Lecture 8. Types of diagrams and generating functionals.

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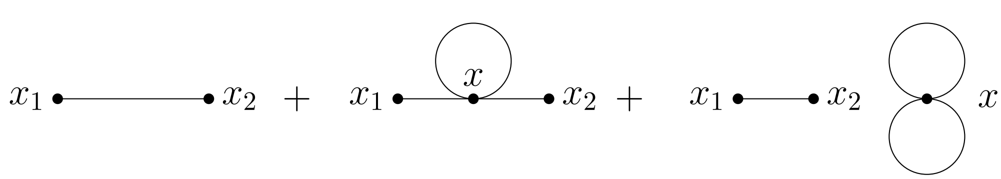
1. Vacuum diagrams in ϕ^4 theory.

1.1 DISCONNECTED AND VACUUM DIAGRAMS IN 2-POINTS FUNCTION.

Let us consider the 2-point correlation function in ϕ^4 theory paying attention to Z-factor. We can write

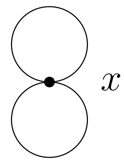
$$< \phi(x_1)\phi(x_2) >= (\frac{Z_0}{Z})(\frac{1}{Z_0}\int [D\phi]\phi(x_1)\phi(x_2)\exp\left(-A_0 - A_I\right)), \tag{1}$$

where Z_0 is the partition function of the KG theory, $A_0 = A_{KG}$, $A_I = \frac{\lambda}{4!}\int d^4x\phi^4(x)$. The diagrams we considered in the previous lecture actually correspond to the second factor from (1):



(2)

The last diagram contains disconnected piece which is



(3)

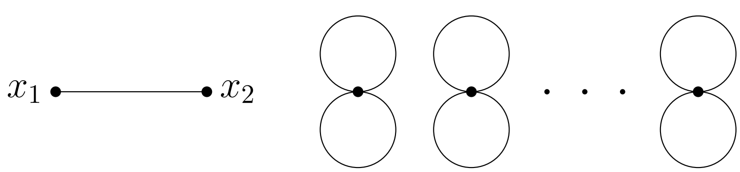
It is not connected to any of the external points x_1, x_2. Such fragments are called **vacuum diagrams**. The value of the last diagram from (2) is the products of its disconnected parts

$$D(x_1 - x_2) \times (-\frac{\lambda}{8} \int d^4x D(x-x)^2). \tag{4}$$

This holds in general: **the value of any disconnected diagram is the product of values of its connected parts** as it follows from Feynmans rules of the theory. In fact, in computing the correlation (1) the **vacuum fragments are cancelled exactly by the first factor in the r.h.s. of (1)**.

1.2. CANCELLATION OF VACUUM DIAGRAMS BY Z FACTOR.

To see how vacuum fragments cancell by the partition function factor Z consider the last diagram from (2). It is clear that this fragment will appear n-times in n-th order of the perturbation theory:

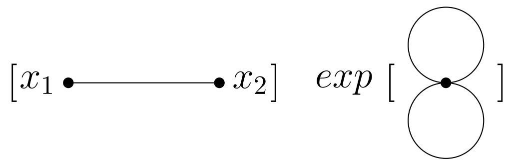


(5)

It is easy to see that the value of n vacuum fragments is

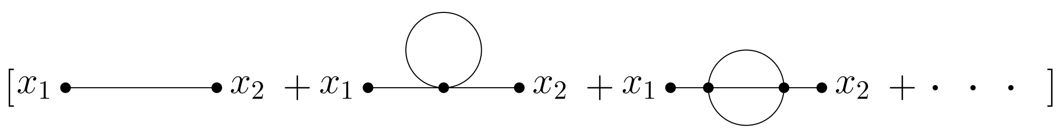
$$\frac{1}{n!}(-\frac{\lambda}{8} \int d^4x D(x-x)^2)^n \tag{6}$$

where $\frac{1}{n!}$ is an extra symmentry factor associated with the permutations of fragments. The sum over n then yelds



(7)

This analysis can be repeated with any more other vacuum fragment and with any other more complicated connected part of the diagram. As a result we obtain the product of two factors



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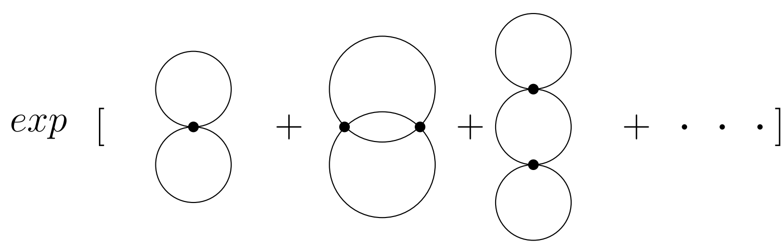


,

(9)

where in the first factor we have a sum of all connected diagrams, while in the second factor we have a sum of all connected vacuum diagrams.

Consider now the first factor from (1):

$$\frac{Z}{Z_0} = \frac{1}{Z_0} \int [D\phi] \exp\left(-A_0 - A_I\right). \tag{10}$$

Expanding this in A_I we find that this quantity is given by the sum of all vacuum diagrams which exponentiates in terms of the connected vacuum diagrams. That is

$$\frac{Z}{Z_0} = \exp\left(\textit{sum of all connected vacuum diagrams}\right). \tag{11}$$

So that this first factor cancellls the contribution of all vacuum diagrams from the second fctor of (1).

1.3. STATISTICAL PHYSICS INTERPRETATION OF VACUUM DIAGRAMS CONTRIBUTION.

One can make the following statistical physics interpretation of (10). The value of the first connected vacuum diagram is

$$-\frac{\lambda}{8}D^2(0)\int d^4x = -\frac{\lambda}{8}D^2(0)V^4. \tag{12}$$

Looking at other diagrams from (10) we can write

$$Z = Z_0 \exp\left(-\epsilon_I V^4\right), \tag{13}$$

where ϵ_I **is the sum of all connected vacuum diagrams with the volume factored out** (this factorization is also in agreement with translation invariance). So, one can interprate the functional integral

$$Z = \int [D\phi] \exp\left(-A\right) \tag{14}$$

as a (configuration) partition function of a classical statistical mechanics. The free energy F of a statistical system is related to the partition function as $Z = \exp\left[-\frac{F}{T}\right]$. We see therefore, that

$$\epsilon_0 + \epsilon_I \ , \ where \ \epsilon_0 = -\frac{\ln Z_0}{V^4} \tag{15}$$

is interpreted as the **specific free energy of this system, where** ϵ_0 **is a specific free energy of free theory and** ϵ_I **incorporates all**

corrections due to the interaction. Because of all vacuum diagrams are divergent at short distancies the short distance divergence of the specific energy is similar to the divergence of vacuum energy in KG theory.

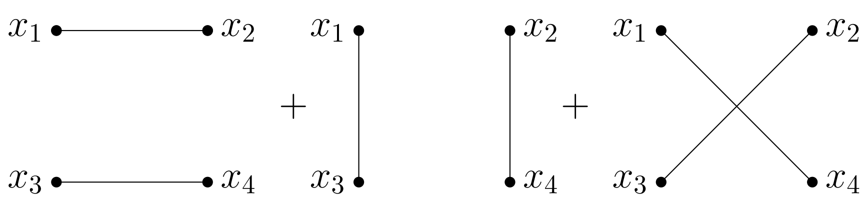
2. Generation functionals.

2.1. DIAGRAMATIC REPRESENTATION OF 4-POINT FUNCTION.

Let us consider the diagram expansion of the 4-point function

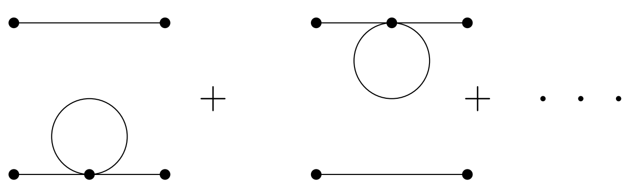
$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) > . \tag{16}$$

First of all the vacuum diagrams are again get cancelled by the expansion of Z in the denominator. However the disconnected diagrams still remain. In zero order we have:



(17)

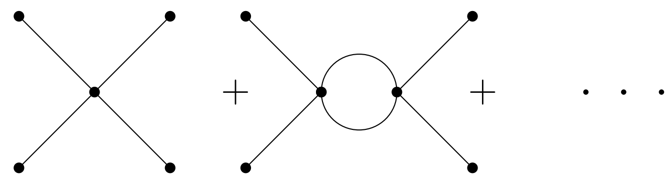
In the first order we obtain:



(18)

These diagrams and similar disconnected diagrams represent corrections to the 2-point correlation functions.

We have also the contributions from the truely connected diagrams:



(19)

It is natural to write thereby

$$
\begin{aligned}
&< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) >= \\
&< \phi(x_1)\phi(x_2) >< \phi(x_3)\phi(x_4) > + < \phi(x_1)\phi(x_3) >< \phi(x_2)\phi(x_4) > + \\
&< \phi(x_1)\phi(x_4) >< \phi(x_2)\phi(x_3) > + \\
&< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) >_c,
\end{aligned}
\tag{20}
$$

where the first 3 terms combine the contributions of the disconnected diagrams. They are expressed in terms of 2-point functions. The connected diagrams are incorporated in the last term which is called the **connected correlation function**.

Similar pattern is observed for the higher-order correlation functions. **There are disconnected diagrams which sum up into the lower order correlation functions, and besides there are truely connected diagrams which sum up into connected correlation function**.

2.2. GENERATING FUNCTIONAL FOR CONNECTED CORRELATION FUNCTIONS.

Let us define the functional

$$
\xi[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 ... d^4x_n < \phi(x_1)...\phi(x_n) > J(x_1)...J(x_n),
\tag{21}
$$

which depends on $J(x)$ and known as **generating functional for correlation functions**. Then, by definition

$$< \phi(x_1)...\phi(x_n) >= \frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}\xi[J]|_{J=0}. \tag{22}$$

It can be written as

$$\xi[J] =< \exp\left(\int d^4x J(x)\phi(x)\right) >= \frac{Z[J]}{Z[0]}, \tag{23}$$

where

$$Z[J] = \int [D\phi] \exp\left(-\int d^4x J(x)\phi(x)\right) \exp\left(-A\right) \tag{24}$$

is the partition function associated to the action

$$A_J = A - \int d^4x J(x)\phi(x). \tag{25}$$

(The denominator $Z[0]$ cancels the vacuum diagrams again).

The source-dependent action above is a particular case of generic theory we disscussed in the previous lecture

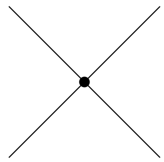
$$A_I = \int d^4x(\lambda_1(x)\phi(x) + \frac{\lambda_2(x)}{2!}\phi^2(x) + \frac{\lambda_3(x)}{3!}\phi^3(x) + \frac{\lambda_4(x)}{4!}\phi^4(x) + ...), \tag{26}$$

where now

$$A_I = \int d^4x(-J(x)\phi(x) + \frac{\lambda}{4!}\phi^4(x)). \tag{27}$$

Therefore its partition function $Z[J]$ (more precisely $\frac{Z[J]}{Z_0}$) is given by a sum of all vacuum diagrams which now contain 2 kinds of vertices

$$= \int d^4x J(x) \tag{28}$$



$$= -\lambda \int d^4x \tag{29}$$

As we have seen from (11)

$$\xi[J] = \frac{Z[J]}{Z[0]} = \exp\left(sum\ of\ all\ connected\ vacuumdiagrams\right) \tag{30}$$

of the theory whose A_I action is given by (30).

It make sense therefore to define the new functional

$$\xi[J] = \exp\left(W[J]\right) \tag{31}$$

whose expansion in terms of J will generate the **connected correlation functions**:

$$W[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 ... d^4x_n W^{(n)}(x_1, ..., x_n) J(x_1) ... J(x_n), \tag{32}$$

where

$$W^{(n)}(x_1, ..., x_n) =< \phi(x_1), ..., \phi(x_n) >_c \tag{33}$$

The functional $W[J]$ is called the generating functional for connected correlation functions.

In the KG theory one can calculate the Gaussian integral (24) for $Z[J]$ explicitly and find

$$W_{KG}[J] = \frac{1}{2} \int d^4x d^4y J(x) D(x-y) J(y), \tag{34}$$

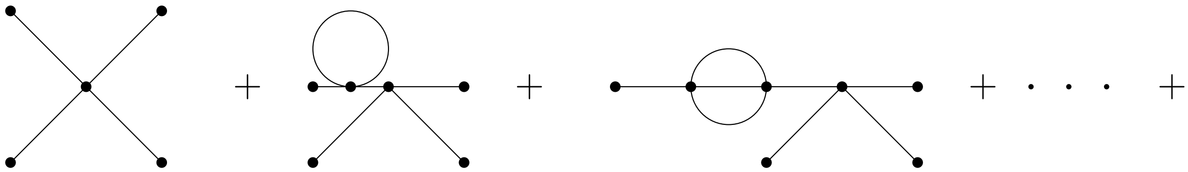
which shows that in free field theory all nonvanishing connected correlation functions are the 2-points only.

2.3.AMPUTATED CORRELATION FUNCTIONS.

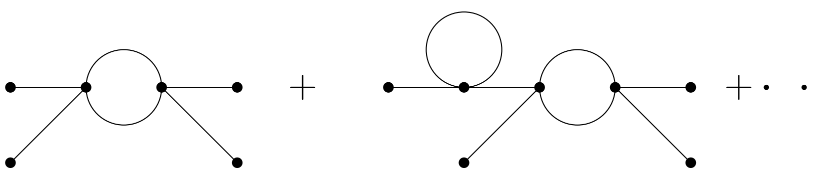
The connected correlation functions can be decomposed in a more elementary blocks.

Let us consider the diagrams for the connected 4-point function

$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) >_c =$$



(35)



(36)

The second, third and fifth diagrams represent the corrections to the propagators (external legs). This shows that the connected correlation functions can be expressed through so-called **amputated** correlation functions. The definition of amputated correlation is given by

$$W^{(n)}(x_1, ..., x_n) = \int [\prod_{i=1}^{n} d^4y_i W(x_i - y_i)] W^{(n)}_{amp}(y_1, ..., y_n), \qquad (37)$$

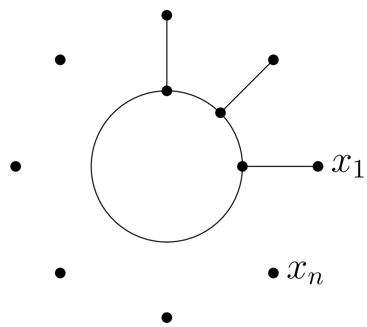
where

$$W(x-y) \equiv W^{(2)}(x,y) =< \phi(x)\phi(y) >_c =< \phi(x)\phi(y) > . \qquad (38)$$

Let us give a graphical representation of this relation. **Denoting the connected n-point correlation function by an empty blob with n**

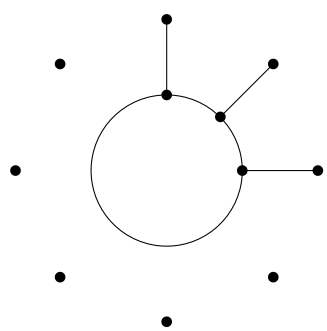
legs

$$< \phi(x_1)...\phi(x_n) >_c= \tag{39}$$

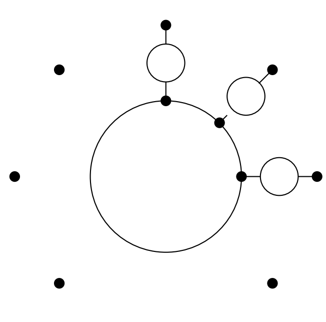


(40)

Then the connected n-point correlation function is



$$= \tag{41}$$



(42)

where **small empty blobs here denote the total 2-points correlation functions** (total propagators).

According to this definition **the amputated correlation functions do not contain neither external legs nor corrections to them**.

Notice that by this definition $W^{(2)}_{amp}(x,y) \equiv S(x-y)$ is inverse to the 2-point function

$$\int d^4 y S(x-y) W(y-z) = \delta^4(x-z). \tag{43}$$

Indeed, as it follows from the definition (37):

$$W(x-x') = W^{(2)}(x,x') = \int d^4 y d^4 y' W(x-y) W^{(2)}_{amp}(y,y') W(y'-x). \tag{44}$$

Therefore $W^{(2)}_{amp}(y,y') = S(y-y')$.

In free field theory this is inverse to the function $D(x-y)$ **i.e. just the euclidean KG operator**

$$S_0(x-y) = (m^2 - \partial_a^2)\delta(x-y). \tag{45}$$

2.4. PROPER VERTICES.

Proper vertices are also known as **one-particle irreducible correlation functions**. For $n>2$ the n**-point proper vertex** $-\Gamma^{(n)}(y_1,...,y_n)$ **is the sum of all diagrams contributing to connected correlation function** $W^{(n)}(y_1,...y_n)$ **which cannot be made disconnected by cutting just one line**.

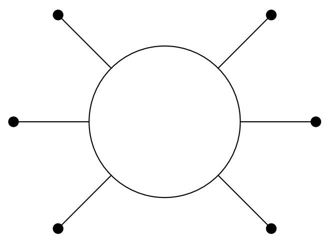
In particular, in ϕ^4 theory

$$\Gamma^{(4)} = -W^{(4)}_{amp}. \tag{46}$$

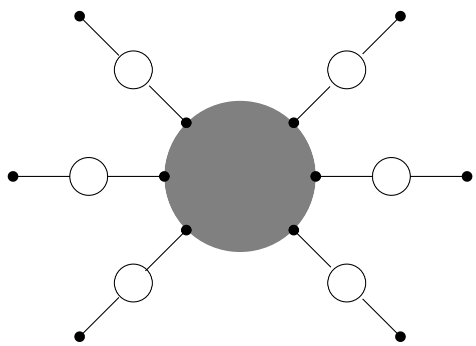
Indeed, $\Gamma^{(4)}$ is the sum of all diagrams contributing to $W^{(4)}(y_1,...,y_4)_{amp}$ which can not be made disconnected by cautting just one internal line. But

for the ϕ^4 the connected diagrams contributing to $W^{(4)}(y_1, ..., y_4)_{amp}$ are precisely the one-particle irreducible diagrams.

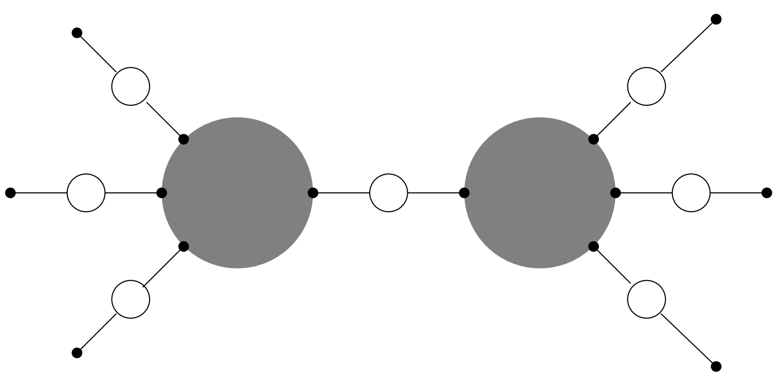
But for the 6-point connected correlation function we have the representation



$$= \tag{47}$$

$-$  $+$

(48)



$$+\ 9\ \textit{similar terms}, \tag{49}$$

where the **shadowed blobs with n points stand for** $\Gamma^{(n)}$ (the last 9 terms are related with different ways of dividing 6 external points into the pairs of 3-points groups).

Two point proper vertex deserves special definition, namely

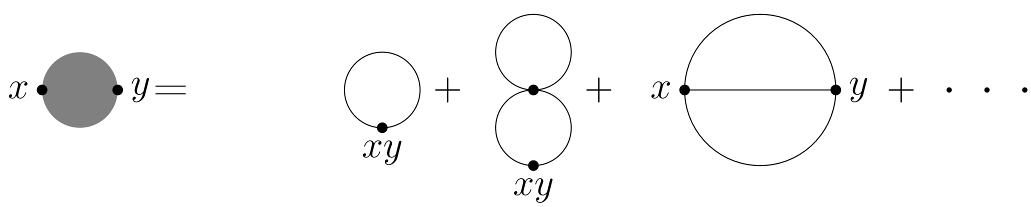
$$\Gamma^{(2)}(x,y) = S(x-y). \tag{50}$$

Its relation to the 1-particle irreducible diagrams is as follows. Write

$$S(x-y) = S_0(x-y) + \Sigma(x-y), \tag{51}$$

where $S_0(x-y)$ is given by (45). The function $\Sigma(x-y)$ is called mass operator.

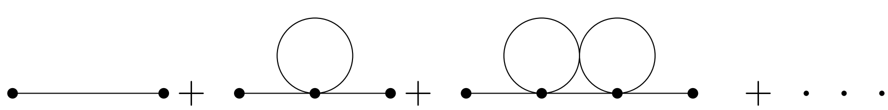
It is the function $-\Sigma(x-y)$ which is equal to the sum of all one-particle irreducible diagrms for the two-point correlation function. Indeed, denoting $-\Sigma$ by the two-point shadowed blob we have



(52)

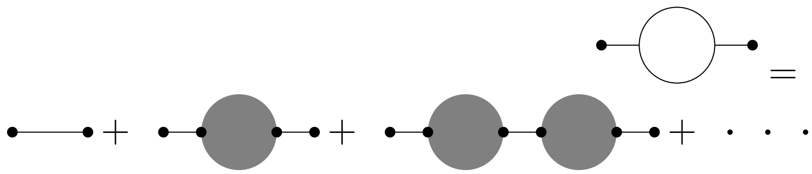
where the xy denotes the amputated legs which connect the vertex with external points x and y.

The full set of diagrams contributing to $W(x-y)$:



(53)

can be obtained by repeating the one-particle irreducible diagrams:



(54)

Explicitly

$$W = D - D * \Sigma * D + D * \Sigma * D * \Sigma * D - ... \tag{55}$$

Here $A * B$ denotes the convolution

$$A * B(x - y) = \int d^4 u A(x - u) B(u - y). \tag{56}$$

Using

$$S_0 * D = I, \tag{57}$$

where $I = \delta(x - y)$ is the unit operator, it is straightforward to check that $S_0 + \Sigma$ is indeed an inverse to $W(x - y)$.

One can also define generating functional for the proper vertices. This functional $\Gamma[\phi_{cl}]$ is called **Effective action** and is given by Legendre transformation

$$\Gamma[\phi_{cl}] = W[J] - \int d^4 x J(x) \phi_{cl}(x), \ \phi_{cl}(x) = \frac{\delta W[J]}{\delta J(x)}. \tag{58}$$

In order to understand this expression recall the statistical physics interpretation of QFT.

More precisely, let us consider the statistical system of magnetic with an external homogeneous magnetic field H. The the partiton function is

$$Z(H) = \int [Ds(x)] \exp{[-\beta \int dx \mathcal{H}(s(x)) - Hs(x)]} = \exp{[-\beta F(H)]}, \tag{59}$$

where $s(x)$ is the spin variable, $\mathcal{H}$ is the Hamiltonian density, $\beta = 1/kT$ and F is the free energy.

By the definition, the magnetiztion M is given by

$$-\frac{\partial F(H)}{\partial H}|_{\beta=const} = \frac{1}{\beta}\frac{\partial}{\partial H}\ln Z =$$
$$\frac{1}{Z}\int dy \int [Ds]s(x) \exp{[-\beta \int dx \mathcal{H}(s(x)) - Hs(x)]]} = \int dy < s(y) >= M. \tag{60}$$

Gibbs free energy is given by Legendre transformation

$$G = F + MH. \tag{61}$$

This value satisfy the equation

$$\frac{\partial G}{\partial M} = \frac{\partial F}{\partial M} + M\frac{\partial H}{\partial M} + H = \frac{\partial H}{\partial M}\frac{\partial F}{\partial H} + M\frac{\partial H}{\partial M} + H = H. \tag{62}$$

If $H = 0$ Gibbs free energy takes an extreme value. Thus, the most stable equilibrium state corresponds to the minimum value of the Gibbs energy.

Hence, our expression for $\Gamma[\phi_{cl}]$ is completely similar to the Gibbs free energy expression where instead of H we have the $J(x)$ and

$$\phi_{cl}(x) = \frac{\delta W[J]}{\delta J(x)} =< \phi(x) >_J \tag{63}$$

is an analog of magnetization. Therefore we can write

$$\frac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}(x)} = -J(x). \tag{64}$$

Let us try to can obtain (43) from the definition Γ. Since

$$\frac{\delta}{\delta J(x)}\frac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}(y)} = -\delta(x-y). \tag{65}$$

Therefore

$$\delta(x-y) = -\int d^4z \frac{\delta\phi_{cl}}{\delta J(x)}\frac{\delta^2\Gamma[\phi_{cl}]}{\delta\phi_{cl}(z)\delta\phi_{cl}(y)} = \int d^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma[\phi_{cl}]}{\delta\phi_{cl}(z)\delta\phi_{cl}(y)} = \int d^4z W(x-z)\frac{\delta^2\Gamma[\phi_{cl}]}{\delta\phi_{cl}(z)\delta\phi_{cl}(y)} \tag{66}$$

which gives (43).

Analogously we find

$$\frac{\delta^3 W}{\delta J(x)\delta J(y)\delta J(z)} = \int d^4u d^4v d^4w W(x-u)W(y-v)W(z-w)\frac{\delta^3\Gamma[\phi_{cl}]}{\delta\phi_{cl}(u)\delta\phi_{cl}(v)\delta\phi_{cl}(w)}, \tag{67}$$

where we have used inverse matrix derivative formula

$$\frac{\delta}{\delta\alpha}(M^{-1})_{xy} = -(M^{-1})_{x\beta}\frac{\delta}{\delta\alpha}M_{\beta\gamma}(M^{-1})_{\gamma y} \tag{68}$$

for the case when $(M^{-1})_{xy} = \frac{\delta^2\Gamma}{\delta\phi_{cl}(x)\delta\phi_{cl}(y)}$ and $W(x-y) = (\frac{\delta^2\Gamma}{\delta\phi_{cl}(x)\delta\phi_{cl}(y)})^{-1}$.

One can find also

$$\frac{\delta^4 W}{\delta J(x)\delta J(y)\delta J(z)\delta J(s)} =$$

$$\int d^4u d^4v d^4w d^4t W(x-u)W(y-v)W(z-w)W(s-t)\Big(-\frac{\delta^4\Gamma[\phi_{cl}]}{\delta\phi_{cl}(u)\delta\phi_{cl}(v)\delta\phi_{cl}(w)\delta\phi_{cl}(t)} +$$

$$\frac{\delta^3\Gamma}{\delta\phi_{cl}(u)\delta\phi_{cl}(v)\delta\phi_{cl}(r)}W(q-r)\frac{\delta^3\Gamma}{\delta\phi_{cl}(q)\delta\phi_{cl}(w)\delta\phi_{cl}(t)} + (v \leftrightarrow w) + (v \leftrightarrow t)\Big). \tag{6}$$

Note that for ϕ^4 theory $\frac{\delta^3\Gamma}{\delta\phi_{cl}(q)\delta\phi_{cl}(w)\delta\phi_{cl}(t)} = 0$, thus we reproduce the relation (46).

Going by this way farther we find that indeed Γ is the generation function for the proper vertices.