## Lecture 11.

## Gaussian fixed point and CS equation.

### Plan.

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*1.1. RG transformation of composite fields.*

Recall the quasilocal action

$$A = \int d^d x [\sum_{k=1}^{\infty} \frac{u_{2k}}{2k!} \phi_0^{2k} + \sum_{k=1}^{\infty} \frac{v_{2k}}{2k!} \phi_0^{2k} (\partial\phi_0)^2 + ...] \tag{1}$$

The variation

$$\begin{aligned} \delta A = \int d^d x [\sum_{k=1}^{\infty} \frac{\delta u_{2k}}{2k!} \phi_0^{2k} + \sum_{k=1}^{\infty} \frac{\delta v_{2k}}{2k!} \phi_0^{2k} (\partial\phi_0)^2 + ...] = \\ = \int d^d x \sum_{\alpha} \delta\lambda_\alpha O_\alpha(x), \\ where\ O_\alpha = \{\phi_0^{2n}, \phi_0^{2n} (\partial\phi_0)^2, ...\}, O_\alpha \in T\Sigma|_A \equiv \mathcal{F} \end{aligned} \tag{2}$$

To find the RG action on composite fields we consider the corr. function

$$< O_\alpha(x) ... >= \frac{1}{Z} \int [D\phi_0] (O_\alpha(x) ...) \exp{(-A_0[\phi_0])} \tag{3}$$

Now we perform the step 1 of RG transformation by splitting the field

$\phi_0(x)$ into its slow and fast parts:

$$\phi_0 = \phi_1 + \tilde{\phi} \Rightarrow$$

$$O_\alpha(\phi_0, \partial\phi_0, ...) = \sum_\beta Y_\alpha^\beta(\tilde{\phi}, \partial\tilde{\phi}, ...) O_\beta(\phi_1, \partial\phi_1, ...) \tag{4}$$

and integrating the fast part $\tilde{\phi}$ out. Therefore

$$\begin{aligned} &\frac{1}{Z}\int_\Delta [D\tilde{\phi}][O_\alpha(\phi_1 + \tilde{\phi}, ...)...] \exp(-A_0[\phi_1 + \tilde{\phi}]) = \\ &\frac{1}{Z_1}[\sum_\beta y_\alpha^\beta(L) O_\beta(\phi_1, \partial\phi_1, ...)...] \exp(-A_1[\phi_1]) \end{aligned} \tag{5}$$

where $\Delta = \{\Lambda_1 < |k| < \Lambda_0\}$,

$$Z_0 = C\int_{|k|<\Lambda_1} [D\phi_1] \exp[-A_1[\phi_1]] = CZ_1 \tag{6}$$

$$y_\alpha^\beta(L) =< Y_\alpha^\beta(\tilde{\phi}, \partial\tilde{\phi}, ...) >_\Delta,\ L = \frac{\Lambda_0}{\Lambda_1} \tag{7}$$

Thus, **in the step 1, any composite field $O_\alpha(x)$ gets replaced by a linear combination of composite fields, with $L$-dependent coefficients.**

At the step 2

$$x \to \frac{x}{L},\ \Lambda_1 \to \Lambda_0$$

$$\begin{aligned} &\phi_1(x) = z^{-\frac{1}{2}}\phi_0(\frac{x}{L}), \\ &\sum_\beta y_\alpha^\beta(L) O_\beta(z^{-\frac{1}{2}}(L)\phi_0(\frac{x}{L}), z^{-\frac{1}{2}}(L)\frac{\partial}{\partial\frac{x}{L}}\phi_0(\frac{x}{L}), ...) = \\ &\sum_\beta z_\alpha^\beta(L) O_\beta(\phi_0(\frac{x}{L}), \frac{\partial}{\partial(\frac{x}{L})}\phi_0(\frac{x}{L}), ...) \end{aligned} \tag{8}$$

We have assumed here that $O_\alpha$ is homogeneous polynom of $\phi_0$ of some dimensions $d_\alpha$ so that

$$z_\alpha^\beta(L) = y_\alpha^\beta z^{-\frac{d_\alpha}{2}} \tag{9}$$

Therefore, under the full RG transformation the composite field $O_\alpha(x)$ gets replaced by the linear combination of composite fields with $L$-dependent coefficients:

$$< O_\alpha(x) ... >_{(A_0,\Lambda_0)} = < \sum_\beta z_\alpha^\beta(L) O_\beta(\frac{x}{L}) ... >_{(A_1,\Lambda_1)} \Leftrightarrow$$
$$RG_l O_\alpha|_{A_0} = \sum_\beta z_\alpha^\beta(L) O_\beta(\frac{x}{L})|_{A_1}, \; L = \exp{(l)} \tag{10}$$

*1.2. RG transformation of composite fields at fixed point.*

Let us assume that

$$A_0 = A_* \tag{11}$$

It means that $RG_{l_1+l_2}A_* = RG_{l_1}RG_{l_2}A_* = A_*$, therefore

$$z_\alpha^\beta(l_1+l_2) = z_\alpha^\gamma(l_2) z_\gamma^\beta(l_1) \Leftrightarrow$$
$$\frac{d}{dl} z_\alpha^\beta(l) = -D_\alpha^\gamma z_\gamma^\beta(l), \; D_\alpha^\gamma = -\frac{d}{dl} z_\alpha^\gamma|_{l=0} \tag{12}$$

The coefficients $D_\alpha^\gamma$ are the matrix elements of the infinitesimal transformation

$$RG_{\delta l} = 1 - \delta l D + ... \tag{13}$$

One can diagonalize the operator $D$ introducing special linear combinations $\Phi_\alpha$ of $O_\alpha$:

$$D\Phi_\alpha = D_\alpha \Phi_\alpha \Rightarrow RG_l \Phi_\alpha = \exp{(-lD_\alpha)}\Phi_\alpha \Rightarrow$$
$$< \Phi_{\alpha_1}(x_1) ... \Phi_{\alpha_n}(x_n) > |_{A_*} =$$
$$L^{-D_{\alpha_1} - ... D_{\alpha_n}} < \Phi_{\alpha_1}(\frac{x_1}{L}) ... \Phi_{\alpha_n}(\frac{x_n}{L}) > |_{A_*} \tag{14}$$

*1.3. K-operator eigenspaces at fixed point.*

At the fixed point one can relate the operator $D$ to the operator $K$ determining $RG$-action on the space of quasilocal actions at the fixed point. Recall that by definition

$$\begin{aligned} RG_{\delta l}(A) &= A + B(A)\delta l + ... \\ B(A) &= \frac{d}{dl} RG_l(A)|_{l=0} \end{aligned} \tag{15}$$

Thus, $K$ is a linearization of the RG flow operator $B(A)$ at the fixed point $A_*$:

$$B(A_* + \delta A) \equiv K\delta A \tag{16}$$

The relation is simple. Any eigenvector $\Psi_\alpha$ of $K$ is an integral

$$\begin{aligned} &K\Psi_\alpha = k_\alpha \Psi_\alpha, \;\; \Psi_\alpha = \int d^d x \Phi_\alpha(x) \Rightarrow \\ &k_\alpha = d - D_\alpha \end{aligned} \tag{17}$$

$d$ is coming from the step 2 of RG transformation because in the integral $\int d^d x \Phi(x)$one needs to make additional transformation of measure:

$$\int d^d x \Phi(\frac{x}{L}) = L^d \int d^d \frac{x}{L} \Phi(\frac{x}{L}) \tag{18}$$

Using the $k_\alpha$ one can solve the RG equation for $A$ at the fixed point $A_*$:

$$\delta A(l) = \sum_\alpha C_\alpha \exp{(k_\alpha l)} \Psi_\alpha \tag{19}$$

Therefore, the tangent space $\mathcal{F}$ at the fixed point can be decomposed by

$$\begin{aligned} &\mathcal{F}_- = \{\Phi_\alpha \; with \; D_\alpha > d\} - \; irrelevant \; fields \\ &\mathcal{F}_0 = \{\Phi_\alpha \; with \; D_\alpha = d\} - \; marginal \; fields \\ &\mathcal{F}_+ = \{\Phi_\alpha \; with \; D_\alpha < d\} - \; relevant \; fields \end{aligned} \tag{20}$$

The subspace $\mathcal{F}_-$ corresponds to the critical surface ($R_c = \infty$). In the case when $\mathcal{F}_0 = 0$ the subspace $\mathcal{F}_+$ is the linear approximation of unstable manifold $U(A_*)$. Recall that $U(A_*)$ (where $A_* \in \Sigma(\infty)$) is, by definition the set of points which lie on RG trajectories, such that being integrated backward, converge to the fixed point $A_*$:

$$A_0 \in U(A_*) \Rightarrow A_l \to A_* \; as \; l \to -\infty \tag{21}$$

But the picture is more complex if $\mathcal{F}_0 \neq 0$. Suppose this space is 1-dimensional: $\mathcal{F}_0 = \{\Phi_M\}$. By the definition, the **relevant (irrelevant) field corresponds to the deformation $\delta A$ which increase (decrease) with the RG flow in the linear approximation**. The deformation

$$\delta A_M = \lambda_M \int d^d x \Phi_M(x) \tag{22}$$

corresponding to marginal field does not change in the linear approximation, but it may very well increase or decrease with RG flow if we go beyond the linear approximation. If $\delta A_M$ **increases** along the RG flow for $\lambda_M$ small enough, the deformationn is called **marginaly relevant**. Similarly, if $\delta A_M$ **decreases** along the RG flow for $\lambda_M$ small enough, the deformationn is called **marginaly irrelevant**.

In many respects the marginaly relevant (irrelevant) field can be treated on the same footing with relevant (irrelevant) ones. Thus if a marginaly relevant field is present, the unstable manifold $U(A_*)$ is generated by the subspace $\hat{\mathcal{F}}$:

$$TU|_{A_*} = \hat{\mathcal{F}} \supset \mathcal{F}_+ \tag{23}$$

**2. Gaussian fixed point.**

## 2.1. RG transformation of the massless free action.

The Gaussian fixed point is given by the massless free action

$$A_G = \int d^d x \frac{1}{2}(\partial\phi_0)^2 \tag{24}$$

Let us check that it is the fixed point indeed.

$$\begin{aligned} \phi_0(x) &= \phi_1(x) + \tilde{\phi}(x), \\ A_G &= \int d^d x (\frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\tilde{\phi})^2) \end{aligned} \tag{25}$$

Therefore the step 1 of RG transformation does not change the action $A_G$:

$$A_1 = \int d^d x \frac{1}{2}(\partial\phi_1)^2 \tag{26}$$

The step 2:

$$\begin{aligned} \phi_1(x) &= z^{-\frac{1}{2}}(L)\phi_0(\frac{x}{L}), \\ \tilde{A} &= z^{-1}\int d^d x \frac{1}{2}(\frac{\partial}{\partial x^\mu}\phi_0(\frac{x}{L}))^2 = \frac{L^{d-2}}{z(L)}\int d^d x \frac{1}{2}(\frac{\partial}{\partial x^\mu}\phi_0(x))^2 \end{aligned} \tag{27}$$

Under the choice

$$z^{\frac{1}{2}}(L) = L^{\frac{d-2}{2}} \Rightarrow \tilde{A} = A_G \tag{28}$$

we get the standard normalization of the kinetic term, so that it is a fixed point indeed.

### *2.2. RG action on composite fields at Gaussian fixed point.*

Now we find RG transformation of composite fields in this fixed-point theory. Consider the correlation function

$$\begin{aligned} &\frac{1}{Z}\int [D\phi_1][D\tilde{\phi}]_\Delta \left[\frac{1}{2M!}(\phi_1(x)+\tilde{\phi}(x))^{2M}...\right] \exp(-A[\tilde{\phi}])\exp(-A[\phi_1]) = \\ &\frac{1}{Z_1}\int [D\phi_1] \\ &\left[(\sum_{m=0}^{M} \frac{\int [D\tilde{\phi}]_\Delta \frac{1}{2m!}\tilde{\phi}^{2m}(x)\exp(-A[\tilde{\phi}])}{\int [D\tilde{\phi}]_\Delta \exp(-A[\tilde{\phi}])} \frac{\phi_1^{2M-2m}(x)}{(2M-2m)!})...\right] \exp(-A[\phi_1]) \end{aligned} \tag{29}$$

Hence, at the step 1 we can write

$$< \frac{1}{2M!}\phi_1^{2M}(x)... > |_{(A_0,\Lambda_0)} =$$
$$\frac{1}{Z_1}\int [D\phi_1]\left[(\sum_{m=0}^{M} < \frac{\tilde{\phi}^{2m}}{2m!} >_\Delta \frac{\phi_1(x)^{2M-2m}}{(2M-2m)!})...\right]\exp\left(-A[\phi_1]\right) \quad (30)$$

so that the step 1 yelds for $O_{2M} = \frac{1}{2M!}\phi_0^{2M}$

$$\frac{1}{2M!}\phi_0^{2M} = \frac{1}{2M!}\phi_1^{2M} + \sum_{m=1}^{M} < \frac{\tilde{\phi}^{2m}}{2m!} >_\Delta \frac{\phi_1(x)^{2M-2m}}{(2M-2m)!} \quad (31)$$

where

$$< \frac{\tilde{\phi}^{2m}}{2m!} >_\Delta = \frac{\Lambda_1^{d-2m}}{2m!(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\frac{L^{d-2m}-1}{d-2m} \quad (32)$$

The step 2 amounts to changing

$$\phi_1 \to z^{-\frac{1}{2}}(L)\phi_0(\frac{x}{L}) = L^{\frac{2-d}{2}}\phi_0(\frac{x}{L}) \quad (33)$$

Therefore under the step 2 the expression (31) takes the form

$$RG_l O_n(x) = L^{n\frac{2-d}{2}}O_n(\frac{x}{L}) + \sum_{k=1}^{n}\frac{< \tilde{\phi}^k >_\Delta}{k!}L^{(n-k)\frac{2-d}{2}}O_{n-k}(\frac{x}{L}) \quad (34)$$

Recalling the definition of matrix $z_\beta^\alpha(L)$ from (10) we find

$$RG_l O_\alpha = \sum_\beta z_\alpha^\beta(L)O_\beta(\frac{x}{L}) \Rightarrow$$
$$z_n^n(L) = L^{n\frac{2-d}{2}},\ z_n^k(L) = \frac{< \tilde{\phi}^k >_\Delta}{k!}L^{(n-k)\frac{2-d}{2}},\ k = 1,...n \quad (35)$$

where it is implied that $< \tilde{\phi}^{2n-1} >_\Delta = 0$. Now one can find the matrix $D_\beta^\alpha$ considering infinitesimal RG transformation $RG_{1+\delta l} = 1 - \delta l D$, $L = 1 + \delta l$. By the definition (12)

$$D_n^n = -\frac{dz_n^n}{dL}|_{L=1} = n\frac{d-2}{2},$$
$$D_n^k = -\frac{d}{dL}(\frac{< \tilde{\phi}^k >_\Delta}{k!}L^{(n-k)\frac{2-d}{2}})|_{L=1} = -\frac{d}{dL}\frac{< \tilde{\phi}^k >_\Delta}{k!}|_{L=1} \equiv -N_k \quad (36)$$

Thus, for the infinitesimal transformation we obtain

$$DO_n = n\frac{d-2}{2}O_n - \sum_{k=1}^{n} N_k O_{n-k},$$
$$where\ N_{2m} = \frac{\Lambda_1^{d-2m}}{2m!(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}, N_{2m-1} = 0 \tag{37}$$

It is not difficult to diagonalize this operator. Proove that

$$\Phi_n = \frac{1}{n!}\phi_0^n - \frac{1}{2!(n-2)!} < \phi_0^2 > \phi_0^{n-2} + ... \equiv \frac{1}{n!} : \phi_0^n :,$$
$$D\Phi_n = D_n\Phi_n,\ D_n = n\frac{d-2}{2} \tag{38}$$

In fact the eigenvalues $D_n$ are found without compution exact form of $\Phi_n$. Indeed, the eq. (37) shows that $D$ acts on the fields $O_n$ as triangle matrix (it mixes $\phi_0^n$ with itself and lower powers of $\phi_0$), and so its eigenvalues are given by the diagonal elements of $D$

$$D_n = n\frac{d-2}{2} \tag{39}$$

We see that **dimensions $D_n$ coinside with the canonical dimensions of the composite fields** $\phi_0^n$, which is not surprising because we have a free massless theory.

Dimensions of the space $\mathcal{F}_+$ of the relevant fields associated with the Gaussian fixed point $A_G$ depends on $d$.

The field $\frac{1}{2} : \phi^2 :$ is always relevant because $D_2 = d - 2 < d$. For $d > 6$ it remains the only relevant field. At $d = 6$ $\Phi_3 = \frac{1}{3!} : \phi^3 :$ is marginal. For $d < 6$ it becomes relevant because $D_3 = \frac{3}{2}(d-2) < d$. The situation with relevant fields is summarized in the table

$$
\begin{aligned}
& d > 6: \ \phi^2, \\
& 6 > d > 4: \ \phi^2, \ \phi^3 \\
& 4 > d > \frac{10}{3}: \ \phi^2, \ \phi^3, \ \phi^4 \\
& .... \\
& \frac{2n}{n-2} > d > \frac{2(n+1)}{n-1}: \ \phi^2, \ \phi^3, \ ... \ \phi^n \\
& d = 2: \phi, \ \phi^2, \ \phi^3, ....
\end{aligned} \tag{40}
$$

Note that the classification above coinsides with our earlier classification of the perturbatively renormalizable scalar field theories. We see that **in generic dimensionality $d$ the perturbatively renormalizable theories correspond to the unstable manifold $U(A_G)$.** (This is because $U(A_*)$ (where $A_* \in \Sigma(\infty)$) is, by definition the set of points which lie on RG trajectories, such that being integrated backward, converge to the fixed point $A_*$:

$$A_0 \in U(A_*) \Rightarrow A_l \to A_* \ as \ l \to -\infty \tag{41}$$

)

Also, we realize now that **this classification of renormalizable field theories is relevant only to the vicinity of the Gaussian fixed point**. If we manage to find more complex fixed point, its own unstable manifold will give rise to a family of local field theories which may have little to do with perturbatively defined renormalizable field theories we considered above.

*2.3. $\phi^4$ is marginaly irrelevant at Gaussian fixed point.*

Exact correspondence between the unstable manifold and the space of perturbatively renormalizable field theories may break down if $d$ takes specccial values such that one of the fields $\Phi_n$ becomes marginal ($D_n = d$).

As at $d = 4$ $\Phi_4$ is marginal we have

$$RG_l(A_G + \lambda \int d^4\Phi_4(x)) = A_G + \lambda \int d^4\Phi_4(x) + O(\lambda^2) \tag{42}$$

i.e. at the order $\lambda$ the perturbed action does not flow at all. However, nontrivial flow can be generated beyond the linear approximation. Let us estimate this flow at the order $\lambda^2$.

**Step 1.**

To perform the step 1 of RG transformation one has to integrate over the fast component $\tilde{\phi}$ in

$$\phi_0 = \phi_1 + \tilde{\phi} \tag{43}$$

Using the definition of diagonalized field $\Phi_4$ (see (38)) we can write for the pertirbed action

$$\begin{aligned} A_0 &= A_G + \lambda \int d^4x\Phi_4(x) = \\ &\int d^4x \left(\frac{1}{2}(\partial\phi_0)^2 + \frac{\lambda}{4!}\phi_0^4 - \frac{\lambda N_2}{4}\phi_0^2 + const\right) \end{aligned} \tag{44}$$

Hence we have

$$\begin{aligned} &C \exp\left(-A_1[\phi_1]\right) = \exp\left(-A_0[\phi_1]\right) \\ &\int [D\tilde{\phi}] \exp\left\{-\int d^4x \left[\frac{1}{2}(\partial\tilde{\phi})^2 + \lambda(\frac{1}{4!}\tilde{\phi}^4 - \frac{N_2}{4}\tilde{\phi}^2 + \frac{1}{6}\tilde{\phi}^3\phi_1 + \frac{1}{4}\tilde{\phi}^2\phi_1^2 + \frac{1}{6}\tilde{\phi}\phi_1^3\right]\right\} \end{aligned} \tag{45}$$

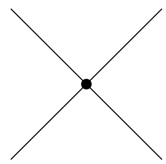
As we have discussed already in Lect.10, the integration over $\tilde{\phi}$ yields

$$A_1 = A_0 - (\ \textit{sum of all connected diagrams}) \tag{46}$$

which are generated by the following vertices



(47)



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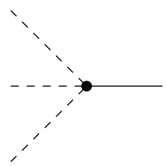

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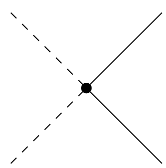

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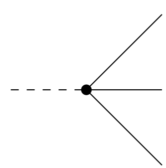
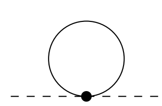


(51)

where the solid lines represent the fast field $\tilde{\phi}(x)$ (and thus, all solid lines must be contracted), the dotted lines represent the field $\phi_1(x)$ which plays the role of an external field (i.e. the diagrams involve no contractions of the dotted lines).

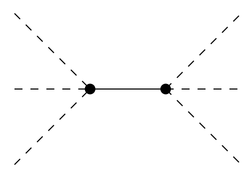
It is not dificult to check that (42) indeed holds to the order $\lambda$ because the constant $N_2$ in the $\phi_1^2$ term in (44) is chosen (see (36), (37)) in such a way that the contribution of the diagram



(52)

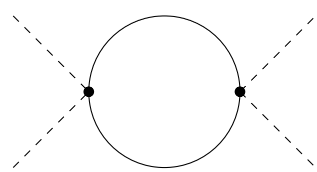
which is proportional to $\phi_1^2$ is compensated exactly by the transformation of the term $\frac{N_2}{4}\phi_1^2$ in the step 2 of the RG procedure.

In the order $\lambda^2$ many new terms in $A_1$ are generated. For example the diagram



(53)

gives rise to the terms $\approx \phi_1^6$. All such fields are irrelevant. The diagrams contributing to the terms $\approx \phi_1^4$ are



(54)

This diagram admits Taylor expansion in the external momenta. The zeroth term of this expansion contribute to the term $\phi_1^4$ in $A_1$, while the higher terms give rise to the irrelevant contributions like $\phi_1^2(\partial\phi_1)^2$. In our approximation we are neglecting the irrelevant terms, because they dye out very fast with the RG flow. In this approximation diagram (54) can be evaluated at zero external momenta only, yielding the contribution to $A_1$

$$\frac{\Delta\lambda}{4!}\int d^4x\phi_1^4(x),$$
$$\frac{\Delta\lambda}{4!} = -\frac{\lambda^2}{16}\int_\Delta \frac{d^4k}{(2\pi)^4}(\frac{1}{k^2})^2 \tag{55}$$

The factor $\frac{\lambda^2}{16}$ appears as follows. The term in (45) responsible for the vertices in this diagram is $\frac{\lambda}{4}\phi_1^2\tilde{\phi}^2$. One has to expand the exponential to the second order to get the diagram (54), this brings in additional factor $\frac{1}{2!}$, a factor 2 comes from combinatorics of contractions in $< \tilde{\phi}^2\tilde{\phi}^2 >_\Delta$. The momentum integral in (55) is easy to evaluate and we obtain

$$\Delta\lambda = -3\frac{\lambda^2}{(4\pi)^2}\ln L \tag{56}$$

**Step 2.**

At this order in $\lambda$ the step 2 is trivial. Indeed, for the marginal term $\int d^4x\phi_1(x)^4$ the factor $z^{-2}(L)$ which comes from the transformation $\phi_1 = z^{-\frac{1}{2}}(L)\phi_0(\frac{x}{L})$ is compensated exactly by the integration measure transformation ($z^{-\frac{1}{2}}(L) = L^{-1}$ but possible corrections to this formula are $\approx \lambda^2$).

Thus, in our approximation the result of the RG transformation of the action (44) again has the form (44) with $\lambda$ replaced by

$$\tilde{\lambda} = \lambda - 3\frac{\lambda^2}{(4\pi)^2}\ln L \tag{57}$$

Taking $L = 1 + \delta l$ we see that $\lambda$ **decreases** with the RG flow:

$$-\frac{d}{dl}\lambda(l) = 3\frac{\lambda(l)^2}{(4\pi)^2} \tag{58}$$

i.e. at $d = 4$ the field $\Phi_4$ is **marginaly irrelevant**.

Note that r.h.s. of (58) coinsides with the $\lambda^2$ term of Callan-Symanzik $\beta$ function of $\phi^4$ theory.

**3. Callan-Symanzik equation.**

RG analysis allows one to rederive the Callan-Symazik equation which was obtained on the basis of renormalized perturbation theory.

Let $A_*$ be a fixed point. Assuming that its unstable manifold is $n$-dimensional, let $\{\lambda^i\}$, $i = 1, ..., n$ be coordinates on $U_{A_*}$, such that $\lambda = 0$ corresponds to $A_*$. This manifold describes $n$-parameter family of finite local field theories. Let us denote by

$$< \phi(x_1)...\phi(x_N) >_\lambda \tag{59}$$

the corresponding correlation function indicating their dependence on $n$ parameters $\lambda^i$.

In these coordinates the RG flow equation can be written as

$$\frac{d}{dl}\lambda^i = -\beta^i(\lambda) \tag{60}$$

where $-\beta^i(\lambda)$ are the components of the vector field $B^i$ on $U_{A_*}$, the minus sign is taken to match the notations of Callan and Symanzik. According to definition of Wilson's RG, if $\lambda(l)$ is the solution of (60) with some initial condition $\lambda(0) = \lambda$, the field theories corresponding to $\lambda$ and $\lambda(l)$ are related by a coordinate scale transformation $x \rightarrow \frac{x}{L}$. In particular, one can write for the correlation functions

$$< \phi(x_1)...\phi(x_N) >_{\lambda(l)} = z^{\frac{N}{2}}(L) < \phi(Lx_1)...\phi(Lx_N) >_{\lambda} \tag{61}$$

where $L = \exp(l)$. For the case of infinitesimal transformation $L = 1 + \delta l$, $\lambda(l) = \lambda - \beta(\lambda)\delta l$ we have

$$\left[ \sum_{k=1}^{N} x_k^a \frac{\partial}{\partial x_k^a} + ND(\lambda) + \sum_{i=1}^{n} \beta^i(\lambda) \frac{\partial}{\partial \lambda^i} \right] < \phi(x_1)...\phi(x_N) >_{\lambda} = 0 \tag{62}$$

where $a = 1, ...d$ and

$$D(\lambda) = L \frac{d}{dL} z^{\frac{1}{2}}(L)|_{L=1} \tag{63}$$

Notice that $D(0) = D_\phi$ is the dimension of $\phi$ at the fixed point $A_*$. $D(\lambda)$ is often written as

$$D(\lambda) = D_\phi + \gamma(\lambda) \tag{64}$$

so that $\gamma(0) = 0$.

The eq. (62) easely generalizes to the corr. functions of composite fields

$O_\alpha(x)$. Repeating the same analysis we obtain

$$\sum_{k=1}^{N} < O_{\alpha_1}(x_1)...(\mathcal{D}O)_{\alpha_k}(x_k)...O_{\alpha_N}(x_N) >_\lambda +$$
$$\sum_{i=1}^{n} \beta^i(\lambda)\frac{\partial}{\partial\lambda^i} < O_{\alpha_1}(x_1)...O_{\alpha_N}(x_N) >_\lambda = 0 \tag{65}$$

where

$$(\mathcal{D}O)_{\alpha_k}(x) = x^a\frac{\partial}{\partial x^a}O_\alpha + D_\alpha^\beta(\lambda)O_\beta(x),$$
$$D_\alpha^\beta(\lambda) = -L\frac{d}{dL}z_\alpha^\beta(L)|_{L=1} \tag{66}$$

$z_\alpha^\beta(L)$ describes the RG transformation of the comopsite fields $O_\alpha(x)$. Again,

$$D_\alpha^\beta = D_\alpha^\beta(\lambda)|_{\lambda=0} \tag{67}$$

coinsides with the matrix of dimensions at $A_*$ whose eigenvalues are the dimensions of the fields $\Phi_\alpha(x)$.

We see that the **CS equation is very general consequence of Wilson's RG theory** and not just the property of the perturbation theory around the Gaussian fixed point.