Lecture 2.

Gauge invariance, paths integral Quantization of Gauge theories.

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1. Paths integral Quantization of Electromagnetic field.

1.1. Paths integral and Gauge invariance, problems with propagator.

Similar to the paths integrals for scalar field and Dirac's field it is natural to consider the following paths integral

$$I = \int DA \exp\,[-\imath \int d^4x \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)]. \qquad (1)$$

Because of the gauge symmetry the action $S[A] = 0$ for the pure gauge fields $A_\mu = \frac{1}{e}\partial_\mu \alpha(x)$ so the paths integral is badly determined. It causes also the problem with propagator. Indeed, one can write

$$S = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} A_\mu(k)(-g^{\mu\nu}k^2 + k^\mu k^\nu) A_\nu(k). \qquad (2)$$

Then for a purely gauge configurations $A_\mu(k) = k_\mu \alpha(k)$

$$k_\mu \alpha(k)(-g^{\mu\nu}k^2 + k^\mu k^\nu) k_\nu \alpha(k) = 0. \qquad (3)$$

It means that the equation determining the propagator

$$(-g_{\mu\nu}k^2 + k_\mu k_\nu)D_F^{\nu\lambda}(k) = \imath\delta_\mu^\lambda \Leftrightarrow$$
$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)D^{\nu\lambda}(x-y) = \imath\delta_\mu^\lambda\delta^4(x-y) \qquad (4)$$

does not have a solution because **the matrix** $(-g_{\mu\nu}k^2 + k_\mu k_\nu)$ **is not invertible**. It could seem that the problem can be cured if one defines the paths integral in phase space, but this is not the case.

1.2. Gauge orbits and gauge fixing.

The gauge invariance problem means that the physical configurations of $A_\mu(x)$ are the gauge orbit classes

$$\tilde{A}_\mu(x) = \{A_\mu(x) \approx A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)\}, \qquad (5)$$

so that **we must integrate in (1) over the space of gauge orbits** $\tilde{A}_\mu(x)$ **instead of the integration over the gauge potentials** $A_\mu(x)$ **themselves**.

How to do it?

One can try to do it picking up in the gauge orbit classes a representative and integrate over these representatives. **This procedure is called a gauge fixing**.

Let us try to determine some surface in the space of gauge fields $A_\mu(x)$ which is given by the equation

$$G(A(x)) = 0. \qquad (6)$$

Ideally, this surface must intersects each class $\tilde{A}(x)$ only at one point. But the question is whether this is possible? **In what follows we will assume that it can be done locally**.

Thus, the paths integral must be equipped with the insertion

$$\prod_x \delta(G(A(x))) \qquad (7)$$

1.3. Faddeev-Popov trick.

By the definition of δ-function

$$1 = \int [DG(A^\alpha(x))]\delta(G(A^\alpha(x))) = \int [D\alpha(x)] det(\frac{\delta G(A^\alpha(x))}{\delta\alpha(x)})\delta(G(A^\alpha(x))) \quad (8)$$

where $A^\alpha(x) = A(x) + \frac{1}{e}\partial_\mu \alpha(x)$ is the gauge orbit of the field $A(x)$. Gauge fixing functional $G(A(x))$ can be considered as introducing some coordinates along the gauge orbits. From the other hand, gauge parameters $\alpha(x)$ can be used as natural coordinates along the gauge orbits. Thus, the second equality in (8) appears due to the change of variables on the gauge orbit:

$$G(A^\alpha(x)) \to \alpha(x) \quad (9)$$

The finite dimensional version of the relation (8) is

$$1 = \int \prod_i d\alpha_i det(\frac{\partial g_j}{\partial \alpha_k})\delta(g(\alpha)). \quad (10)$$

After the insertion (8) into the paths integral we obtain

$$I = \int [DA][D\alpha] \exp{[-\imath \int d^4x \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)]} det(\frac{\delta G(A^\alpha)}{\delta\alpha})\delta(G(A^\alpha)) \quad (11)$$

One can do the inverse gauge transformation of gauge field $A(x)$ and use the invariance of the action and the mesure $[DA^{-\alpha}] = [DA]$:

$$I = \int [D\alpha] \int [DA] \exp{[-\imath \int d^4x \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)]} det(\frac{\delta G(A)}{\delta\alpha})\delta(G(A)) = Vol(G) \int [DA] \exp{[-\imath \int d^4x \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)]} det(\frac{\delta G(A)}{\delta\alpha})\delta(G(A)), \quad (12)$$

where $Vol(G)$ is a volume of the group of gauge transformations.

Let us consider Lorentz gauging

$$G(A(x)) = \partial_\mu A^\mu(x). \tag{13}$$

Then

$$det(\frac{\delta G(A^\alpha)}{\delta \alpha}) = det(\frac{1}{e}\partial^2),$$

$$I = Vol(G)det(\frac{1}{e}\partial^2)\int [DA] \exp[-\imath \int d^4x \frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x)]\delta(\partial_\mu A^\mu)) \tag{14}$$

Thus, we obtain a path integral that takes into account only the classes of gauge orbits (at least locally).

One can generalize the Lorentz gauging:

$$G(A(x)) = \partial_\nu A^\nu - f(x) \tag{15}$$

where $f(x)$ is an arbitrary function:

$$I = Vol(G)det(\frac{1}{e}\partial^2)\int [DA] \exp[-\imath \int d^4x \frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x)]\delta(\partial_\mu A^\mu - f(x)) \tag{16}$$

It is helpful to consider the following superposition of the integrals (16)

$$I = N(\xi)\int [Df] \exp[-\imath \int d^4x \frac{f^2}{2\xi}] Vol(G)det(\frac{1}{e}\partial^2)$$
$$\int [DA] \exp[-\imath \int d^4x \frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x)]\delta(\partial_\mu A^\mu - f(x)). \tag{17}$$

Integrating over $f(x)$ we obtain

$$I = N(\xi)Vol(G)det(\frac{1}{e}\partial^2)$$
$$\int [DA] \exp[-\imath \int (d^4x \frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2\xi}(\partial_\mu A^\mu)^2]. \tag{18}$$

As one can see the gauge degrees of freedom enter the path integral as a factor so that they can be absorbed by the normalization factor.

1.4. ξ-gauge, correlation functions and photon's propagator.

Now one can consider paths integral representation for Green's functions of gauge invariant operators

$$< \Omega|T\hat{O}_1(A)...\hat{O}_n(A)|\Omega >= \\ lim_{T\to\infty(1-\imath\epsilon)} \frac{\int[DA]O_1(A)...O_n(A)\exp\left[\imath\int_{-T}^{T}d^4x(L_{EM}-\frac{1}{2\xi}(\partial_\mu A^\mu)^2)\right]}{\int[DA]\exp\left[\imath\int_{-T}^{T}d^4x(L_{EM}-\frac{1}{2\xi}(\partial_\mu A^\mu)^2)\right]} \quad (19)$$

The gauge invariance condition for $O(A)$ is important in order to the nominator be gauge invariant.

It gives the foton propagator because **one can inverse the differential operator from this new ξ-dependent action for** $A_\mu(x)$:

$$-g_{\mu\nu}k^2+(1-\frac{1}{\xi})k_\mu k_\nu)D_F^{\nu\sigma}(k)=\imath\delta_\mu^\sigma \Leftrightarrow \\ D_F^{\mu\nu}(k)=-\frac{\imath}{k^2+\imath\epsilon}(g^{\mu\nu}-(1-\frac{1}{\xi})\frac{k_\mu k_\nu}{k^2}) \quad (20)$$

2.1. Gauge symmetry in non-Abelian case.

The gauge group of QED is $U(1)$:

$$\psi(x)\to\exp\left[\imath\alpha(x)\right]\psi(x),\ A_\mu(x)\to A_\mu(x)+\frac{1}{e}\partial_\mu\alpha(x) \quad (21)$$

Faddeev-Popov trick in this case gives just a constant factor $Vol(\hat{G})det(\frac{1}{e}\partial^2)$. For the non-Abelian groups the situation is more interesting.

Non-Abelian group gauge transformations, covariant derivatives and gauge invariant action:

$$A_\mu(x) \equiv A^a_\mu t_a \to U^{-1}(x)A_\mu(x)U(x) + U^{-1}(x)\frac{\imath}{g}\partial_\mu U(x), U(x) \in SU(N),$$
$$D_\mu \equiv \partial_\mu + \imath g A_\mu \to U^{-1}(x)D_\mu U(x),$$
$$F_{\mu\nu}(x) \equiv D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \imath g[A_\mu, A_\nu] \to U^{-1}(x)F_{\mu\nu}U(x) \Rightarrow$$
$$S_{YM} = -\frac{1}{4}\int d^4x Tr(F_{\mu\nu}F^{\mu\nu}) \quad (22)$$

2.2. Paths integral, gauge fixing and F-P ghosts.

Let us consider the paths integral for non-Abelian YM gauge fields

$$I = \int [DA] \exp\left[-\frac{\imath}{4}\int d^4x Tr(F_{\mu\nu}F^{\mu\nu})\right]. \quad (23)$$

The natural metric in the space of YM fields is given by:

$$ds^2 = \int d^4x d^4y \delta(x-y) Tr(\delta A_{1\mu}(x)\delta A^\mu_2(y)). \quad (24)$$

Thus, the mesure in the paths integral is

$$[DA] = \prod_x \prod_{a,\mu} dA^a_\mu(x). \quad (25)$$

As in the Abelian case, we have the problem of integrating over the classes of gauge orbits, so we need to fix the gauge $G(A(x))$ by some of the ways and use Faddeev-Popov trick. As a result we find

$$I = \int [DA] \exp\left[-\frac{\imath}{4}\int d^4x Tr(F_{\mu\nu}F^{\mu\nu})\right] =$$
$$Vol(\hat{G}) \int [DA] \exp[\imath S_{YM}] \delta(G(A)) det(\frac{\delta G(A^\alpha)}{\delta\alpha}) \quad (26)$$

Now we choose Lorentz gauging

$$G^a(A) = \partial^\mu A^a_\mu. \quad (27)$$

For an infinitesimal gauge transformation $U = 1 + \alpha(x) = 1 + \alpha^a(x)t^a$

$$\delta A_\mu = \frac{1}{g}\partial_\mu \alpha + \imath[A_\mu, \alpha] \Leftrightarrow$$
$$\delta A^a_\mu = \frac{1}{g}\partial_\mu \alpha^a + f^{abc}A^b_\mu \alpha^c. \tag{28}$$

Then

$$\delta G^a(A) = \partial^\mu(\frac{1}{g}\partial_\mu \alpha^a) + f^{abc}\partial^\mu(A^b_\mu \alpha^c) =$$
$$\frac{1}{g}\partial^\mu(\delta^{ac}\partial_\mu + gf^{abc}A^b_\mu)\alpha^c = \frac{1}{g}\partial^\mu D^{ac}_\mu \alpha^c$$
$$\Leftrightarrow$$
$$\frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{g}\partial^\mu D_\mu. \tag{29}$$

We already know that in a finite-dimensional space the matrix determinant can be represented as a result of integration over Grassmann variables

$$\int \prod_i d\theta_i d\theta^*_i \exp(-\theta^\dagger A\theta) = detA. \tag{30}$$

Here, one can use the Grassmann variables $c^a(x)$, $\bar{c}^a(x)$ (Faddeev-Popov ghosts) to write

$$det(\frac{\delta G(A^\alpha)}{\delta\alpha}) = \int [Dc][D\bar{c}] \exp\left[-\imath \int d^4x \bar{c}^a(x)\partial^\mu D^{ab}_\mu c^b(x)\right] =$$
$$\int [Dc][D\bar{c}] \exp\left[-\imath \int d^4x \bar{c}^a(x)(\partial^\mu\partial_\mu c^a + gf^{abc}\partial^\mu(A^b_\mu c^c))\right] =$$
$$\int [Dc][D\bar{c}] \exp\left[-\imath \int d^4x L_{gh}(x)\right] \tag{31}$$

where the ghost's Lagrangian is

$$L_{gh} = \partial^\mu \bar{c}^a \partial_\mu c^a - gf^{abc}(\partial^\mu \bar{c}^a)c^b A^c_\mu. \tag{32}$$

The ghost's fields are Lorentz scalars but they are in adjoint representation of the gauge group G and interact with the gauge potential.

Similarly to the EM fields one can consider more general gauge fixing

$$G^a(A) = \partial^\mu A^a_\mu(x) - f^a(x) \tag{33}$$

and integrate out over the all possible $f^a(x)$ with the weight $\exp[-\imath \int d^4x \frac{f^2}{2\xi}]$ so that one gets

$$I = \int [DA][Dc][D\bar{c}] \exp[\imath \int d^4x (-\frac{1}{4} Tr(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2\xi} Tr((\partial^\mu A_\mu)^2) - L_{gh})]. \tag{34}$$

The integral allows to extract propagator for the gauge fields

$$< \Omega|T A^a_\mu(x) A^b_\nu(y)|\Omega >= \int \frac{d^4k}{(2\pi)^4} \frac{-\imath\delta^{ab}}{k^2 + \imath\epsilon} (g_{\mu\nu} - (1 - \frac{1}{\xi}) \frac{k_\mu k_\nu}{k^2}) \exp(-\imath k(x-y)). \tag{35}$$

As well as the ghosts fields propagators

$$< \Omega|T c^a(x) \bar{c}^b(y)|\Omega >= \int \frac{d^4k}{(2\pi)^4} \frac{\imath\delta^{ab}}{k^2 + \imath\epsilon} \exp(-\imath k(x-y)). \tag{36}$$

2.3. BRST symmetry and physical states in Gauge theory.

The integral (34) possesses some important fermionic symmetry called BRST-symmetry.

Let us introduce additional auxiliary field $B^a(x)$

$$I = \int [DA][Dc][D\bar{c}][DB] \exp[\imath S[A, c, \bar{c}, B]],$$

$$S[A, c, \bar{c}, B] = \int d^4x [-\frac{1}{4} Tr(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2\xi} Tr(B^2) + Tr(B\partial^\mu A_\mu) - \partial^\mu \bar{c}^a (\partial_\mu c^a - g f^{abc} c^b A^c_\mu)]. \tag{37}$$

Because of the field $B(x)$ is not dynamical it can be integrated out from the paths integral above giving back the integral (34). But the auxiliary field B allows us to see the symmetry of the action $S[A, c, \bar{c}, B]$ under the transformations generated by Grassmann (fermionic) parameter ϵ:

$$\delta A_\mu^a = \epsilon (D_\mu c)^a, \ \delta c^a = -\frac{1}{2} g \epsilon f^{abc} c^b c^c, \ \delta \bar{c}^a = \epsilon B^a, \\ \delta B^a = 0. \tag{38}$$

Let us define the BRST operator Q acting on an arbitrary field $\Phi(x)$ from the set $A, c, \bar{c}, B$ by the formula

$$\epsilon Q \Phi(x) = \epsilon \delta \Phi(x) \Leftrightarrow \\ Q A_\mu^a = \epsilon (D_\mu c)^a, \ Q c^a = -\frac{1}{2} g \epsilon f^{abc} c^b c^c, \ Q \bar{c}^a = \epsilon B^a, \\ Q B^a = 0. \tag{39}$$

One can check that

$$Q^2 = 0. \tag{40}$$

That is Q is nilpotent operator. Therefore in the space of states generated by the creation operators of fields $A(x), c(x), \bar{c}(x), B(x)$ one can define the BRST cohomology by the rule

$$\mathcal{H} = \frac{KerQ}{ImQ}. \tag{41}$$

They are the functions of physical states in YM field theory.