Lecture 9. Interaction representation and Wick theorem.

Plan.

1. Interaction picture for the Green's functions.

1.1. Interaction picture and Heisenberg field operator.

In the preceding lecture we have obtained the way of evaluating of the n-points Green functions

$$< \Omega|T(\phi(x_1)...\phi(x_n))|\Omega > \tag{1}$$

in the theory with ϕ^4 interaction. We used the functional representation for the correlation functions in euclidean space

$$< \phi(x_1)...\phi(x_N) >= \frac{1}{Z}\int [D\phi]\phi(x_1)...\phi(x_N)\exp\left(-A[\phi]\right). \tag{2}$$

and applied the perturbation theory approach to represent the correlation functions in interacting theory in terms of the correlation functions of free theory. Then we obtained the Green's function (1) in Minkowski space as a result of analitic continuation of the correlation function determined in euclidean space (with imaginary time $\tau = \imath t$). **Here we reproduce these results considering the theory in Minkowski space and using Hamiltonian approach.**

In the theory with interaction the Heisenberg field operators $\phi(x) \equiv \phi(\vec{x}, t)$ satisfy, by definition the equation

$$\imath \frac{\partial}{\partial t} \phi(\vec{x}, t) = [H, \phi(\vec{x}, t)] \tag{3}$$

For the ϕ^4-theory we have

$$H = H_0 + H_I = H_{KG} + \frac{\lambda}{4!} \int d^4 x \phi^4(x) \tag{4}$$

We are going to find a representation for the Green's function as a perturbation series over the powers of λ. **According to Feynman, this perturbation series can be understood as the interaction processes in space-time.**

Notice that the interaction Hamiltonian H_I appears not only in the definition of Heisenberg operator (69) but also in the definition of vacuum state $|\Omega>$.

The idea behind the perturbation theory is to express the Heisenberg field $\phi(x)$ and vacuum state $|\Omega>$ in terms of the free (KG) Heisenberg field and free field (KG) vacuum state $|0>$.

For any fixed t_0, using translation invariance, we can expand the Heisenberg field in the interation theory as

$$\phi(\vec{x}, t_0) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{E_{\vec{p}}}} (a_{\vec{p}} \exp(\imath \vec{p}\vec{x}) + a^\dagger_{\vec{p}} \exp(-\imath \vec{p}\vec{x})) \tag{5}$$

where the operators $a_{\vec{p}}, a^{\dagger}_{\vec{p}}$ satisfy the standard commutation relations

$$[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] = (2\pi)^3\delta(\vec{p} - \vec{p}') \tag{6}$$

The operators $a_{\vec{p}}, a^{\dagger}_{\vec{p}}$ are expressed by the canonical variables similar to the KG field. Therefore they satisfy (6) at the moment t_0, though they do not diagonalize the Hamiltonian. In fact the commutation relations (6) will formally hold for any other moment t because

$$\begin{aligned} a_{\vec{p}}(t) &= \exp\left[\imath(t - t_0)H\right]a_{\vec{p}}\exp\left[-\imath(t - t_0)H\right], \\ a^{\dagger}_{\vec{p}}(t) &= \exp\left[\imath(t - t_0)H\right]a^{\dagger}_{\vec{p}}\exp\left[-\imath(t - t_0)H\right]. \end{aligned} \tag{7}$$

Then for an arbitrary $t \neq t_0$

$$\begin{aligned} \phi(\vec{x}, t) = \exp\left(\imath H(t - t_0)\right)\phi(\vec{x}, t_0)\exp\left(-\imath H(t - t_0)\right) = \\ \exp\left(\imath H(t - t_0)\right)\exp\left(-\imath H_0(t - t_0)\right)\exp\left(\imath H_0(t - t_0)\right)\phi(\vec{x}, t_0) \\ \exp\left(-\imath H_0(t - t_0)\right)\exp\left(\imath H_0(t - t_0)\right)\exp\left(-\imath H(t - t_0)\right) = \\ \exp\left(\imath H(t - t_0)\right)\exp\left(-\imath H_0(t - t_0)\right)\phi_I(\vec{x}, t)\exp\left(\imath H_0(t - t_0)\right)\exp\left(-\imath H(t - t_0)\right) \end{aligned} \tag{8}$$

where the field

$$\begin{aligned} \phi_I(\vec{x}, t) = \exp\left(\imath H_0(t - t_0)\right)\phi(\vec{x}, t_0)\exp\left(-\imath H_0(t - t_0)\right) = \\ \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_{\vec{p}}}}\left(a_{\vec{p}}\exp\left(\imath\vec{p}\vec{x} - \imath E_{\vec{p}}(t - t_0)\right) + a^{\dagger}_{\vec{p}}\exp\left(-\imath\vec{p}\vec{x} + \imath E_{\vec{p}}(t - t_0)\right)\right) \end{aligned} \tag{9}$$

is called **interaction picture field**. It is clear that $\phi_I(x)$ satisfy KG equation of motion.

If the coupling constant λ is small the field $\phi_I(\vec{x}, t)$ determines the main contribution to the total Heisenberg field $\phi(\vec{x}, t)$ of the ϕ^4-theory.

Now we can write for the Heisenberg field

$$\phi(\vec{x}, t) = U^{\dagger}(t, t_0)\phi_I(\vec{x}, t)U(t, t_0) \tag{10}$$

where the unitary operator

$$U(t, t_0) = \exp\left(\imath H_0(t - t_0)\right)\exp\left(-\imath H(t - t_0)\right) \tag{11}$$

which is known as **evolution operator in interaction picture** has been introduced.

Although this operator is written in terms of the $\phi(\vec{x}, t)$, it is natural to expect that for small λ it can be expressed in terms of $\phi_I(x)$. This can be done noticing that $U(t, t_0)$ is the unique solution of the equation

$$\begin{gathered}
\imath\frac{\partial}{\partial t}U(t, t_0) = \exp\left(\imath H_0(t - t_0)\right)(H - H_0)\exp\left(-\imath H(t - t_0)\right) = \\
\exp\left(\imath H_0(t - t_0)\right)H_I\exp\left(-\imath H_0(t - t_0)\right)\exp\left(\imath H_0(t - t_0)\right)\exp\left(-\imath H(t - t_0)\right) = \\
H_I(t)U(t, t_0) \\
\Leftrightarrow \\
\imath\frac{\partial}{\partial t}U(t, t_0) = H_I(t)U(t, t_0)
\end{gathered} \tag{12}$$

where $U(t_0, t_0) = 1$ and

$$\begin{gathered}
H_I(t) = \exp\left(\imath H_0(t - t_0)\right)H_I\exp\left(-\imath H_0(t - t_0)\right) = \\
\frac{\lambda}{4!}\int d^4x\exp\left(\imath H_0(t - t_0)\right)\phi^4(\vec{x}, t_0)\exp\left(-\imath H_0(t - t_0)\right) = \\
\frac{\lambda}{4!}\int d^4x\phi_I^4(x)
\end{gathered} \tag{13}$$

is the **interaction Hamiltonian written in interaction picture**.

The solution of (12) can be represented as a series

$$U(t,t_0) = 1 + (-\imath)\int_{t_0}^{t} dt_1 H_I(t_1) + (-\imath)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + (-\imath)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_I(t_1)H_I(t_2)H_I(t_3) + ... \quad (14)$$

where the operators $H_I(t_i)$ are time-ordered. It can be written in a more compact form

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-\imath)^n}{n!} \int_{t_0}^{t} dt_1...dt_n T(H_I(t_1)...H_I(t_n)) = T\exp\left(-\imath \int_{t_0}^{t} dt' H_I(t')\right) \quad (15)$$

where the time-ordered exponent is determined by the Taylor series of time-ordered terms. **Now the Heisenberg field $\phi(x)$ of the interacting theory is written in terms $\phi_I(x)$ which is just a KG Heisenberg field**.

We can consider more general evolution operator

$$U(t,t') = T\exp\left(-\imath \int_{t'}^{t} dt'' H_I(t'')\right) \ , \ \imath\frac{\partial}{\partial t}U(t,t') = H_I(t)U(t,t'), \ t \geq t' \quad (16)$$

with the initial condition $U(t',t') = 1$.

This is a unitary operator because the solution of the equation above is

$$U(t,t') = \exp\left(\imath H_0(t-t_0)\right)\exp\left(-\imath H(t-t')\right)\exp\left(-\imath H_0(t'-t_0)\right). \quad (17)$$

One can check also that

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3) \quad (18)$$

for any $t_1 \geq t_2 \geq t_3$.

1.2. Vacuum state in interaction picture.

One has to find also a representation for the vacuum state $|\Omega>$ in interacting theory. **The statement is**

$$|\Omega>= \lim_{T\to\infty(1-\imath\epsilon)} (\exp{(-\imath E_0 T < \Omega|0>)})^{-1} \exp{(-\imath HT)}|0> \tag{19}$$

To proove this one can write

$$\begin{gathered}\exp{(-\imath HT)}|0>= \sum_{n=0}^{\infty} \exp{(-\imath E_n T)}|n>< n|0>= \\ \exp{(-\imath E_0 T)}|\Omega>< \Omega|0> + \sum_{n\neq 0}^{\infty} \exp{(-\imath E_n T)}|n>< n|0>\end{gathered} \tag{20}$$

where $|n>$ and E_n are the eigenstates and eigenvalues of the H. **We must assume of course that $<\Omega|0>\neq 0$, otherwise the H_I can not be considered as a small perturbation**.

In the limit $T \to \infty(1-\imath\epsilon)$ only the vacuum state contribution survives in the formula above because of $E_n > E_0$ for $n \neq 0$.

1.3. Green's function formula.

Now one can find the expression for the Green's functions in interacting theory in terms of the Green's functions of free field theory.

Let us consider for example 2-point Green's function $<\phi(x)\phi(y)>$, where $x^0 > y^0$. Then

$$
\begin{aligned}
< T(\phi(x)\phi(y)) >=< \Omega|T(\phi(x)\phi(y))|\Omega =< \Omega|\phi(\vec{x},x^0)\phi(\vec{y},y^0)|\Omega >= \\
< \Omega|U^{-1}(x^0,t_0)\phi_I(\vec{x},x^0)U(x^0,t_0)U^{-1}(y^0,t_0)\phi_I(\vec{y},y^0)U(y^0,t_0)|\Omega >= \\
< \Omega|U^{-1}(x^0,t_0)\phi_I(\vec{x},x^0)U(x^0,y^0)U(y^0,t_0)U^{-1}(y^0,t_0)\phi_I(\vec{y},y^0)U(y^0,t_0)|\Omega >= \\
< \Omega|U^{-1}(x^0,t_0)\phi_I(\vec{x},x^0)U(x^0,y^0)\phi_I(\vec{y},y^0)U(y^0,t_0)|\Omega >
\end{aligned}
\tag{21}
$$

If one substitutes the expression (19) for the vacuum $|\Omega >$ and notices taht

$$
\begin{aligned}
\exp(-\imath H(t_0-(-T))|0 >= \exp(-\imath H(t_0-(-T))\exp(\imath H_0(t_0-(-T))|0 > \\
= U(t_0,-T])|0 >
\end{aligned}
\tag{22}
$$

one can obtain

$$
\begin{aligned}
< \Omega|\phi(x)\phi(y)|\Omega >= \\
\lim_{T\to\infty(1-\imath\epsilon)}(\exp(-\imath E_0(T-t_0) < \Omega|0 >))^{-1} < 0|U(T,t_0) \\
U^\dagger(x^0,t_0)\phi_I(x)U(x^0,t_0)U^\dagger(y^0,t_0)\phi_I(y)U(y^0,t_0) \\
U(t_0,-T)|0 > (\exp(-\imath E_0(t_0-(-T))) < \Omega|0 >)^{-1} = \\
\lim_{T\to\infty(1-\imath\epsilon)}(| < 0|\Omega > |^2\exp(-\imath E_0 2T))^{-1} \\
< 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0 >
\end{aligned}
\tag{23}
$$

where $y^0 > t_0$. If one devides this expression by

$$
1 =< \Omega|\Omega >= (| < 0|\Omega > |^2\exp(-\imath E_0 2T))^{-1} < 0|U(T,t_0)U(t_0,-T)|0 >
\tag{24}
$$

we get the desired expression

$$< \Omega|\phi(x)\phi(y)|\Omega >= \lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0 >}{< 0|U(T,-T)|0 >} \tag{25}$$

where $U(T,x^0)$, $U(x^0,y^0)$, $U(y^0,-T)$ are given by (16). Since all the fields in this expression are time-ordered we can write in general

$$< \Omega|T(\phi(x)\phi(y))|\Omega >= \lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\phi_I(x)\phi_I(y)\exp{(-\imath\int_{-T}^{T} dtH_I(t)))}|0 >}{< 0|T(\exp{(-\imath\int_{-T}^{T} dtH_I(t)))}|0 >} \tag{26}$$

It is clear that for the n-point Green's function we have similar expression

$$< \Omega|T(\phi(x_1)...\phi(x_n))|\Omega >= \lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\phi_I(x_1)...\phi_I(x_n)\exp{(-\imath\int_{-T}^{T} dtH_I(t)))}|0 >}{< 0|T(\exp{(-\imath\int_{-T}^{T} dtH_I(t)))}|0 >} \tag{27}$$

Expanding the exponentials in the powers of coupling constant λ we get the representation of the Green's function of the interacting theory as a series of the Green's functions of free theory. This is what we have found in the previous lecture, considering the perturbation theory for the correlation functions in euclidean space and making then analitic continuation to the imaginary values of euclidean time τ.

2. Normal ordering and Wick's theorem.

The formula (27) gives the way to evaulate the Green's functions in interracting theory in terms of Green's functions of the fields $\phi_I(x)$ of free (KG) theory. Indeed, if we expand the interaction Hamiltonian H_I in a series the r.h.s of (27) will contain the Green's functions of free theory so one can apply Wick's theorem (in operator formulation) again.

2.1. Normal ordering of operators.

To this end let us consider first the 2-point Green's function $< 0|T(\phi_I(x)\phi_I(y))|0 >$ in KG theory and expand the Heisenberg field operators by the positive and negative frequences parts

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x),$$
$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}a_{\vec{p}}\exp\left(-\imath px\right)$$
$$\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}a^\dagger_{\vec{p}}\exp\left(\imath px\right) \tag{28}$$

Suppose that $x^0 > y^0$ in $T(\phi_I(x)\phi_I(y))$. Then

$$T(\phi_I(x)\phi_I(y)) = \phi_I(x)\phi_I(y) =$$
$$\phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) =$$
$$\phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + [\phi_I^+(x), \phi_I^-(y)] \tag{29}$$

In each term of this expression, excluding the commutator, the annihilation operators $a_{\vec{p}}$ are to the right of the creation operators $a^\dagger_{\vec{p}}$. This way to order the fields in $T(\phi_I(x)\phi_I(y))$ is convenient because by the vacuum definition

$$\phi_I^+(x)|0 >= 0 =< 0|\phi_I^-(x) \tag{30}$$

so that the all terms in $< 0|T(\phi_I(x)\phi_I(y))|0 >$ except the commutator vanish. The way to arrange the creation-annihilation operators in the operator product when the annihilation operators stand to the right of the creation operators is called **the normal ordering of operators**. The normal ordering of operators $a_{\vec{q}}...a^{\dagger}_{\vec{p}}...$ is denoted usualy as

$$: a_{\vec{q}} \, ... a^{\dagger}_{\vec{p}} \, ... : \tag{31}$$

For example

$$: a_{\vec{s}} a^{\dagger}_{\vec{p}} a_{\vec{q}} a_{\vec{r}} a^{\dagger}_{\vec{t}} := a^{\dagger}_{\vec{p}} a^{\dagger}_{\vec{t}} a_{\vec{s}} a_{\vec{q}} a_{\vec{r}} \tag{32}$$

Suppose now that $y^0 > x^0$ **in** $T(\phi_I(x)\phi_I(y))$. Then

$$\begin{aligned} T(\phi_I(x)\phi_I(y)) = \phi_I(y)\phi_I(x) = \\ \phi_I^+(y)\phi_I^+(x) + \phi_I^+(y)\phi_I^-(x) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(y)\phi_I^-(x) = \\ \phi_I^+(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(y)\phi_I^-(x) + [\phi_I^+(y), \phi_I^-(x)] \end{aligned} \tag{33}$$

and hence all terms in $< 0|T(\phi_I(x)\phi_I(y))|0 >$ are vanishing except the last commutator. It makes sense therefore to define the operation which is called **contraction**:

$$\begin{aligned} \overbrace{\phi(x)\phi}(y) = [\phi^+(x), \phi^-(y)] \; if \; x^0 > y^0 \\ and \\ \overbrace{\phi(x)\phi}(y) = [\phi^+(y), \phi^-(x)] \; if \; y^0 > x^0 \end{aligned} \tag{34}$$

But the contraction so defined coincides with the definition of Feynman's propagator

$$\overbrace{\phi(x)\phi}(y) = D_F(x-y) \tag{35}$$

Hence we can write

$$T(\phi_I(x)\phi_I(y)) =: \phi_I(x)\phi_I(y) : + \overbrace{\phi(x)\phi}(y) \tag{36}$$

2.2. Wick's theorem.

Wick's Theorem:

$$
\begin{aligned}
T(\phi_I(x_1)...\phi_I(x_N)) =: \phi_I(_1)...\phi_I(x_N): + \\
sum\ of\ \ :\phi_I(_1)...\phi_I(x_N):\ \ with\ all\ possible\ contractions\ inside
\end{aligned}
\tag{37}
$$

Example

$$
\begin{aligned}
T(\phi_I(x_1)...\phi_I(x_4)) =: \phi_I(x_1)...\phi_I(x_4): + : \wick{\c1 \phi_I(x_1) \c1 \phi_I(x_2)}...\phi_I(x_4): + \\
: \wick{\c1 \phi_I(x_1)\phi_I(x_2) \c1 \phi_I(x_3)}\phi_I(x_4): + : \wick{\c1 \phi_I(x_1)\phi_I(x_2)\phi_I(x_3) \c1 \phi_I(x_4)}: + \\
: \phi_I(x_1)\wick{\c1 \phi_I(x_2) \c1 \phi_I(x_3)}\phi_I(x_4): +...+ : \phi_I(x_1)\phi_I(x_2)\wick{\c1 \phi_I(x_3) \c1 \phi_I(x_4)}: + \\
: \wick{\c1 \phi_I(x_1) \c1 \phi_I(x_2) \c2 \phi_I(x_3) \c2 \phi_I(x_4)}: +...+ : \wick{\c1 \phi_I(x_1) \c2 \phi_I(x_2) \c2 \phi_I(x_3) \c1 \phi_I(x_4)}:
\end{aligned}
\tag{38}
$$

Therefore

$$
\begin{aligned}
< 0|T(\phi_I(x_1)...\phi_I(x_4))|0 > =< 0|\wick{\c1 \phi_I(x_1) \c1 \phi_I(x_2) \c2 \phi_I(x_3) \c2 \phi_I(x_4)}|0 > +... + \\
< 0|\wick{\c1 \phi_I(x_1) \c2 \phi_I(x_2) \c2 \phi_I(x_3) \c1 \phi_I(x_4)}|0 > = \\
D_F(x_1 - x_2)D_F(x_3 - x_4) + ... + D_F(x_1 - x_4)D_F(x_2 - x_3).
\end{aligned}
\tag{39}
$$

To prove the theorem we use induction method. For the 2-point function the theorem is proved already. Suppose it is also proved for $N-1$-point function. One can choose without restrictions the ordering in the form $x_1^0 > ... > x_N^0$ (if it's not the case one can relabel the points to get the ordering we choose). Then we can write

$$
\begin{aligned}
T(\phi(x_1)...\phi(x_N)) = \phi(x_1)...\phi(x_N) = \\
\phi(x_1) : (\phi(x_2)...\phi(x_N) + \\
\textit{sum of } \phi(_2)...\phi(x_N) \textit{ with all possible contactions}) := \\
(\phi^+(x_1) + \phi^-(x_1))(: \phi(x_2)...\phi(x_N) : + \\
\textit{sum of } \; : \phi(x_2)...\phi(x_N) : \; \textit{with all possible contactions inside})
\end{aligned}
\tag{40}
$$

(here and in what follows the label I will be omitted).

One needs to put the field $(\phi^+(x_1)+\phi^-(x_1))$ inside the normal ordering. It is easy to put the term $\phi^-(x_1)$ inside the normal ordering because it is anough to put it from the left in all terms in the sum over the contractions. To put the second term $\phi^+(x_1)$ one needs to take into account all commutators which appear while we are moving $\phi^+(x_1)$ to put it from the right of all the operators $\phi(x_i)$, $i = 2, ..., N$. For example

$$
\begin{aligned}
\phi^+(x_1) : \phi(x_2)...\phi(x_N) := \\
: \phi(x_2)...\phi(x_N) : \phi^+(x_1) + : [\phi^+(x_1), \phi(x_2)]\phi(x_3)...\phi(x_N) : +... \\
: \phi(x_2)...\phi(x_{N-1})[\phi^+(x_1), \phi(x_N)] := \\
: \phi^+(x_1)\phi(x_2)...\phi(x_N) : + : \contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2)\phi(x_3)...\phi(x_N) : + \\
: \contraction{}{\phi}{(x_1)\phi(x_2)}{\phi}\phi(x_1)\phi(x_2)\phi(x_3)...\phi(x_N) : +...+ : \contraction{}{\phi}{(x_1)\phi(x_2)\phi(x_3)...}{\phi}\phi(x_1)\phi(x_2)\phi(x_3)...\phi(x_N) :
\end{aligned}
\tag{41}
$$

Thus we have

$$
\begin{aligned}
\phi(x_1) : \phi(x_2)...\phi(x_N) :=: \phi(x_1)\phi(x_2)...\phi(x_N) : + \\
: \contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2)\phi(x_3)...\phi(x_N) : +...+ : \contraction{}{\phi}{(x_1)\phi(x_2)\phi(x_3)...}{\phi}\phi(x_1)\phi(x_2)\phi(x_3)...\phi(x_N) :
\end{aligned}
\tag{42}
$$

Similarly doing with all other terms from (40) we obtain the statement of the theorem.

2.3. Green's functions and Feynman diagrams.

When we calculate Green's functions, Wick's theorem, we just have proved, leads again to the Feynman diagrams.

Let us consider for example the 2-points Green's function $< T(\phi(x)\phi(y)) >$. According to (26) the numerator is

$$
\begin{aligned}
\lim_{T\to\infty(1-\imath\epsilon)} < 0|T(\phi_I(x)\phi_I(y)\exp{(-\imath\int_{-T}^{T} dt H_I(t)))}|0>= \\
\lim_{T\to\infty(1-\imath\epsilon)} [< 0|T(\phi_I(x)\phi_I(y))|0> + \\
\frac{-\imath\lambda}{4!} < 0|T(\phi_I(x)\phi_I(y)\int_{-T}^{T} dt \int d^3u\phi_I^4(\vec{u},t))|0> + \\
\frac{1}{2!}(\frac{-\imath\lambda}{4!})^2 < 0|T(\phi_I(x)\phi_I(y)\int_{-T}^{T} dtd^3u\phi_I^4(\vec{u},t)\int_{-T}^{T} dt'd^3v\phi_I^4(\vec{v},t'))|0> + ...]
\end{aligned}
\tag{43}
$$

At zero perturbation order the Wick's theorem gives

$$
< 0|T(\phi_I(x)\phi_I(y))|0> =< 0|(:\phi(x)\phi(y): + D_F(x-y))|0> = D_F(x-y)
\tag{44}
$$

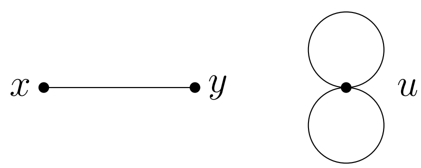
Now we consider the first perturbation order

$$
\begin{aligned}
\frac{-\imath\lambda}{4!}\int d^4u < 0|T(\phi_I(x)\phi_I(y)\phi_I^4(u))|0> = \\
\frac{-\imath\lambda}{4!}\int d^4u(< 0| : \phi(x)\phi(y)\phi(u)\phi(u)\phi(u)\phi(u) : |0> + \\
< 0| : \phi(x)\phi(y)\phi(u)\phi(u)\phi(u)\phi(u) : |0> + \\
4 < 0| : \phi(x)\phi(y)\phi(u)\phi(u)\phi(u)\phi(u) : |0> + ... \\
+3 < 0|\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)\phi(u)|0> + \\
12 < 0|\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)\phi(u)|0>)
\end{aligned}
\tag{45}
$$

Due to the normal ordering definition, only the last two terms give nonzero contribution:

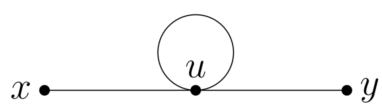
$$
\begin{aligned}
&\frac{-\imath\lambda}{4!}\int d^4u < 0|T(\phi_I(x)\phi_I(y)\phi_I^4(u))|0 >= \\
&\frac{-\imath\lambda}{4!}\int d^4u(3 < 0|\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)\phi(u)|0 > + \\
&12 < 0|\phi(x)\phi(y)\phi(u)\phi(u)\phi(u)\phi(u)|0 >) = \\
&\frac{-\imath\lambda}{4!}\int d^4u(3D_F(x-y)D_F(u-u)^2 + 12D_F(x-u)D_F(y-u)D_F(u-u))
\end{aligned}
\tag{46}
$$

These terms can be represented by the following diagrams: 3 diagrams of type



(47)

plus 12 diagrams of the type

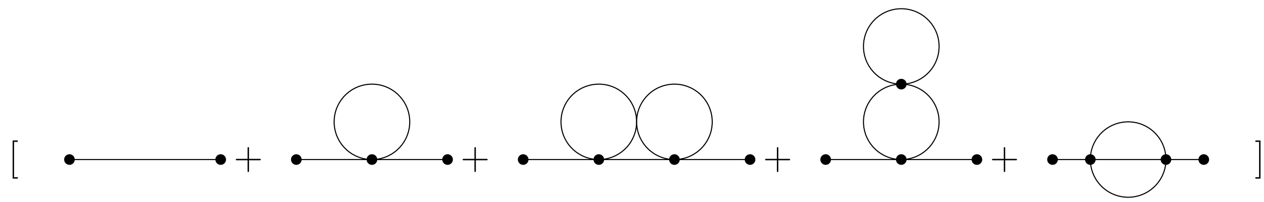


(48)

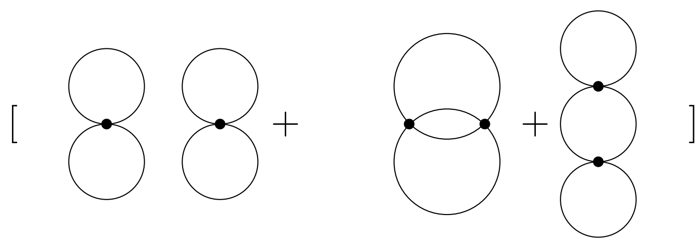
The second perturbation order is

$$
\frac{1}{2!}(\frac{-\imath\lambda}{4!})^2\int d^4ud^4v < 0|T(\phi(x)\phi(y)\phi^4(u)\phi^4(v))|0 >
\tag{49}
$$

According to Wick's theorem the nonzero contribution is given by the terms where all Heisenberg operators are contracted. It gives the sum of diagrams consisting of product of connected digrams and vacuum diagrams:



(50)

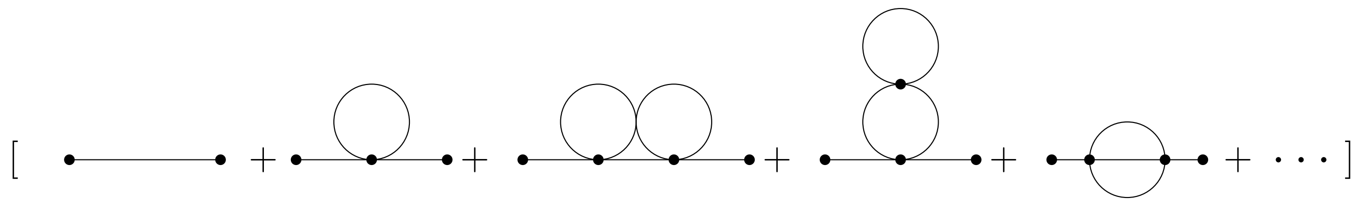


(51)

where the product of diagrams from the first factor and diagrams from the second factor is taken in such a way to have second order perturbation factor. Again we see here the contribution of vacuum diagrams.

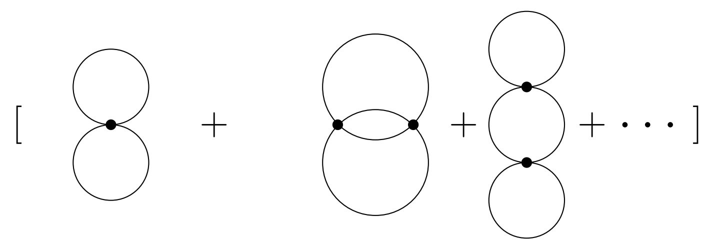
Notice also that the vacuum diagram we have had at the first order appears here to the power of 2 and with the factor $\frac{1}{2!}$. One may also find that this diagram will appear at third perturbation order to the power of 3 and with the factor $\frac{1}{3!}$. It is easy to see that the contribution of this diagram at higher orders is given by the exponential. The same is true for other vacuum diagrams which appeared at the second order. One can see in fact, that the **vacuum diagrams contributions exponentiate when we consider higher perturbation theory orders**, so that we can write

$$\lim_{T\to\infty(1-\imath\epsilon)} <0|T(\phi_I(x)\phi_I(y)\exp{(-\imath\int_{-T}^{T} dtH_I(t)))}|0>=$$



(52)

exp



(53)

The denominator of the Green's 2-point function can be analysed similarly, so that we find

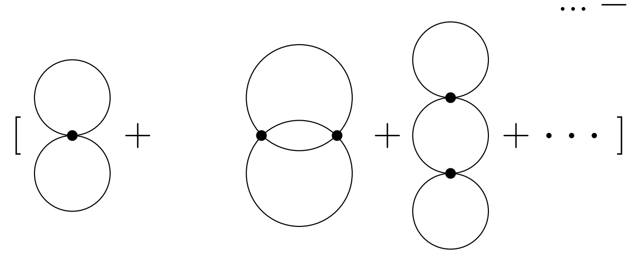
$$\lim_{T\to\infty(1-\imath\epsilon)} <0|T(\exp{(-\imath\int_{-T}^{T} dtH_I(t)))}|0>=$$

$$1+\frac{-\imath\lambda}{4!}\int d^4u <0|T(\phi(u)\phi(u)\phi(u)\phi(u))|0>+$$

$$\frac{1}{2!}\frac{(-\imath\lambda}{4!)^2}\int d^4ud^4v <0|T(\phi(u)\phi(u)\phi(u)\phi(u)\phi(v)\phi(v)\phi(v)\phi(v))|0>+$$

$$...=$$

exp

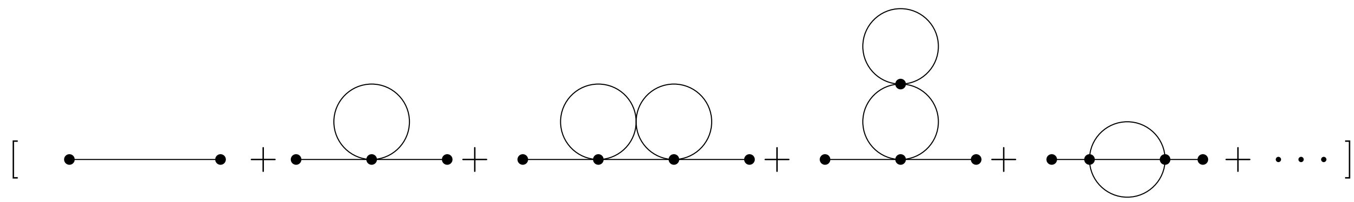


(54)

Hence the denominator cancels the vacuum diagrams contribution from numerator. Thus, **the 2-points Green's function in ϕ^4 theory is given**

by the contribution of all connected diagrams with two external points:

$$< \Omega|T(\phi(x)\phi(y)|\Omega >=$$



(55)

The multi-points Green's functions can be analysed similarly.

3. Wick's theorem for fermions and Yukawa model.

3.1. Normal ordering and Wick's theorem for fermions.

We start with the calculation of $< 0|T(\psi(x)\bar{\psi}(y))|0 >$ for the Dirac field. As one knows this Green's function is Feynman's propagator for Dirac fermions. But now we calculate it directly using creation-anihilation operators.

In the Hesenberg picture the field operators are

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s (a^s_{\vec{p}}u^s(\vec{p})\exp{(-\imath px)} + b^{s\dagger}_{\vec{p}}v^s(\vec{p})\exp{(\imath px)})$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s ((a^s_{\vec{p}})^\dagger\bar{u}^s(\vec{p})\exp{(\imath px)} + (b^s_{\vec{p}})\bar{v}^s(\vec{p})\exp{(-\imath px)}) \quad (56)$$

where the creation-annihilation operators satisfy the anti-commutators relations

$$[a^s_{\vec{p}}, a^{r\dagger}_{\vec{q}}]_+ = [b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}]_+ = (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta^{s,r} \quad (57)$$

Let us introduce the decompositions

$$\psi^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s a^s_{\vec{p}} u^s(\vec{p}) \exp(-\imath px),$$
$$\bar{\psi}^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s b^s_{\vec{p}} \bar{v}^s(\vec{p}) \exp(-\imath px),$$
$$\psi^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s b^{s\dagger}_{\vec{p}} v^s(\vec{p}) \exp(\imath px),$$
$$\bar{\psi}^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s a^{s\dagger}_{\vec{p}} \bar{u}^s(\vec{p}) \exp(\imath px) \tag{58}$$

Suppose that $x^0 > y^0$ **in** $T(\psi(x)\bar{\psi}(y))$. Then

$$T(\psi(x)\bar{\psi}(y)) = (\psi^+(x)+\psi^-(x))(\bar{\psi}^+(y)+\bar{\psi}^-(y)) =$$
$$\psi^+(x)\bar{\psi}^+(y) - \bar{\psi}^-(y)\psi^+(x) + \psi^-(x)\bar{\psi}^+(y) + \psi^-(x)\bar{\psi}^-(y) + [\psi^+(x), \bar{\psi}^-(y)]_+ \tag{59}$$

where we have taken into account Fermi statistics of the Dirac field.

In each term of this expression, excluding the anti-commutator, the annihilation operators $a^s_{\vec{p}}$, $b^s_{\vec{p}}$ are to the right of the creation operators $a^{s\dagger}_{\vec{p}}$, $b^{s\dagger}_{\vec{p}}$. This way to order the fields in $T(\psi(x)\bar{\psi}(y))$ is convenient because by the vacuum definition

$$\psi^+(x)|0>= 0 = \bar{\psi}^+(x)|0>, \; <0|\psi^-(x) = 0 =<0|\bar{\psi}^-(x) \tag{60}$$

so that the all terms in $<0|T(\psi(x)\bar{\psi}(y))|0>$ except the anti-commutator vanish. **The way to arrange the creation-annihilation operators in the operator product when the annihilation operators stand to the right of the creation operators is called the normal ordering**

of operators. The normal ordering of operators $a^s_{\vec{q}}...b^r_{\vec{p}}...a^{t\dagger}_{\vec{k}}...b^{u\dagger}_{\vec{l}}$ is denoted usualy as

$$: a^s_{\vec{q}}...b^r_{\vec{p}}...a^{t\dagger}_{\vec{k}}...b^{u\dagger}_{\vec{l}} : \tag{61}$$

The only difference from the bosonic case in the definition of normal ordering of fermions is the sign factor (due to Fermi statistics). For example

$$: a^s_{\vec{q}} a^{r\dagger}_{\vec{p}} b^t_{\vec{k}} b^{u\dagger}_{\vec{l}} :=-a^{r\dagger}_{\vec{p}} b^{u\dagger}_{\vec{l}} a^s_{\vec{q}} b^t_{\vec{k}} \tag{62}$$

Suppose now that $y^0 > x^0$ **in** $T(\psi(x)\bar{\psi}(y))$. Then

$$T(\psi(x)\bar{\psi}(y)) = -\bar{\psi}(y)\psi(x) =$$
$$-\bar{\psi}^+(y)\psi^+(x) + \psi^-(x)\bar{\psi}^+(y) - \bar{\psi}^-(y)\psi^+(x) - \bar{\psi}^-(y)\psi^-(x) - [\bar{\psi}^+(y), \psi^-(y)]_+ \tag{63}$$

where the Fermi statistics has been taken into account again.

The all terms in $< 0|T(\psi(x)\bar{\psi}(y))|0 >$ are vanishing except the last anti-commutator. It makes sense therefore to define the operation which is called **contraction** for fermions:

$$\overbracket{\psi(x)\bar{\psi}}(y) = [\psi^+(x), \bar{\psi}^-(y)]_+ \; if \; x^0 > y^0$$
$$and$$
$$\overbracket{\psi(x)\bar{\psi}}(y) = -[\bar{\psi}^+(y), \psi^-(x)]_+ \; if \; y^0 > x^0$$
$$\overbracket{\psi(x)\psi}(y) = \overbracket{\bar{\psi}(x)\bar{\psi}}(y) = 0. \tag{64}$$

But the contraction so defined coincides with the definition of

Feynman's propagator for Dirac fermions

$$\overbrace{\psi(x)\bar{\psi}}(y) = S_F(x-y),$$
$$S_F(x-y) =$$
$$\Theta(x^0 - y^0) < 0|\psi_a(x)\bar{\psi}_b(y)|0 > -\Theta(y^0 - x^0) < 0|\bar{\psi}_b(y)\psi_a(x)|0 >=$$
$$\int \frac{d^4p}{(2\pi)^4} \frac{\imath(p_\mu\gamma^\mu + m)}{p^2 - m^2 + \imath\epsilon} \exp\left(-\imath p(x-y)\right) \quad (65)$$

Hence, we can write

$$T(\psi(x)\bar{\psi}(y)) =: \psi(x)\bar{\psi}(y) : + \overbrace{\psi(x)\bar{\psi}}(y) \quad (66)$$

It allows to prove by induction **Wick's Theorem:**

$$T(\psi(x_1)\bar{\psi}(x_2)...\psi(x_N)) =: \psi(x_1)\bar{\psi}(x_2)...\psi(x_N) : +$$
$$sum\ of\ \ : \psi(x_1)\bar{\psi}(x_2)...\psi(x_N) :\ \ with\ all\ possible\ contractions\ inside. \quad (67)$$

3.2. Interaction picture for Yukawa model.

The Yukawa model can be considered as a simplifying version of QED where the foton is replaced by a scalar particle. The Lagrangian is the sum

$$\mathcal{L}_Y = \mathcal{L}_{KG} + \mathcal{L}_{Dir} + \mathcal{L}_{int},$$
$$\mathcal{L}_{KG} = \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2],$$
$$\mathcal{L}_{Dir} = \bar{\psi}(\imath\gamma^\mu\partial_\mu - m)\psi,$$
$$\mathcal{L}_{int} = -g\phi\bar{\psi}\psi. \quad (68)$$

Similarly to the ϕ^4 model one can develop interaction picture for Yukawa model assuming that coupling constant g is small and introducing Heisenberg fields operators $\psi_I(x)$, $\phi_I(x)$ in **interaction picture**. By the definition, these fields obey the free field theory equations of motion:

$$\imath\frac{\partial}{\partial t}\phi_I(\vec{x},t) = [H_{KG}, \phi_I(\vec{x},t)]$$
$$\imath\frac{\partial}{\partial t}\psi_I(\vec{x},t) = [H_{Dir}, \psi_I(\vec{x},t)] \tag{69}$$

Then we can express the Heisenberg fields $\psi(x)$, $\phi(x)$ and the vacuum state $|\Omega >$ of Yukawa theory in terms of the Heisenberg fields $\psi_I(x)$, $\phi_I(x)$ and vacuum $|0>$ of free theory similarly to the case of ϕ^4 theory. Doing this waywe introduce the **evolution operator in interaction picture**

$$U(t,t_0) = \exp\left(\imath(t-t_0)H_0\right)\exp\left(\imath(t-t_0)H\right) \tag{70}$$

where $H_0 = H_{KG} + H_{Dir}$, $H = H_0 + H_{int}$, which allows to express the Heisenberg interaction fields in terms of free fields ϕ_I, ψ_I

$$\phi(\vec{x},t) = U^\dagger(t,t_0)\phi_I(\vec{x},t)U(t,t_0),\ \psi(\vec{x},t) = U^\dagger(t,t_0)\psi_I(\vec{x},t)U(t,t_0). \tag{71}$$

This operatorsatisfy the differential equation

$$\imath\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0) \tag{72}$$

which can be solved as

$$U(t,t_0) = T\exp\left(-\imath\int_{t_0}^{t} dt' H(t')\right),\ U(t_0,t_0) = 1, \tag{73}$$

where

$$H_I(t) = g\int d^3x\phi_I(\vec{x},t)\psi_I(\vec{x},t)\psi(\vec{x},t). \tag{74}$$

It allows to obtain the Green's function formula like this

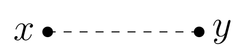
$$< \Omega|T(\psi(x_1)...\psi(x_n)\bar{\psi}(y_1)...\bar{\psi}(y_n)\phi(z_1)...\phi(z_m))|\Omega >=$$
$$\lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\psi_I(x_1)...\psi_I(x_n)\bar{\psi}_I(y_1)...\bar{\psi}_I(y_n)\phi_I(x_1)...\phi_I(x_n)\exp{(-\imath\int_{-T}^{T}dtH_I(t))})|0 >}{< 0|T(\exp{(-\imath\int_{-T}^{T}dtH_I(t))})|0 >}. \tag{75}$$

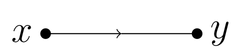
3.3. Green's functions and Feynmam diagrams.

Consider first the 2-points Green's functions

$$< \Omega|T(\phi(x)\phi(y))|\Omega >= \lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\phi_I(x)\phi_I(y)\exp{(-\imath\int_{-T}^{T}dtH_I(t))})|0 >}{< 0|T(\exp{(-\imath\int_{-T}^{T}dtH_I(t))})|0 >},$$
$$< \Omega|T(\psi(x)\bar{\psi}(y))|\Omega >= \lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\psi_I(x)\bar{\psi}(y)\exp{(-\imath\int_{-T}^{T}dtH_I(t))})|0 >}{< 0|T(\exp{(-\imath\int_{-T}^{T}dtH_I(t))})|0 >}. \tag{76}$$

One can use perturbation expansion in order to calculate these functions. At zero perturbation order the numerators give the free propagators

$D_F(x-y) =$  (77)

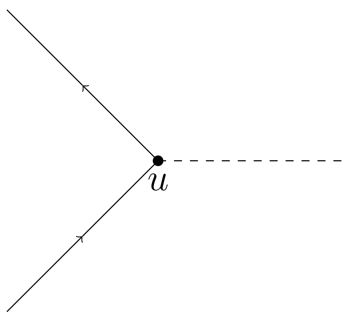
$S_F(x-y) =$  (78)

As usual, there are certainly highest order corrections which contain connected diagrams as well as unconnected fragments of vacuum diagrams, but these fragments are cancelled by the vacuum diagrams from denominator.

The interaction vertex appears when we consider 3-point Green's function $< \Omega|T(\psi(x)\bar{\psi}(y)\phi(z))|\Omega >$ at first order. In this case Wick's theorem applied to the numerator gives

$$
\begin{aligned}
-\imath g \int d^4u < 0|T(\psi(x)\bar{\psi}(y)\phi(z)\bar{\psi}(u)\psi(u)\phi(u))|0 >= \\
-\imath g \int d^4u (< 0| : \psi(x)\bar{\psi}(y)\phi(z)\bar{\psi}(u)\psi(u)\phi(u) : |0 > + ... \\
+ < 0| : \psi(x)\bar{\psi}(y)\phi(z)\bar{\psi}(u)\psi(u)\phi(u) : |0 >) = \\
-\imath g \int d^4u < 0|(-1)\psi(x)\bar{\psi}(u)(-1)\psi(u)\bar{\psi}(y)\phi(z)\phi(u)|0 >= \\
-\imath g \int d^4u S_F(x-u) S_F(u-y) D_F(z-u)
\end{aligned}
\tag{79}
$$

Thus, we find the vertex diagram



$$
= -\imath g \int d^4u \tag{80}
$$

The diagrams of propagators (77), (78) and the vertex diagram (80) generate all the diagrams in Yukawa theory. Therefore, one can calculate any Green's function of the theory using these Feynman rules.