Lecture 6.

Higgs effect in non abelian Gauge Theory.

Plan.

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1. Yang-Mills theory. Gauge invariance. Lagrangian.

1.1. Nonabelean generalization of gauge symmetry. $SU(2)$ example.

Instead of the $U(1)$ group with gauge transformation rule for Dirac fermion $\psi(x) \to \exp(\imath\alpha(x))\psi(x)$, one can consider nonabelian group, $SU(2)$ and instead of Dirac fermion one can consider a dublet of Dirac fermions $\Psi(x) = (\psi^1(x), \psi^2(x))$ with the gauge action determined by

$$\Psi(x) \to \exp(\imath \sum_i \alpha_k \frac{\sigma_k}{2})\Psi(x) \equiv V(x)\Psi(x),\ V(x) \in SU(2) \tag{1}$$

where σ_k are the Pauli matrices.

Let us define the parallel transport operator from the point x to the point y by a unitary matrix $U(y,x)$ satisfying the equations

$$\begin{gathered} U(y,x) \to V(y)U(y,x)V^\dagger(x), \\ U(x,x) = 1,\ U(y,x)U(x,y) = 1 \end{gathered} \tag{2}$$

for each gauge transformation $V(x) \in SU(2)$. Then, in the case the point

y is close to the point x we can write

$$\begin{aligned} y &= \epsilon n(x) + x, \\ U(\epsilon n(x)+x,x) &= 1 + \imath g \epsilon n^\mu(x) A^k_\mu(x)\frac{\sigma_k}{2} + ... \end{aligned} \quad (3)$$

Using (28) one can find the gauge field $A_\mu(x)$ transformation rule

$$\begin{aligned} U(\epsilon n(x)+x,x) = 1 + \imath g \epsilon n^\mu(x) A^k_\mu(x)\frac{\sigma_k}{2} + ... \Rightarrow \\ V(\epsilon n(x)+x)(1 + \imath g \epsilon n^\mu(x) A^k_\mu(x)\frac{\sigma_k}{2})V^{-1}(x) + ... = \\ 1 + \imath g \epsilon n^\mu(x) \tilde{A}^k_\mu(x)\frac{\sigma_k}{2} + ... \Leftrightarrow \\ \tilde{A}^k_\mu(x)\frac{\sigma_k}{2} = V(x)A^k_\mu(x)\frac{\sigma_k}{2}V^{-1}(x) + \frac{\imath}{g}V(x)\partial_\mu V^{-1}(x) \end{aligned} \quad (4)$$

Define the covariant derivative

$$n^\mu D_\mu \Psi(x) \equiv lim_{\epsilon\to 0}\frac{1}{\epsilon}(\Psi(x+\epsilon n(x)) - U(\epsilon n(x)+x,x)\Psi(x)) \quad (5)$$

It then gives

$$D_\mu\Psi(x) = (\partial_\mu - \imath g A^k_\mu(x)\frac{\sigma_k}{2})\Psi(x) \quad (6)$$

Under the gauge transformations $\Psi(x) \to V(x)\Psi(x)$, $A_\mu(x) \to V(x)A_\mu V^{-1}(x) + \frac{\imath}{g}V(x)\partial_\mu V^{-1}(x)$ we find that

$$\begin{aligned} D_\mu\Psi(x) = (\partial_\mu - \imath g A^k_\mu(x)\frac{\sigma_k}{2})\Psi(x) \to \\ (\partial_\mu - \imath g(VA^k_\mu(x)\frac{\sigma_k}{2}V^{-1} + \frac{\imath}{g}V\partial_\mu V^{-1}))V(x)\Psi(x) = \\ VD_\mu\Psi(x) \end{aligned} \quad (7)$$

In order to find kinetic term for the gauge field consider the parallel transport operator along small rectangle $x \to x+\epsilon e_1 \to x+\epsilon e_1+\epsilon e_2 \to x+\epsilon e_2 \to x$, generated by the vectors $e_{1,2}$. Using the property $U(z,y)U(y,x) =$

$U(z, x)$ we find

$$U(x \to x) = U(x, x + \epsilon e_1)U(x + \epsilon e_1, x + \epsilon e_1 + \epsilon e_2)$$
$$U(x + \epsilon e_1 + \epsilon e_2, x + \epsilon e_2)U(x + \epsilon e_2, x) =$$
$$1 - \imath g^2 \epsilon^2 (e_1)^\mu (e_2)^\mu F_{\mu\nu}(x) + ...$$
$$F_{\mu\nu} = \imath g[D_\mu, D_\nu] = \partial_\nu A_\mu - \partial_\mu A_\nu + \imath g[A_\mu, A_\nu] \tag{8}$$

One can check that under the gauge transformations $V(x)A^k_\mu(x)\frac{\sigma_k}{2}V^{-1}(x) + \frac{\imath}{g}V(x)\partial_\mu V^{-1}(x)$

$$F_{\mu\nu} \to V(x)F_{\mu\nu}(x)V^{-1} \tag{9}$$

It follows from Lorentz invariance, gauge invariance, locality and renormalizibility that the only possible Lagrangian is

$$L(A, \Psi) = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\Psi}\imath\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi \tag{10}$$

1.2. Nonabelean gauge theory in general case.

$$\Psi(x) = (\psi^1(x), ...\psi^n(x)),$$
$$\Psi(x) \to V(x)\Psi(x), V(x) \in U(n),$$
$$V(x) = \exp(\imath\alpha^a(x)t^a),\ [t^a, t^b] = \imath f^{abc}t^c,$$
$$D_\mu = \partial_\mu - \imath g A^a_\mu t^a \tag{11}$$

2. Higgs effect in $SU(2)$ YM theory.

2.1. $SU(2)$-dublet scalar fields.

Gauge symmetry transformations for $SU(2)$- dublet scalar fields $\Phi(x) = (\phi^1(x), \phi^2(x))$:

$$\Phi(x) \to U(x)\Phi(x),\ U(x) = \exp(\imath\alpha^a(x)\frac{\sigma^a}{2}). \tag{12}$$

The potential for Φ is

$$V(\Phi) = -\mu^2\Phi^\dagger\Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2,$$
$$\textit{at the point } \Phi_0 = \frac{1}{\sqrt{2}}(0, v), \ \frac{\partial V}{\partial \phi^i} = 0 \tag{13}$$

The covariant derivative is

$$D_\mu \Phi(x) = (\partial_\mu - \imath g A^a_\mu \frac{\sigma^a}{2})\Phi(x) \tag{14}$$

Then

$$\begin{aligned}
\Phi(x) \equiv \Phi_0 + \phi(x), \\
|D_\mu\Phi|^2 = [(D_\mu\phi - \imath g A_\mu \Phi_0)^\dagger (D^\mu\phi - \imath g A^\mu \Phi_0)] = \\
|D_\mu\phi|^2 - \imath g((D_\mu\phi)^\dagger A^\mu \Phi_0) + \imath g((\Phi_0^\dagger A_\mu^\dagger) D^\mu \phi) + \\
g^2(\Phi_0^\dagger A_\mu^\dagger A^\mu \Phi_0) = \\
|D_\mu\phi|^2 - \imath g((D_\mu\phi)^\dagger A^\mu \Phi_0) + \imath g((\Phi_0^\dagger A_\mu^\dagger) D^\mu \phi) + \\
\frac{g^2 v^2}{8} A^a_\mu A^{a\mu} \Rightarrow \\
m_A = \frac{gv}{2}
\end{aligned} \tag{15}$$

There are no massless gauge bosons because there is no nontrivial $SU(2)$ subgroup leaving the vacuum Φ_0 fixed.

2.2. $SU(2)$-triplet of real scalar fields.

Gauge symmetry transformations for $SU(2)$- triplet of real scalar fields $\Phi(x) = (\phi^1(x), \phi^2(x), \phi^3)$:

$$\Phi(x) \to U(x)\Phi(x), \ U(x) = \exp(\imath \alpha^a(x) t^a) \tag{16}$$

The covariant derivative is

$$D_\mu \phi^a(x) = \partial_\mu \phi^a + g\epsilon^{abc} A^b_\mu \phi^c(x) \tag{17}$$

The potential for Φ is

$$V(\Phi) = -\mu^2\phi^a\phi^a + \frac{\lambda}{2}(\phi^a\phi^a)^2,$$
$$\textit{and at the point } \Phi_0 = (0,0,v),\ \frac{\partial V}{\partial\phi^a} = 0. \tag{18}$$

Then

$$\Phi(x) \equiv \Phi_0 + \varphi(x), \tag{19}$$

Then

$$D_\mu\phi^a D^\mu\phi^a = D_\mu\varphi^a D^\mu\varphi^a + 2g\epsilon^{abc}D^\mu\varphi^a A^b_\mu\Phi^c_0 + g^2v^2(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}) \Rightarrow m_{1,2} = \sqrt{2}gv,\ m_3 = 0 \tag{20}$$

$< \phi^3 >= v$ leaves the symmetry of rotations around the direction 3 unbroken but brokes the other symmetries from $SU(2)$.

2.3. GWS model (part 1).

It is given by YM field theory with gauge group $SU(2)\times U(1)$ interacting with a dublet of complex scalar fields $\Phi(x) = (\phi^1(x), \phi^2(x))$ with the following rule of gauge transformations

$$\Phi(x) \to \exp{(\imath\alpha^a(x)\frac{\sigma^a}{2} + \imath\frac{\beta(x)}{2})}\Phi(x) \tag{21}$$

Suppose that Φ acquires the vacuum expectation value

$$\Phi_0 = \frac{1}{\sqrt{2}}(0, v). \tag{22}$$

It is easy to find the subgroup of matrices which leaves the vacuum vector fixed:

$$\exp{(\imath\beta(x)(\frac{\sigma^3+1}{2}))}\Phi_0 = \Phi_0 \tag{23}$$

(This is a $U(1)$ subgroup). Therefore we have one massless gauge boson while 3 other bosons becomes massive. Indeed

$$D_\mu(\Phi_0+\phi(x)) = (\partial_\mu - \imath g A^a_\mu \frac{\sigma^a}{2} - \imath\frac{\acute{g}}{2}B_\mu)(\Phi_0+\phi(x)) =$$
$$D_\mu\phi(x) - (\imath g A^a_\mu\frac{\sigma^a}{2} + \imath\frac{\acute{g}}{2}B_\mu)\Phi_0,$$
$$\Phi_0^\dagger(gA^a_\mu\frac{\sigma^a}{2}+\frac{\acute{g}}{2}B_\mu)(gA^{a\mu}\frac{\sigma^a}{2}+\frac{\acute{g}}{2}B^\mu)\Phi_0 =$$
$$\frac{1}{2}\frac{v^2}{4}(g^2(A^1_\mu)^2+g^2(A^2_\mu)^2+(-gA^3_\mu+\acute{g}B_\mu)^2) \tag{24}$$

(notice that we have two independent coupling constants g and $\acute{g}$ because of the gauge group is not semi-simple). It make sense to introduce the following combinations of gauge bosons

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1 \mp \imath A^2)_\mu,\ m_W = \frac{1}{gv},$$
$$Z_\mu = \frac{1}{\sqrt{g^2+\acute{g}^2}}(gA^3-\acute{g}B)_\mu,\ m_Z = \sqrt{g^2+\acute{g}^2}\frac{v}{2}$$
$$A_\mu = \frac{1}{\sqrt{g^2+\acute{g}^2}}(\acute{g}A^3+gB)_\mu,\ m_A = 0 \tag{25}$$

For the case of general representation of the gauge group $SU(2)\times U(1)$

$$D_\mu = \partial_\mu - \imath g A^a_\mu T^a - \imath\acute{g}YB_\mu =$$
$$\partial_\mu - \imath\frac{g}{\sqrt{2}}(W^+T^+ + W^-T^-)_\mu - \frac{\imath}{\sqrt{g^2+\acute{g}^2}}Z_\mu(g^2T^3-\acute{g}^2Y)$$
$$-\frac{\imath g\acute{g}}{\sqrt{g^2+\acute{g}^2}}A_\mu(T^3+Y) \tag{26}$$

where $T^\pm = T^1 \pm \imath T^2$.

It is natural to identify the EM gauge potential coupling to the charge of electron

$$e = \frac{g\acute{g}}{\sqrt{g^2+\acute{g}^2}} \tag{27}$$

and determine the electric charge operator as

$$Q = T^3 + Y \tag{28}$$

It is also convenient to introduce the mixing angle Θ_W by the relation of two basic fields

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} cos\Theta_W & -sin\Theta_W \\ sin\Theta_W & cos\Theta_W \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix} \Leftrightarrow$$

$$cos\Theta_W = \frac{g}{\sqrt{g^2 + \acute{g}^2}}, \ sin\Theta_W = \frac{\acute{g}}{\sqrt{g^2 + \acute{g}^2}} \tag{29}$$

Then we will have

$$D_\mu = \partial_\mu - \imath \frac{g}{\sqrt{2}}(W^+T^+ + W^-T^-)_\mu - \imath \frac{g}{cos\Theta_W} Z_\mu (T^3 - sin^2\Theta_W Q) - \imath e A_\mu Q,$$
$$g = \frac{e}{sin\Theta_W} \tag{30}$$

Experimental data: $m_W = 80 Gev$, $m_Z = 91 Gev$, $m_H = 126 Gev$ (2012). Thus the coupling constants g, $\acute{g}$ introduced initialy are related to the electric charge e and mixing angle Θ_W, which in turn are given by m_W and m_Z.