Lecture 12.

Universality of critical behaviour. WF fixed point.

Plan.

1. Scaling relations and universality class.

1.1. LG approximation and Curie critical point.

It is possible to establish direct relation between the singular behavior of thermodynamic quantities near critical points and fixed points of the Wilson's RG.

We consider a magnet. The long-range fluctuations of local magnetization $M(x)$ can be taken into account by considering the statistical integral

$$Z = \int [D\phi_0] \exp\left[-A_0[\phi_0]\right], \tag{1}$$

where

$$\phi_0(x) = C(T)M(x) \tag{2}$$

with some temeperature-dependent coefficient. It is assumed that the functional integral has the cutoff parameter Λ such that

$$\Lambda^{-1} >> \textit{ atomic scale } a \tag{3}$$

The action A_0 combines statistical contributions of all microscopic fluctuations and depend significantly on all details of the microscopic interactions. One can take Landau-Ginzburg approximation

$$A_0 = \int d^d x (\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0(T)^2}{2}\phi_0^2 + \frac{\lambda_0(T)}{4!}\phi_0^4) \tag{4}$$

where $m_0(T)^2$, $\lambda_0(T)$ are unknown functions of the temerature T, but any other combinations of powers of ϕ_0 and its derivatives can be added to the action. Because of all coefficients are functions of T, we have one parametric family of actions Σ describing physical system, with all its microscopic details, depending on the temperature. This one dimensional manifold will be called "physical curve".

Let us assume that there is a fixed point A_* out there in Σ. At this point we actually will consider a subspace $\Sigma_{sym} \in \Sigma$ which contains only actions invariant under $\phi_0 \to -\phi_0$. We assume that there is a fixed point $A_* \in \Sigma_{sym}$ such that it has only one (symmetric) relevant field Φ_0 with dimension $D_0 < d$. In other words, the unstable manifold in Σ_{sym} is one -dimensional and correspondingly the critical surface Σ_{crit} has codimension one. This means that physical curve generally intersects Σ_{crit} at some point $T = T_c$.

The large distance properties of the system can be understood if we can follow the RG flow which starts at any point on the physical curve. Let us consider a typical pattern of the RG flow in this situation. Besides the critical fixed point A_* there are two non-critical ($R_c = 0$) fixed points P_h and P_l. At the point P_h we have the probability distribution in the form

$$\prod_x \delta(\phi_0(x)) \tag{5}$$

At the point P_l we have

$$\prod_x (\delta(\phi_0(x) - \bar{\phi}) + \delta(\phi_0(x) + \bar{\phi})) \tag{6}$$

A trajectory starting at any point on the physical curve at $T > T_c$ and far enough from T_c, converges quickly to the fixed point P_h. Similarly, starting at any point with $T < T_c$ one quickly reaches P_l.

The only trajectory which behaves qualitatively different way is the one that starts at the point T_c itself. This trajectory C lays entirely at the critical surface and converges to the critical point A_* as $l \to -\infty$. **This means that the large-distance physics exactly at the critical temperature is completely determined by the fixed-point action A_*.** In particular, the two-point correlation function will have the scale-invariant form

$$< M(x)M(y) > |_{T=T_c} = \frac{Const}{|x-y|^{2D_\phi}} \tag{7}$$

for $|x-y| >> a \exp(l_0)$, where l_0 is some amount of the RG "time" needed to arrive closely to A_* and D_ϕ is the dimension of ϕ at the fixed point.

1.2. RG trajectory behavior near the fixed point and critical exponents.

Consider now the trajectory C_τ which starts at some point T very close to T_c,

$$\tau = \frac{T_c - T}{T_c} << 1. \tag{8}$$

Let us assume, for example that $T < T_c$. There are 3 distinct stages of the RG evolution for such trajectory:

1. it goes very close to the surface Σ_{crit} approaching A_*, untill it comes very close this fixed point;

2. it then stays near A_* for a very long RG time l;

3. it departs towards P_l staying close to the unstable manifold U.

The stage 2 is most important one, that is where the critical singularities are formed. Because at this stage the trajectory stays close to the fixed point, we can use linear analysis.

Let us assume that in some finite amount l_0 of RG time the trajectory C_τ comes close to A_* where the corresponding action can be written as

$$A = A_* + y_0 \int d^d x \Phi_0(x) + \sum_{\alpha \neq 0} y_\alpha \int d^d x \Phi_\alpha(x) \tag{9}$$

where y_0 depends on τ. **This dependence is analytical because it comes from the integrating the RG flow equations for finite time l_0.** As for $\tau = 0$ one must stay at the critical surface, which in the linear approximation corresponds to $y_0 = 0$, for small τ, we can assume

$$y_0 = Const(T_c - T) \tag{10}$$

and neglect the τ-dependence of all y_α, for $\alpha \neq 0$.

In the second stage the RG flow is described by the equations

$$\frac{dy_0}{dl} = k_0 y_0, \ \frac{dy_\alpha}{dl} = k_\alpha y_\alpha \tag{11}$$

where

$$k_0 = d - D_0 > 0,\ k_\alpha = d - D_\alpha < 0,\ for\ \alpha \neq 0. \tag{12}$$

During this second stage all irrelevant terms in (9) dye out, while the coefficient y_0 increases as

$$y_0(l) = y_0 \exp\left[k_0(l - l_0)\right]. \tag{13}$$

Recall that if R_c is the correlation lenght corresponding to the initial action on the physical curve, the correlation lenght corresponding to the time l along the trajectory is

$$R_c(l) = R_c \exp\left(-l\right). \tag{14}$$

From this and (13) one can finds

$$R_c \approx y_0^{-\nu},\ \nu = \frac{1}{k_0} = \frac{1}{d - D_0}, \tag{15}$$

and from the linear dependence (10):

$$R_c \approx |T_c - T|^{-\nu} \tag{16}$$

Thus, **the correlation length is diverges near T_c, with the exponent determined by the dimension D_0 of the relevant field Φ_0**. This divergence is in full agreement with the idea that is **the large scale fluctuations which are responsible for the critical behavior**.

Similar way one can relate other exponents to the dimensions of various fields at the fixed point A_*. For example, the corresponding magnetization

$$\bar{M} \approx < \phi_0 > \tag{17}$$

must satisfy the CS equation, which in this case reads

$$(D_\phi - k_0 y_0 \frac{\partial}{\partial y_0})\bar{M} = 0, \Leftrightarrow \bar{M} \approx y_0^{\frac{D_\phi}{k_0}}. \tag{18}$$

Again, taking into account (10) we find ($T < T_c$)

$$\bar{M} \approx (T_c - T)^\beta,\ \beta = \frac{D_\phi}{d - D_0}. \tag{19}$$

Note that (16), (18) can be obtained by simple dimensional counting, if one assumes that the coupling constants y_0 has the dimension k_0 and the field ϕ_0 has dimension D_ϕ. This follows from the CS equation with the linear approximation for the RG functions β. In fact it is valid beyond the linear approximation, as it follows from the existence of the canonical coordinates.

As another example of this dimensional counting let us find some other critical exponent in terms of dimensions. If, at $T = T_c$, one adds an external field

$$H \int d^d x \phi_0(x) \tag{20}$$

the dimensional counting implies that H has dimension $d - D_\phi$, and so the magnetization as the function of H must have the following singular behavior

$$\bar{M} \approx H^{\frac{1}{\delta}},\ \delta = \frac{d - D_\phi}{D_\phi} \tag{21}$$

(as it comes from the corresponding CS equation

$$(D_\phi - (d - D_\phi) H \frac{\partial}{\partial H})\bar{M} = 0). \tag{22}$$

Similarly, one finds specific heat at $H = 0$

$$C \approx |T_c - T|^{\alpha}, \alpha = 2 - d\nu \tag{23}$$

Note that all these exponents are expressed through just 2 numbers, the dimensions of the fields ϕ_0 and Φ_0. Therefore no matter what the values of this dimensions are, **the critical exponents must obey certain scaling relations**, like (23).

1.3. Universality class of critical behavior.

Note that the above analysis relies on the properties of the fixed point A_*, as well as on general topological properties of RG. **Very little indeed depends on the details of the original actions on the physical curve, and hence on the detailes of the microscopic structure of the magnet**. One expects therefore that the most important characteristics of the critical singularities must not be different for phase transitions in different magnetic materials. One fixed point may represent the criticality in the whole class of physical systems. It is said **it describes a universality class of critical behavior**.

This all shows that understandingthe fixed points is of crucial importance both in statistical mechanics and in QFT.

2. Wilson-Fisher fixed point.

2.1. Geometry of RG flow near A_{*G} and A_{*WF} fixed points.

We have studied one explicit example of the fixed point-the gaussian fixed point corresponding to free massless scalar field. It is easy to argue, however, that the **gaussian fixed point cannot be responsible for the ordinary transition in a magnet in 3 dimensions**. Indeed, it was very important for the above arguments that the critical surface of

A_* has codimension 1 in Σ_{Sym}, i.e. there is only 1 relevant field which respects the symmetry ($\phi \to -\phi$). However, the gaussian fixed point has more that 1 symmetric relevant fields for $d < 4$. In particular, for $d = 3$ the fields ϕ^2, ϕ^4 are relevant. Therefore it is not the gaussian fixed point that describes universality class of the magnetic phase transition. The best theoretical evidence for its existance was given by Wilson and Fisher.

At $d < 4$ the topology of the RG flow relating the fixed points A_{*G} and A_{*WF} is likely to be the following. The unstable manifold $U(A_{*G})$ at $d < 4$ has dimension 2. It is generated by 2 relevant fields ϕ^2 and ϕ^4. The fixed point A_{*WF} is likely to lay in $U(A_{*G})$, so that the RG flow on this unstable manifold would look qualitatively as follows.

There is a trajectory C going along the critical surface Σ_c from the fixed point A_{*G} towards A_{*WF}. Therefore, A_{*G} lays on the critical surface Σ_c which also passes through A_{*WF}. The unstable manifold $U(A_{*WF})$ is then 1-dimensional.

The whole unstable manifold $U(A_{*G})$ in this picture, being an element of the $\Sigma(\infty)$, represent local field theories- ϕ^4 field theories. These theories can be described by 2 parameters, the renormalized mass parameter M^2 and the renormalized coupling constant λ. These parameters, can be taken therefore as the coordinates in the above picture. Although there is an ambiguity in the choice of these coordinates (the scheme dependence), the critical surface corresponding to $M^2 = 0$ is defined by the scheme-independent condition

$$\Gamma^2(p^2)|_{p^2=0} = 0. \tag{24}$$

In these coordinates the fixed point A_{*WF} lays at $M^2 = 0$ and $\lambda = \lambda_*$, where the value of λ_* depends on the scheme.

One can consider the ϕ^4 theory treating the dimensions of space d as a continous parameter.

What happens when d approaches 4? We know that at $d = 4$ the field ϕ^4 becomes marginally irrelevant and for $d > 4$ it is strictly irrelevant, so that for $d \geq 4$ the gaussian fixed point has only one relevant field ϕ^2. This behavior can be explained if one assumes that when d approaches 4 from below the fixed point A_{*WF} gets very close to the gaussian fixed point and at $d = 4$ these 2 points merge. At $d > 4$ λ_* becomes negative and it does not correspond to field theory in usual sense.

This idea leads to the patterns of the RG flow at $d = 4$ and at $d > 4$ is shown in the pictures.

Wilson and Fisher **have assumed that if one take $d = 4 - \epsilon$ with small ϵ the fixed point A_{*WF} can be observed by the renormalized perturbation theory**.

2.2. Renormalized perturbation theory at $d = 4 - \epsilon$.

We start with the bare action

$$A = \int d^d x (\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4),$$
$$[\phi_0] = \frac{d-2}{2} = 1 - \frac{\epsilon}{2},\ [\lambda_0] = 4 - d = \epsilon > 0. \tag{25}$$

and rewrite this action in terms of renormalized parameters:

$$A = \int d^d x (\frac{1}{2}(\partial\phi)^2 + \frac{M^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta M^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4), \tag{26}$$

where the counterterms are determined from suitably chosen normalization conditions.

It makes sense to recall here that for $d < 4$ the diagrams describing the field and the coupling constant renormalizations do not diverge, the only divergent renormalization is the one of the mass parameter. Therefore the counterterms δZ, and $\delta\lambda$ have finite limit when the cutoff parameter $\Lambda \to \infty$. **Therefore at $d < 4$ these renormalizations are introduced not for the purpose of exterminating the divergences, which are absent, but in order to introduce more suitable coordinate λ on the $U(A_{*G})$ instead of λ_0.**

As the fixed point A_{*WF} is expected to be located on the critical surface Σ_c, it suffices to study the massless ϕ^4 theory which is defined by the condition

$$\Gamma^2(p^2)|_{p^2=0} = 0. \tag{27}$$

As before, choose also the normalization conditions at some normalization scale μ:

$$\begin{aligned} \frac{d}{dp^2}\Gamma^2(p^2)|_{p^2=\mu^2} &= 1, \\ \Gamma^4(p_1, ..., p_4)|_{p_{12}^2=p_{13}^2=p_{23}^2=...=4\mu^2} &= \lambda. \end{aligned} \tag{28}$$

The first one fixes the normalization of ϕ while the second gives the definition of the renormalized coupling constant λ.

Now we write CS equation

$$(x\frac{\partial}{\partial x} + ND(\lambda) + \beta(\lambda)\frac{\partial}{\partial\lambda}) < \phi(x_1)...\phi(x_N) >_\lambda = 0. \tag{29}$$

where $D(\lambda) = \frac{d-2}{2} + \gamma(\lambda)$, and $\gamma(0) = 0$, because canonical dimension of ϕ is $(d-2)/2$. **We are going to determine the RG functions $\beta(\lambda)$ and $\gamma(\lambda)$ for $d < 4$.**

2.3. Calculation of $\gamma(\lambda)$ and $\beta(\lambda)$ at $d < 4$.

Since the RG flow along the massless part of $U(A_{*G})$ (towards the point A_{*WF} along Σ_c) we have

$$\frac{d}{dl}\lambda = -\beta(\lambda). \tag{30}$$

We use perturbation theory to calculate the correlation functions as a series in λ and then find β and γ from the CS equation (29) itself.

It is useful to introduce $\bar{\lambda}$:

$$\lambda = \mu^{4-d}\bar{\lambda} \tag{31}$$

Then by dimensional counting the N-point correlation function has the form

$$< \phi(x_1)...\phi(x_N) >_\lambda = \mu^{N\frac{d-2}{2}} C^N(\mu x_1, ..., \mu x_N|\bar{\lambda}) \tag{32}$$

and therefore the CS eq. takes the form

$$(\mu\frac{\partial}{\partial\mu} + N\bar{\gamma}(\bar{\lambda}) + \bar{\beta}(\bar{\lambda})\frac{\partial}{\partial\bar{\lambda}}) < \phi(x_1)...\phi(x_N) >_{\bar{\lambda}} = 0 \tag{33}$$

where $\bar{\beta}$ and $\bar{\gamma}$ are the dimensionless functions defined by

$$\beta(\lambda) = \mu^\epsilon\bar{\beta}(\frac{\lambda}{\mu^\epsilon}),\ \gamma(\lambda) = \bar{\gamma}(\frac{\lambda}{\mu^\epsilon}) \tag{34}$$

In what follows we will omit all these bars. The CS equation can be written for proper vertices in momentum space

$$(\mu\frac{\partial}{\partial\mu} + N\gamma(\lambda) + \beta(\lambda)\frac{\partial}{\partial\lambda})\Gamma^N(p_1, ..., p_N) = 0. \tag{35}$$

We can now repeat the calculations of the leading contributions to the proper vertices within dimensional regularization method and obtain

$$\Gamma^2 = p^2 + O(\lambda^2),$$

$$\Gamma^4(p_1, ..., p_4) = \mu^{4-d}(\lambda - \frac{1}{2}\frac{\lambda^2}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})\Gamma^2(\frac{d}{2}-1)}{\Gamma(d-2)}((p_{12}^2)^{\frac{d-4}{2}} + (p_{13}^2)^{\frac{d-4}{2}} + (p_{14}^2)^{\frac{d-4}{2}} - 3(2\mu)^{d-4}) + O(\lambda^3))(36)$$

where the counterterm is fixed by the condition

$$\Gamma^4(p_1, ..., p_4)|_{p_{12}^2 = p_{13}^2 = p_{23}^2 = ... = 4\mu^2} = \mu^{4-d}\lambda. \tag{37}$$

Substitution of (36) into CS eq. (35) we find

$$\gamma(\lambda) = O(\lambda^2),$$

$$\beta(\lambda) = (d-4)\lambda + 3\frac{\lambda^2}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(3-\frac{d}{2})\Gamma^2(\frac{d}{2}-1)}{\Gamma(d-2)} + O(\lambda^3) \tag{38}$$

The term $(d-4)\lambda$ represents the canonical dimension of the coupling constant λ in the action (26).

2.4. WF fixed point.

Let us assume that $d = 4 - \epsilon$ for small ϵ, then we can write

$$\beta(\lambda) = -\epsilon\lambda + 3\frac{\lambda^2}{(4\pi)^2} + O(\lambda^3). \tag{39}$$

For small ϵ the RG flow equation

$$\frac{d}{dl}\lambda = -\beta(\lambda) \tag{40}$$

has nontrivial fixed point

$$\lambda_* = \frac{(4\pi)^2}{3}\epsilon. \tag{41}$$

The higher order corrections to the $\beta(\lambda)$ for small ϵ can shift the position of this point

$$\lambda_* = \frac{(4\pi)^2}{3}\epsilon + O(\epsilon^2) \tag{42}$$

but these corrections cannot destroy the fixed point. The point (41) is called the **Wilson-Fisher fixed point.**

2.5. ϵ-expansion and critical exponents.

The main assumption of the approach known as the ϵ-expansion is that the **general features of the RG flow observed at small ϵ remain qualitatively the same at $\epsilon = 1$, i.e. at $d = 3$.** Under this asuumption, the characteristics of the WF fixed point can be calculated perturbatively, as the power series of ϵ.

To find the critical exponents one must calculate the dimensions of various relevant fields at the WF point. Note that $\Phi_0 = \phi^2$ is the only symmetric relevant field at A_{*WF}. The dimensions of the fields Φ_0 and ϕ at this point can be calculated as

$$D_0 = D_{\phi^2} = d - 2 + \gamma_{\phi^2}(\lambda_*),\ D_\phi = \frac{d-2}{2} + \gamma(\lambda_*) \tag{43}$$

In the leading order in ϵ we can just borrow the results from the $d = 4$ calculation

$$\begin{aligned} \gamma_{\phi^2}(\lambda) &= \frac{\lambda}{(4\pi)^2} + O(\lambda^2), \\ \gamma(\lambda) &= \frac{1}{12}\frac{\lambda^2}{(4\pi)^2} + O(\lambda^2) \end{aligned} \tag{44}$$

and substitute $\lambda = \lambda_*$ from (42). This yelds the critical exponents

$$\frac{1}{\nu} = d - D_0 = 2 - \frac{\epsilon}{3} + ...,$$
$$\alpha = 2 - \frac{d}{d - D_0} = \frac{\epsilon}{6} + ... \tag{45}$$

For $\epsilon = 1$ this gives $\nu = 0.60$, $\alpha = 0.16$ to be compared with $\nu = 0.63$, $\alpha = 0.12$ found in experiment.