Lecture 4.

Spontaneously broken symmetries, Goldstone Theorem.

Plan.

1. Sponataneously broken symmetries and Goldsone theorem.

1.1. Sponatneously broken $\mathbb{Z}_2$-symmetry in scalar ϕ^4 model.

Let us consider the simplest case of one scalar field with the lagrangian

$$L(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

Now we change the mass term $m^2 \to -\mu^2$, $\mu^2 > 0$

$$L(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2}$$

One can rewrite the potential adding a constant (which is not important)

$$L(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}(\phi^2 - \frac{6\mu^2}{\lambda})^2 \tag{3}$$

and consider the hamiltonian

$$H = \int d^3x[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4] \approx$$
$$\int d^3x[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{\lambda}{4!}(\phi^2 - \frac{6\mu^2}{\lambda})^2] \tag{4}$$

The lagrangian and hamiltonian are invariant under the $\mathbb{Z}_2$-symmetry chage $\phi \to -\phi$. The classical minimal energy configurations are given by the homogeneous configurations:

$$\phi_0(t,\vec{x}) = \pm v = \pm\sqrt{\frac{6}{\lambda}}\mu \tag{5}$$

Hence, the $\phi_0 =< \phi >$ can be considered as a nonzero vacuum averege which sponatneously breaks $\mathbb{Z}_2$-symmetry $\phi \to -\phi$. (The vacuum is unique, $\phi = 0$ when $\mu^2 < 0$).

Near the vacuum configuration one can introduce the fluctuating field $\sigma(x)$:

$$\phi(x) = v + \sigma(x) \tag{6}$$

and substitute this parametrization into the lagrangian:

$$L = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{2\mu^2}{2}\sigma^2 - \sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4 \tag{7}$$

We get the massive scalar field lagrangian with the mass $\sqrt{2}\mu$ and self-interraction terms σ^3 and σ^4. The relic of broken $\mathbb{Z}_2$-symmetry is the relation between the constants in the last 3 terms.

Notice that in quantum mechanics we would have a tunneling between the vaccua (5) but it is not in QFT.

1.2. Sponataneously broken $U(1)$-symmetry.

It is interesting to see what happens if the continuous symmetry is broken. Let us consider the symplest case of $U(1)$ symmetry. Suppose that $U(1)$ symmetry is acting on the complex scalar field:

$$\phi(x) = \phi^1(x) + \imath\phi^2(x) \to \exp(\imath\alpha)\phi(x) \tag{8}$$

Then the Lagrangian (12) can be rewritten in the form

$$L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \bar{\phi}) + \frac{\mu^2}{2}\phi\bar{\phi} - \frac{\lambda}{4}(\phi\bar{\phi})^2 \tag{9}$$

The classical minimum energy configuration is given by

$$\frac{\partial V}{\partial \phi} = 0 \Leftrightarrow \ \phi\bar{\phi} = \frac{\mu^2}{\lambda} \tag{10}$$

If we choose some particular vacuum configuration

$$\phi_0 = \phi_0^1 + \imath\phi_0^2 \tag{11}$$

the only subgroup of $U(1)$ leaving this ϕ_0 fixed is the trivial one ($O(1)$). Hence we conclude that the **whole $U(1)$ is spontaneously broken** and the set of vacuum configurations can be represented as the coset space $U(1)/1 = S^1$ which is nothing else but a circle.

1.3. Sponataneously broken $O(N)$-symmetry in linear σ-model.

The case of continuous symmetry breaking is more interesting and important. Let us consider the generalization of the previous Lagrangian

$$L = \frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a) + \frac{\mu^2}{2}\phi^a\phi^a - \frac{\lambda}{4}(\phi^a\phi^a)^2 \tag{12}$$

The Hamiltonian (energy) is

$$\int d^3x[\frac{1}{2}\dot{\phi}^a\dot{\phi}^a + \frac{1}{2}(\nabla\phi^a)^2 - \frac{\mu^2}{2}(\phi^a\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a)^2] \tag{13}$$

where, $a = 1, ...N$. The Lagrangian and Hamiltonian are invariant w.r.t. $O(N)$-transformations

$$\phi^a(x) \to R^{ab}\phi^b(x), a, b = 1, ..., N \tag{14}$$

In $D > 2$ dimensional space, the classical minimum energy configurations are given by the homogeneous fields ϕ_0^a minimazing the potential energy

$$\frac{\partial V}{\partial \phi^a} = -\frac{\mu^2}{2}2\phi^a + \frac{\lambda}{2}(\phi^b\phi^b)^2 2\phi^a = 0,$$

$$\Leftrightarrow \phi_0^a\phi_0^a = \frac{\mu^2}{\lambda} \Leftrightarrow \phi_0 \in S^{N-1} \tag{15}$$

Thus, we have a set of vacuum configurations parametrizing the the points of $N-1$-dimensional sphere. One can choose obviously the coordinates in such way that

$$\phi_0 = (0, ..., v), \ v = \frac{\mu}{\sqrt{\lambda}} \tag{16}$$

so that the remaining unbroken symmetry is given by $O(N-1)$-rotations leaving the vector (16) fixed. In other words, the set of vaccua is given by the coset

$$O(N)/O(N-1) \approx S^{N-1} \tag{17}$$

Near the vacuum configuration (16) one can introduce the set of fluctuating fields

$$\phi(x) = \phi_0 + (\pi^1(x), ...\pi^{N-1}(x), \sigma(x)) \tag{18}$$

Then the lagrangian (12) takes the form

$$L = \frac{1}{2}(\partial_\mu \pi^a)^2 + \frac{1}{2}(\partial_\mu \sigma)^2$$
$$-\frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\sigma(\pi^a)^2 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\sigma^2(\pi^a)^2 - \frac{\lambda}{4}((\pi^a)^2)^2 \tag{19}$$

We get a set of $N-1$ massless scalar fields π^a and massive field σ interracting to each other and themselvs. Initial $O(N)$ symmetry is broken to $O(N-1)$ acting by rotations on the fields π^a. These fields describe fluctuations along $O(N)/O(N-1)$ manifold of vaccua, while $\sigma(x)$ describes fluctuations in radial direction.

1.4. Sponataneously broken $SU(2)$-symmetry in fundamental representation.

One can generalize the $U(1)$ example above for the case of $SU(2)$ group. Suppose we have the dublet of complex scalar fields $\Phi(x) = (\Phi^1(x), \Phi^2(x))$ which are in fundamental representation of $SU(2)$:

$$\Phi(x) \to U\Phi(x), \ U = \exp(t^a \tau^a) \in SU(2) \tag{20}$$

The Lagrangian is given by

$$L = \frac{1}{2}(\partial_\mu \Phi^\dagger)(\partial^\mu \Phi) + \frac{\mu^2}{2}\Phi^\dagger \Phi - \frac{\lambda}{4}(\Phi^\dagger \Phi)^2 \tag{21}$$

The classical minimum energy configuration is given by

$$\frac{\partial V}{\partial \Phi} = (\frac{\lambda}{2}\Phi^\dagger \Phi - \frac{\mu^2}{2})\Phi^\dagger = 0 \Leftrightarrow \ \Phi^\dagger \Phi = \frac{\mu^2}{\lambda} \tag{22}$$

If we choose some particular vacuum configuration

$$\Phi_0 = (\Phi_0^1, \Phi_0^2) \tag{23}$$

the only subgroup of $SU(2)$ leaving this Φ_0 fixed is the trivial one. Hence we conclude that the **whole $SU(2)$ is spontaneously broken in this case**.

1.5. Sponataneously broken $SU(2)$-symmetry in adjoint representation.

Let us consider the triplet of real scalar fields $\phi(x) = (\phi^1(x), ..., \phi^3(x)) = \phi(x) \equiv \phi^a \tau^a$ which are in adjoint representation of $SU(2)$:

$$\phi(x) \to U^{-1}(x)\phi(x)U(x), \ U(x) = \exp(t^a \tau^a) \in SU(2) \tag{24}$$

In this case the adjoint action of the group is equivalent to the $SO(3)$ rotations of the vectors in $\mathbb{R}^3$. Thus, we are in the situation of $O(N)$ example considered above where $N = 3$. The minimum energy configuration is given by (15) where the particular vacuum solution is given by (16). Hence, **in the adjoint representation we have unbroken $U(1) \approx O(2)$ subgroup of $SU(2)$**.

1.6. Goldstone's theorem.

Goldsone's theorem states that each spontaneously broken symmetry must lead to the massless particle called Goldstone's boson.

It can be proven looking at the Lagrangian of a number of scalar fields $\phi^a(x)$

$$L = f(\partial_\mu \phi^a) + V(\phi) \tag{25}$$

Suppose that ϕ_0 is the local minimum of $V(\phi)$:

$$\frac{\partial V}{\partial \phi^a}|_{\phi=\phi_0} = 0 \tag{26}$$

Then, near ϕ_0 we can expand

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^a(\phi - \phi_0)^b(\frac{\partial^2 V}{\partial \phi^a \partial \phi^b})|_{\phi_0} + ... \tag{27}$$

The second derivatives term is a symmetric matrix whose eigenvalues are equal to the masses of fluctuating modes near the minimum ϕ_0:

$$(\frac{\partial^2 V}{\partial \phi^a \partial \phi^b})|_{\phi_0} = m^2_{ab} \tag{28}$$

They are nonegative because of ϕ_0 is a minimum of the potential. One needs to show that each continuous symmetry of the Lagrangian, shifting the minimum ϕ_0 corresponds to zero eigenvalue of the matrix (28).

The symmetry transformation with parameter t can be written as

$$\phi^a \to \phi^a + t\epsilon^a(\phi) \tag{29}$$

Because of we are looking for minimal energy configurations it is natural to think they are given by **homogeneous fields**, so that derivative terms in the lagrangian vanish. Then $V(\phi)$ must be invariant:

$$\epsilon^a(\phi)\frac{\partial V}{\partial \phi^a} = 0 \Rightarrow$$

$$0 = \frac{\partial}{\partial \phi^b}(\epsilon^a(\phi)\frac{\partial V}{\partial \phi^a})|_{\phi_0} = \frac{\partial \epsilon^a}{\partial \phi^b}|_{\phi_0}\frac{\partial V}{\partial \phi^a}|_{\phi_0} + \epsilon^a(\phi_0)(\frac{\partial^2 V}{\partial \phi^a \partial \phi^b})|_{\phi_0} = \epsilon^a(\phi_0)(\frac{\partial^2 V}{\partial \phi^a \partial \phi^b})|_{\phi_0} \tag{30}$$

For the transformations leaving the vacuum ϕ_0 fixed ($\epsilon^a(\phi_0) = 0$) we have in the theory an unbroken symmetry subgroup ($O(N-1)$). By the definition, for the spontaneously broken symmetry $\epsilon^a(\phi_0) \neq 0$, hence, this vector is zero-eigenvalue vector and corresponds to the massless particle.