Lecture 1.

Plan.

1. Why do we need in Quantum FT?

1.1 Lagrangian, action and equations of motion in Classical mechanics and FT.

1.2 Lorentz invariance, locality and causuality.

1.3 Symmetries of the action and Noether theorem.

1.4 Stress-energy tensor.

1.5 KG theory.

2. Hamiltonian formalism.

2.1.Canonical variables and Hamiltonian in Classical Mechanics.

2.2.Canonical variables and Hamiltonian in Classical FT.

2.3. Canonical variables and Hamiltonian in KG theory.

3. Quantization procedure in Hamiltonian formalism.

3.1. Quantization procedure in mechanics.

3.2. Quantization of KG field in Shrödinger picture.

3.3. KG as a set of harmonic oscillators.

3.4. Vacuum energy, normal ordering and particle interpretation.

3.5. Vacuum energy regularization and QFT at small distances.

3.6. Creation anihilitation operators as a coserved charges.

1. Why do we need in Quantum FT?

1.1 Lagrangian, action and equations of motion in Classical mechanics and FT.

The equations of motion in Classical Mechanics follow from extremal action principle: we have an action

$$S = \int L(q(t), \dot{q}(t))dt \tag{1}$$

where the Lagrangian $L(q(t), \dot{q}(t))$ is a function of coordinate of a particle

$q(t)$ and its velocity $\dot{q}(t)$. The equations of motion follow if we demand that trajectory is extremal

$$\delta S = 0 \tag{2}$$

In Classical FT an analog of $q(t)$ is a field $\phi(\vec{x}, t)$, we have an action

$$S = \int \Lambda(\phi(\vec{x}, t), \dot{\phi}(\vec{x}, t), \nabla\phi(\vec{x}, t)) d^3x dt \tag{3}$$

and the equations of motion follow if we demand that "'trajectory"' $\phi(\vec{x}, t)$ is extremal

$$\delta S = 0 \tag{4}$$

1.2 Lorentz invariance, locality and causuality.

The action (3) will be Lorentz invariant if the Lagrangian density Λ is Lorentz invariant because the mesure d^3xdt is Lorentz invariant clearly.

The action must be local. It means that Lagrangian desity is local. This in turn means that all the quantities $\phi(\vec{x}, t)$, $\dot{\phi}(\vec{x}, t)$, $\nabla\phi(\vec{x}, t)$ are taken in one point $(\vec{x}, t)$. This reflects (Faraday's) principle of short-range action:

the field degree of freedom $\phi(\vec{y}, t)$ does not interract immediately with $\phi(\vec{x}, t)$ if $|\vec{y} - \vec{x}|$ =finite.

Allowing such interractions would lead to possible terms in Lagrangian like

$$\int F(\phi(\vec{x}, t), \phi(\vec{y}, t)) d^3x d^3y \tag{5}$$

But Lorentz invariance of the action S would require also the terms which are nonlocal in time as well, like

$$\int G(\phi(\vec{x}, t), \phi(\vec{y}, t')) d^3x d^3y dt dt' \tag{6}$$

which evidently **violate causuality**. The state in the future affects the dynamics at present. Locality and causuality are **deeply connected**.

1.3 Symmetries of the action and Noether theorem.

The Lagrangian approach opens up a natural way to relate the symmetries of the action to the consrvation lows. It is given by

Noether theorem: suppose we have a continued set of transformations

$$\phi(x) \to \tilde{\phi}(x) = F_s(x, \phi(x)) \tag{7}$$

parametrized by a parameter s, such that $F_s(x, \phi(x)) = \phi(x)$ and the action is invariant

$$S[\phi(x)] = S[\tilde{\phi}(x)] \tag{8}$$

Consider the infinitesimal transformation

$$\begin{gathered} \phi(x) \to \phi(x) + \epsilon E(x, \phi(x)) \Rightarrow \\ \Lambda(\tilde{\phi}, \partial_\mu \tilde{\phi}) = \Lambda(\phi, \partial_\mu \phi) + \epsilon[\frac{\partial \Lambda}{\partial \phi} E + \frac{\partial \Lambda}{\partial(\partial_\mu \phi)} \partial_\mu E)] \end{gathered} \tag{9}$$

Because of action is invariant

$$\begin{gathered} \frac{\partial \Lambda}{\partial \phi} E + \frac{\partial \Lambda}{\partial(\partial_\mu \phi)} \partial_\mu E = \partial_\mu K^\mu \Leftrightarrow \\ \partial_\mu (\frac{\partial \Lambda}{\partial(\partial_\mu \phi)} E + (\frac{\partial \Lambda}{\partial \phi} - \partial_\mu \frac{\partial \Lambda}{\partial(\partial_\mu \phi)}) E = \partial_\mu K^\mu \end{gathered} \tag{10}$$

But the second term on the r.h.s. is zero due to the equations of motion and we obtain the conservation low

$$\begin{aligned} \partial_\mu (\frac{\partial \Lambda}{\partial(\partial_\mu \phi)} E) - \partial_\mu K^\mu \equiv \partial_\mu J^\mu = 0 \\ J^\mu = \frac{\partial \Lambda}{\partial(\partial_\mu \phi)} E(x, \phi) - K^\mu(\phi, \partial \phi) \end{aligned} \tag{11}$$

This leads to the conserved charges for the equations of motion solutions (on shell):

$$Q_{t_1} \equiv \int d^3x J^0(\vec{x}, t_1) = Q_{t_2} \equiv \int d^3x J^0(\vec{x}, t_2) \tag{12}$$

where it is implyed that classical solution is such that $\phi(\vec{x},t) \to 0$ when $\vec{x} \to \infty$. The equation (12) is proved by Gauss theorem.

1.4 Stress-energy tensor.

The most general symmetry taking place in FT is **translational symmetry**. If the Lagrangian density

$$\Lambda(\phi(x), \partial\phi(x)) \tag{13}$$

has no explicit dependence on x_μ the action is invariant w.r.t the shifts by a constant vector a:

$$x \to \tilde{x} = x + a,$$
$$\phi(x) \to \phi(\tilde{x}) = \phi(x + a) \tag{14}$$

The infinitesimal version of the shift is

$$x \to \tilde{x} = x + da,$$
$$\phi(x) \to \tilde{\phi}(x) = \phi(x) + da^\nu \partial_\nu \phi(x) \tag{15}$$

so that $E(\phi, \partial\phi) = \partial_\nu \phi(x)$ is vector and

$$K_\mu = \partial_\nu \Lambda \tag{16}$$

Now the expression (11) gives the stress-energy tensor conservation low

$$T^\mu_\nu = \frac{\partial \Lambda}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \Lambda \delta^\mu_\nu, \ \partial_\mu T^\mu_\nu = 0 \tag{17}$$

The conservation low (17) leads to 4 conserved quantyties:

$$E = \int T^{00} d^3x, \ P^i = \int T^{0i} d^3x \Leftrightarrow P^\mu = \int g^{\mu\nu} T^0_\nu d^3x,$$
$$i = 1, ..., 3, \ \nu, \mu = 0, ..., 3, \ g^{\mu\nu} = diag(1, -1, -1, -1) \tag{18}$$

E is an energy and $\vec{P}$ is a momentum.

1.5 KG theory.

It is a single component (scalar) field $\phi(\vec{x}, t)$ with the action

$$S = \int \frac{1}{2}[(\dot{\phi})^2 - (\nabla\phi)^2 - m^2\phi^2]d^3xdt =$$
$$\int \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2]d^4x \tag{19}$$

The action is Lorentz invariant because the scalar field transforms under the Lorentz tralsformation $x^\mu \to \tilde{x}^\mu = R^\mu_\nu x^\nu$ as $\tilde{\phi}(x) = \phi(\tilde{x})$. The equation of motion:

$$\delta S = 0 \Leftrightarrow \partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{20}$$

so that m^2 is a mass of the field ϕ. The stress-energy tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\phi\partial^\lambda\phi - m^2\phi^2) \tag{21}$$

Hence the energy and momentum densityes are given by

$$\epsilon = \frac{1}{2}(\dot{\phi})^2 + (\nabla\phi)^2 + m^2\phi^2),$$
$$\vec{p} = \dot{\phi}\nabla\phi \tag{22}$$

2. Hamiltonian formalism.

2.1.Canonical variables and Hamiltonian in Classical Mechanics.

In order to pass from Lagrangian formalism to Hamiltonian formalism in classical mechanics, one needs to introduce canonical momenta

$$p^i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i} \tag{23}$$

and exludes $\dot{q}$ in favor of p: $\dot{q} = f(q, p)$. Then the Hamiltonian function appears as the Legandre transform

$$H(p, q) = \sum_i p^i\dot{q}_i - L(q, p) \tag{24}$$

Then the equations of motion take the Hamiltonian form

$$\dot{q}_i = \frac{\partial H}{\partial p^i} = \{q_i, H\},$$
$$\dot{p}^i = -\frac{\partial H}{\partial q_i} = \{p^i, H\},$$
$$\{f, g\} \equiv \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} \tag{25}$$

2.2.Canonical variables and Hamiltonian in Classical FT.

In the field theory we have $\vec{x}$ instead of i, hence the canonical momenta is

$$\pi(\vec{x}) = \frac{\delta L}{\delta \dot{\phi}} \tag{26}$$

where

$$L = \int \Lambda(\phi, \partial\phi) d^3x \tag{27}$$

2.3. Canonical variables and Hamiltonian in KG theory.

Applying the definition above to the case of KG Lagrangian we find

$$\pi(\vec{x}) = \dot{\phi}(\vec{x}) \tag{28}$$

Hence the Hamiltonian is

$$H = \int d^3x \pi(\vec{x})\dot{\phi}(\vec{x}) - L =$$
$$\int d^3x \frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2) \tag{29}$$

3. Quantization procedure in Hamiltonian formalism.

3.1. Quantization procedure in mechanics.

Classical mechanics: phase space parametrized by the canonical coordinates p^i, q_i, endowed with the canonical Poisson brackets

$$\{q_i, p^j\} = \delta_i^j \tag{30}$$

Quantum mechanics: Hilbert space of states, $\mathcal{H}$, $\hat{p}^i, \hat{q}_i$-operators with canonical bracets

$$[q_i, p^j] = \imath\delta_i^j \tag{31}$$

Classical mechanics: Hamiltonian $H(p,q)$- function on the phase space.

Quantum mechanics: Hamiltonian $\hat{H}(\hat{p},\hat{q})$-operator acting on the space of states.

Classical mechanics: equations of motion

$$\dot{q}_i = \{q_i, H\},\ \dot{p}^i = \{p^i, H\} \tag{32}$$

Quantum mechanics: Heisenberg's equations of motion

$$\dot{\hat{q}}_i = [\hat{q}_i, H],\ \dot{\hat{p}}^i = [\hat{p}^i, H] \tag{33}$$

or Schr$\ddot{o}$dinger's representation: $\mathcal{H}$ is the space of quadraticaly integrable functions $\Psi(q)$ such that

$$\hat{q}_i\Psi(q) = q_i\Psi(q),\ \hat{p}^i\Psi(q) = \imath\frac{\partial}{\partial q^i}\Psi(q)$$
$$\imath\frac{\partial}{\partial t}\Psi(q) = \hat{H}\Psi \tag{34}$$

3.2. Quantization of KG field in Schrödinger picture.

An analog of quantum mechanical wave function is a functional $\Psi[\phi(x)]$ which is an element of the Hilbert space of states $\mathcal{H}$ of KG QFT $\Psi[\phi(x)] \in \mathcal{H}$. The scalar product in $\mathcal{H}$ is determined by the functional integral

$$(\Psi_1, \Psi_2) = \int D[\phi(x)]\Psi_1(\phi)\Psi_2^*(\phi) \tag{35}$$

By definition, the functional $\Psi[\phi]$ satisfy the equations

$$\hat{\phi}(\vec{x})\Psi = \phi(\vec{x})\Psi,\ \hat{\pi}(\vec{x})\Psi = -\imath\frac{\delta}{\delta\phi(\vec{x})}\Psi \tag{36}$$

where the operators $\hat{\phi}(\vec{x})$, $\hat{\pi}(\vec{x})$ does not depend on time and satisfy the canonical commutation relations

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{x})] = \imath\delta^3(\vec{x} - \vec{y}) \tag{37}$$

The evolution of $\Psi[\phi]$ in time is given by the Schrödinger equation

$$\imath\frac{\partial}{\partial t}\Psi = \hat{H}\Psi \tag{38}$$

where the Haniltonian is

$$\hat{H} = \int d^3x\frac{1}{2}(\hat{\pi}(\vec{x})^2 + (\nabla\phi(\vec{x}))^2 + m^2\phi(\vec{x})^2) \tag{39}$$

3.3. KG as a set of harmonic oscillators.

The idea is to treat the KG field as a set of harmonic oscillators. Indeed, the KG Hamiltonian (29) is similar to the harmonic oscillator Hamiltonian

$$H_{osc} = \frac{2}{2}(p^2 + \omega^2 q^2) \tag{40}$$

The difference is that in case of KG field we have a set of harmonic oscillators numbered by a points $\vec{x}$ of space.

For the harmonic oscillator the operators

$$a = \frac{\imath}{\sqrt{2\omega}}(p - \imath\omega\phi),\ \ a^* = -\frac{\imath}{\sqrt{2\omega}}(p + \imath\omega\phi) \tag{41}$$

diagonalize the Hamiltonian

$$H_{osc} = \frac{\omega}{2}(a^*a + aa^*) = \omega(a^*a + \frac{1}{2}) \tag{42}$$

where the last expression follows from the commutation relation

$$[a, a^*] = 1 \tag{43}$$

which in turn follows from the canonical commutator for p and q. Then it is easy to check that the states $(a^*)^n|0>$ are H_{osc}-eigenstates

$$H_{osc}(a^*)^n|0>= (n + \frac{1}{2})\omega(a^*)^n|0> \tag{44}$$

and the space of states of harmonic oscillator is generated by these eigenstates.

For the KG field one can consider an analog of the operators (41)

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_{\vec{p}} + a^*_{-\vec{p}})\exp(\imath\vec{p}\vec{x})$$
$$\pi(\vec{x}) = -\imath\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}(a_{\vec{p}} - a^*_{-\vec{p}})\exp(\imath\vec{p}\vec{x}) \tag{45}$$

They are introduced in such way to have

$$[a_{\vec{p}}, a^*_{\vec{p}'}] = (2\pi)^3\delta(\vec{p}-\vec{p}')$$
$$\hat{H} = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}(a^*_{\vec{p}}a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}a^*_{\vec{p}}]) \tag{46}$$

Because of the relations

$$[\hat{H}, a_{\vec{p}}] = -\omega_{\vec{p}}a_{\vec{p}},\ [\hat{H}, a^*_{\vec{p}}] = \omega_{\vec{p}}a^*_{\vec{p}} \tag{47}$$

one can easy to build the space of states $\mathcal{H}$.

We demand that the energy spectrum is bounded from below: $E \geq E_0$. It means that there is a state $|0>$ with minimal energy E_0 such that

$$a_{\vec{p}}|0>= 0 \tag{48}$$

Then the other states are given by creation operators

$$|\vec{p}_1, ..., \vec{p}_N >= a^*_{\vec{p}_1}...a^*_{\vec{p}_N}|0>,$$
$$\hat{H}|\vec{p}_1, ..., \vec{p}_N >= (\omega_{\vec{p}_1} + ... + \omega_{\vec{p}_N} + E_0)|\vec{p}_1, ..., \vec{p}_N > \tag{49}$$

The ground state energy

$$E_0 = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\frac{1}{2}[a_{\vec{p}}a^*_{\vec{p}}] = \int d^3p\frac{\omega_{\vec{p}}}{2}\delta(0) \tag{50}$$

is divergent. But the expression for E_0 can be rewritten as follows

$$E_0 = \int d^3p\frac{\omega_{\vec{p}}}{2}\int d^3x \exp(\imath(\vec{p}-\vec{p}')\vec{x})\delta(\vec{p}-\vec{p}') = \epsilon_0 V^3, \tag{51}$$

where

$$\epsilon_0 = \int d^3p \frac{\omega_{\vec{p}}}{2} \tag{52}$$

is the density of vacuum energy.

3.4. Vacuum energy, normal ordering and particle interpretation.

Though the infinite vacuum energy can cause the problems, it can be ignored as long as the difference between the energy of a given state and vacuum energy matters. Therefore it makes sense to redefine $\hat{H}$ by subtracting E_0:

$$: \hat{H} := \hat{H} - E_0 = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} a^*_{\vec{p}} a_{\vec{p}} \tag{53}$$

and do similar subtraction for the momentum operator

$$: \hat{\vec{P}} := \int \frac{d^3p}{(2\pi)^3} \vec{p} a^*_{\vec{p}} a_{\vec{p}} \tag{54}$$

It allows us to interprate the operator $a^*_{\vec{p}}$ as creating a particle with the energy $\omega_{\vec{p}}$ and momentum $\vec{p}$ so that the vector $|\vec{p}_1, ..., \vec{p}_N >= a^*_{\vec{p}_1} ... a^*_{\vec{p}_N} |0>$ is an N-particles state with the momenta $\vec{p}_1, ..., \vec{p}_N$ because of the correct relation $\omega_{\vec{p}_i} = \sqrt{\vec{p}_i^2 + m^2}$ between the momentum and energy of each particle.

3.5. Vacuum energy regularization and QFT at small distances.

Though the subtraction (53), (54) is very convenient and allows to study the spectrum of excited states it does not solve the vacuum energy problem. E_0 is invisible in infinite flat space-time but it is impodtant when we consider gravity. That is vacuumenergy contributes to the cosmological constant which as we know is astronomically small.

As one can see from (52) comes from small distances (big momenta). It can be assumed that the theory does not apply to very small scales and must be modifyed so that instead of (52) we would have

$$\epsilon_0 = \int d^3p \frac{\omega_{\vec{p}}}{2} \Phi(\frac{\vec{p}^2}{\lambda^2}), \;\; \Phi(0) = 1 \tag{55}$$

where λ is large momentum, where a new physics emerge. If we assume that $\Phi(x) \to 0$ as $x \to \infty$ sufficiently fast, the integral above would converge. But $\Phi(x)$ and λ must be determined by small scales physics. Thus, the vacuum energy divergence problem is a physical problem.

3.6. Creation anihilitation operators as a coserved charges.

The creation-anihilation operators introduced in (45) can be considered as a coserved charges of some infinite symmetry which manifests itself in KG theory. Indead, under the infinitesimal field transformation

$$\phi(x) \to \tilde{\phi}(x) = \phi(x) + f(x) \tag{56}$$

where the function $f(x)$ is an arbitrary KG equation equation solution, the Lagrangian density changes by

$$\Lambda(\bar{\phi}) = \Lambda(\phi) + \partial_\mu K^\mu_f, \;\; K^\mu_f = \phi \partial_\mu f \tag{57}$$

Hence one can use the Noether theorem to conclude that

$$\partial_\mu J^\mu_f = 0, \;\; J^\mu_f = f\partial^\mu \phi - \phi \partial^\mu f \tag{58}$$

So the corresponding conserved charges (integrals of motion) are given by

$$A_f = \int d^3x(\dot{\phi} f - \dot{f}\phi) = \int d^3x(\pi f - \dot{f}\phi) \tag{59}$$

The last formula allows to calculate the Poisson brackets

$$\{A_f, A_g\} = \int d^3x(\dot{f} g - f\dot{g}) \tag{60}$$

One can easy to see that the charges form the algebra of creation-anihilation operators w.r.t Poisson brackets if we use the plane waves basis

$$\begin{aligned} A_{\vec{p}} \equiv A_{f_{\vec{p}}}, \;\; f_{\vec{p}} = \exp\left(\imath(\omega_{\vec{p}} t - \vec{p}\vec{x})\right), \\ A^*_{\vec{p}} \equiv A_{f^*_{\vec{p}}}, \;\; f^*_{\vec{p}} = \exp\left(-\imath(\omega_{\vec{p}} t - \vec{p}\vec{x})\right), \\ \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2} \end{aligned} \tag{61}$$

$$\{A_{\vec{p}}, A^*_{\vec{p'}}\} = -\imath(2\pi)^3 2\omega_{\vec{p}}\delta(\vec{p}-\vec{p'}),$$
$$\{A_{\vec{p}}, A_{\vec{p'}}\} = \{A^*_{\vec{p}}, A^*_{\vec{p'}}\} = 0 \tag{62}$$

Thus, in quantum theory

$$A_{\vec{p}} \to \hat{A}_{\vec{p}},\ \ A^*_{\vec{p}} \to \hat{A}^*_{\vec{p}}$$
$$[\hat{A}_{\vec{p}}, \hat{A}^*_{\vec{p'}}] = (2\pi)^3 2\omega_{\vec{p}}\delta(\vec{p}-\vec{p'}),$$
$$[\hat{A}_{\vec{p}}, \hat{A}_{\vec{p'}}] = [\hat{A}^*_{\vec{p}}, \hat{A}^*_{\vec{p'}}] = 0 \tag{63}$$

Thus one can identify

$$\hat{A}_{\vec{p}} = \sqrt{2\omega_{\vec{p}}}a_{\vec{p}},\ \ \hat{A}^*_{\vec{p}} = \sqrt{2\omega_{\vec{p}}}a^*_{\vec{p}} \tag{64}$$

It is important to note that operators $\hat{A}_{\vec{p}}$, $\hat{A}^*_{\vec{p}}$ are Lorntz invariants if one uses the covariant form of their definition

$$A_{\vec{p}} = \int d\Sigma_\mu (f_{\vec{p}}\partial^\mu\phi - \phi\partial^\mu f_{\vec{p}}) \tag{65}$$

where Σ is any space-like 3-dim. surface. Therefore the quantization of KG theory represented above is Lorentz covariant.