarks generation the Lagrangian is

$$L_q = \bar{Q_L}^i(\imath\gamma^\nu D_\nu)Q_L^i + \bar{u_R}(\imath\gamma^\nu D_\nu)u_R + \bar{d_R}(\imath\gamma^\nu D_\nu)d_R -$$
$$\lambda_d\bar{Q}_L^i\Phi^i d_R - \lambda_d\bar{d}_R(\Phi^i)^\dagger Q_L^i - \lambda_u\epsilon^{ij}\bar{Q}_L^i\Phi^j u_R - \lambda_u\epsilon^{ij}\bar{u}_R(\Phi^j)^\dagger Q_L^i,$$
$$D_\nu Q_L^i = \partial_\nu Q_L^i - \frac{\imath g}{2}A_\nu^a(\sigma^a)_j^i Q_L^j - \frac{\imath \acute{g}}{2}B_\nu(Y_L)_j^i Q_L^j,$$
$$D_\nu u_R = \partial_\nu u_R - \frac{\imath \acute{g}}{2}B_\nu Y_{Ru}u_R,$$
$$D_\nu d_R = \partial_\nu d_R - \frac{\imath \acute{g}}{2}B_\nu Y_{Rd}d_R. \tag{66}$$

By the Higgs effect the quarks get the following masses:

$$\Delta L_q = -\lambda_d\bar{Q}_L^i\Phi^i d_R - \lambda_d\bar{d}_R(\Phi^i)^\dagger Q_L^i - \lambda_u\epsilon^{ij}\bar{Q}_L^i\Phi^j u_R - \lambda_u\epsilon^{ij}\bar{u}_R(\Phi^j)^\dagger Q_L^i =$$
$$-\frac{\lambda_d v}{\sqrt{2}}(\bar{d}_L d_R + \bar{d}_R d_L) - \frac{\lambda_u v}{\sqrt{2}}(\bar{u}_L u_R + \bar{u}_R u_L)... \Rightarrow$$
$$m_d = \frac{\lambda_d v}{\sqrt{2}},\ m_u = \frac{\lambda_u v}{\sqrt{2}}, \tag{67}$$

where $\lambda_{u,d}$ are renormalizable constants so that they are the parameters of the model.

Now we find

$$Y_L = Q_{em} - T^3 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix} \tag{68}$$

The values of u, d quarks electric charges follow from the fact that proton has electric charge $+1$ and it is a bound state of two u-quarks and one d-quark (in order to be colorless), while neutron has electric charge 0 and it is a bound state of two d-quarks and one u-quark:

$$p = \epsilon_{\alpha\beta\gamma}u^\alpha u^\beta d^\gamma,\ n = \epsilon_{\alpha\beta\gamma}d^\alpha d^\beta u^\gamma \tag{69}$$

(Recall also the electric charge quantization phenomenon.)

For the right-handed quarks we find

$$Y_R = Q_{em} = (\frac{2}{3}, -\frac{1}{3}) \tag{70}$$

Appendix 2. Gell-Mann matrices.

Gell-Mann matrices:

$$t^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; t^2 = \begin{pmatrix} 0 & -\imath & 0 \\ \imath & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; t^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$t^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; t^5 = \begin{pmatrix} 0 & 0 & -\imath \\ 0 & 0 & 0 \\ \imath & 0 & 0 \end{pmatrix}, \; t^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$t^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\imath \\ 0 & \imath & 0 \end{pmatrix}, \; t^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{71}$$