Lecture 6. Quantization of Dirac field. Spin-statistics relation.

Plan.

1. Action, Lagrangian, canonical variables and Hamiltonian of Dirac field.

1.1 DIRAC'S CONJUGATION AND LORENTZ INVARIANT LAGRANGIAN.

To build Lorentz invariant Lagrangian we must be able to construct scalars from the Dirac's spinors. The naive combination

$$\psi^\dagger\psi \tag{1}$$

is invariant w.r.t. space rotations:

$$\begin{aligned} \psi &\rightarrow \exp{(-\frac{\imath}{2}\omega_{ij}\Sigma^{ij})}\ ,\ i,j=1,2,3 \\ \psi^\dagger &\rightarrow \psi^\dagger \exp{(-\frac{\imath}{2}\omega_{ij}\Sigma^{ij})}^\dagger . \end{aligned} \tag{2}$$

Because of

$$\Sigma^{ij} = \frac{\imath}{4}[\gamma^i,\gamma^j] = \frac{\imath}{4}[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{3}$$

we find

$$\exp{(-\frac{\imath}{2}\omega_{ij}\Sigma^{ij})}^\dagger = \exp{(\frac{\imath}{2}\omega_{ij}(\Sigma^{ij})^\dagger)} = \exp{(\frac{\imath}{2}\omega_{ij}\Sigma^{ij})}. \tag{4}$$

Hence

$$\psi^\dagger\psi \rightarrow \psi^\dagger \exp{(\frac{\imath}{2}\omega_{ij}\Sigma^{ij})}\exp{(-\frac{\imath}{2}\omega_{ij}\Sigma^{ij})}\psi = \psi^\dagger\psi. \tag{5}$$

But for the boosts we have

$$\Sigma^{0i} = \frac{\imath}{4}[\gamma^0,\gamma^i] = -\frac{\imath}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \tag{6}$$

and hence

$$\begin{aligned} \exp{(-\imath\omega_{0i}\Sigma^{0i})}^\dagger &= \exp{(-\imath\omega_{0i}\Sigma^{0i})} \Rightarrow \\ \psi^\dagger\psi &\rightarrow \psi^\dagger \exp{(-2\imath\omega_{0i}\Sigma^{0i})}\psi. \end{aligned} \tag{7}$$

The correct construction of a scalar is given by a Dirac conjugation:

$$\bar{\psi}\psi \;, \bar{\psi} = \psi^{\dagger}\gamma^0. \tag{8}$$

Indeed

$$\bar{\psi} \rightarrow \psi^{\dagger} \exp{(-\imath\omega_{0j}\Sigma^{0j} - \frac{\imath}{2}\omega_{ij}\Sigma^{ij})}^{\dagger}\gamma^0 =$$
$$\psi^{\dagger} \exp{(-\imath\omega_{0i}\Sigma^{0i} + \frac{\imath}{2}\omega_{ij}\Sigma^{ij})}\gamma^0 = \bar{\psi} \exp{(\frac{\imath}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})}. \tag{9}$$

The Lorentz invariant Lagrangian density leading to Dirac eqaution is given by

$$\Lambda = \bar{\psi}(\imath\gamma^{\mu}\partial_{\mu} - m)\psi. \tag{10}$$

1.2. CANONICAL VARIABLES AND HAMILTONIAN.

The canonical momentum for the Dirac field is

$$\pi_{\psi} = \frac{\partial\Lambda}{\partial(\partial_0\psi)} = \imath\psi^{\dagger}. \tag{11}$$

Hence, the canonical Poisson brackets has to be postulated as

$$\{\psi_a(\vec{x}, t), \psi_b^{\dagger}(\vec{y}, t)\} = \delta_{ab}\delta(\vec{x} - \vec{y}). \tag{12}$$

The Hamiltonian is given by

$$\int d^3x\bar{\psi}(-\imath\gamma^{\imath}\partial_i + m)\psi = \int d^3x\psi^{\dagger}(-\imath\gamma^0\vec{\gamma}\nabla + m\gamma^0)\psi. \tag{13}$$

2. Quantization of Dirac field.

2.1. DIAGONALIZATION OF HAMILTONIAN.

We want to quantize the Dirac's field and construct the space of states.

In order to build the space of states we can diagonalize the Hamiltonian. We have found the plane wave solutions for the Dirac equation

$$(\imath\gamma^0\partial_0 + \imath\vec{\gamma}\nabla - m)u^s(p)\exp{(-\imath px)} = 0, \tag{14}$$

where $s = 1, 2$ labels the independent solutions. An arbitrary solution can be decomposed by this basis of solutions

$$\psi_c(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} (a^s_{\vec{p}} u^s_c(\vec{p}) + b^s_{-\vec{p}} v^s_c(-\vec{p})) \exp{(-\imath \vec{p}\vec{x})},$$

$$\psi^\dagger_c(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} ((a^s_{\vec{p}})^\dagger (u^s)^*_c(\vec{p}) + (b^s)^\dagger_{-\vec{p}} (v^s)^*_c(-\vec{p})) \exp{(\imath \vec{p}\vec{x})}. \tag{15}$$

The Hamiltonian is given by

$$H = \int d^3x \bar{\psi}(-\imath \vec{\gamma} \cdot \vec{\nabla} + m)\psi. \tag{16}$$

In the basis of solutions above the Hamiltonian takes the diagonal form

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} E_{\vec{p}}((a^s_{\vec{p}})^\dagger a^s_{\vec{p}} - (b^s_{\vec{p}})^\dagger b^s_{\vec{p}}), \tag{17}$$

where the relations

$$(u^s)^\dagger_{\vec{p}} u^r_{\vec{p}} = 2E_{\vec{p}}\delta^{sr} \ , \ (v^s)^\dagger_{\vec{p}} v^r_{\vec{p}} = -2E_{\vec{p}}\delta^{sr} \tag{18}$$

have been used.

2.2. THE WRONG (NAIVE) QUANTIZATION.

According to the quantization prescription one must impose the commutation relations for the canonical variables instead of Poisson brakets (12)

$$[\psi(\vec{x}, t)_a, \psi^\dagger(\vec{y}, t)_b] = \delta_{ab}\delta(\vec{x} - \vec{y}) \Leftrightarrow$$

$$[\psi(\vec{p})_a, \psi^\dagger(\vec{q})_b] = (2\pi)^3 \delta_{ab}\delta(\vec{p} - \vec{q}), \ a, b = 1, ..., 4. \tag{19}$$

In view of (15) it leads to the following commutation relations

$$[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}] = [b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{rs}\delta(\vec{p} - \vec{q}) \tag{20}$$

(the other commutators are zero). It is natural also to take the vacuum vector as defined by

$$a^s_{\vec{p}}|0>= b^s_{\vec{p}}|0>= 0. \tag{21}$$

But this definition makes the spectrum of energies unrestricted from below!

The problem seems coming from the last term in the expression for the Hamiltonian (17). Let us change the definition of vacuum to try to avoid this catastrophy

$$a^s_{\vec{p}}|0>= (b^s_{\vec{p}})^\dagger|0>= 0. \tag{22}$$

(One could change the commutators

$$[b_{\vec{p}}, b^\dagger_{\vec{p}}] = 1 \rightarrow [b^\dagger_{\vec{p}}, b_{\vec{p}}] = 1 \tag{23}$$

but it is a bad idea because it breaks the commutation relations between the canonical variables (19)).

2.3. CAUSALITY BREAKING BY WRONG QUANTIZATION.

Inspite of we get the spectrum which is restricted from below (due to (22)), the commutators (20) break the causality.

In order to consider this problem let us go to the Heisenberg picture

$$\begin{aligned}
\exp{(\imath Ht)} a^s_{\vec{p}} \exp{(-\imath Ht)} &= a^s_{\vec{p}} \exp{(-\imath E_{\vec{p}} t)} \\
\exp{(\imath Ht)} b^s_{\vec{p}} \exp{(-\imath Ht)} &= b^s_{\vec{p}} \exp{(\imath E_{\vec{p}} t)} \Rightarrow \\
\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} &(a^s_{\vec{p}} u^s(\vec{p}) \exp{(-\imath px)} + b^s_{\vec{p}} v^s(\vec{p}) \exp{(\imath \vec{p} x)}) \\
\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} &((a^s_{\vec{p}})^\dagger \bar{u}^s(\vec{p}) \exp{(\imath px)} + (b^s_{\vec{p}})^\dagger \bar{v}^s(\vec{p}) \exp{(-\imath \vec{p} x)})
\end{aligned} \tag{24}$$

and consider the commutator of Heisenberg's fields

$$
\begin{aligned}
&[\psi_a(x), \bar{\psi}_b(y)] = \\
&\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \sum_{s=1,2} (u^s_a(p)\bar{u}^s_b(p) \exp(-\imath p(x-y)) + v^s_a(p)\bar{v}^s_b(p) \exp(\imath p(x-y))) = \\
&\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} ((p_\mu\gamma^\mu + m)_{ab} \exp(-\imath p(x-y)) + (p_\mu\gamma^\mu + m)_{ab} \exp(\imath p(x-y))) = \\
&(\imath\gamma^\mu\partial_\mu + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (\exp(-\imath p(x-y)) - \exp(\imath p(x-y))) = \\
&(\imath\gamma^\mu\partial_\mu + m)_{ab}[\phi(x), \phi(y)].
\end{aligned}
\tag{25}
$$

Therefore it vanishes outside the light-cone.

Moreover, for the vacuum determined by (21) (this is the case when the spectrum does not restricted from below) we obtain

$$
\begin{aligned}
< 0|[\psi_a(x), \bar{\psi}_b(y)]|0 > = < 0|\psi_a(x)\bar{\psi}_b(y)|0 > - < 0|\bar{\psi}_b(y)\psi_a(x)|0 > = \\
< 0|\psi_a(x)\bar{\psi}_b(y)|0 > .
\end{aligned}
\tag{26}
$$

This is strange because the cancelation outside the light-cone comes from the first term only. Thus, the cancelation occurs between the particles with positive energy and the particles with negative energy traveling from y to x.

Recall that in KG theory

$$
\begin{aligned}
[\hat{\phi}(x), \hat{\phi}(y)] = D_-(x-y) - D_+(x-y) \\
where \\
D_-(x-y) = [\hat{\phi}_-(x), \hat{\phi}_+(y)], \\
D_+(x-y) = [\hat{\phi}_-(y), \hat{\phi}_+(x)] = D_-(y-x).
\end{aligned}
\tag{27}
$$

So, we had the cancellation between particles with positive energy going from y to x and anti particles going from x to y.

We would like get rid of the negative energy particles believing that $< 0|\psi_a(x)\bar{\psi}_b(y)|0 >$ describes the propagation of positive energy particles from y to x. It means that one has to consider the state $\bar{\psi}(x)|0 >$ as the positive energy particles states only.

To this end one needs to define the vacuum as in (22), so that we have a spectrum restricted from below.

Then

$$< 0|\psi(x)\bar{\psi}(y)|0 >= \\ < 0| \int \frac{d^3pd^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \sum_r a^r_{\vec{p}} u^r(p) \exp{(-\imath px)} \sum_s (a^s_{\vec{p}})^\dagger \bar{u}^r(p) \exp{(\imath qy)}|0 > . \tag{28}$$

To calculate $< 0|a^r_{\vec{p}}(a^s_{\vec{p}})^\dagger|0 >$ we use Poincare invariance arguments. So we do not exactly fix the right-hand sides of the commutators (20) but try to find them from general considerations of Poincare invariance.

First of all we can take into account the invariance of the vacuum w.r.t. the translations $\exp{(\imath\vec{P}\vec{x})}|0 >= |0 >$. It gives

$$< 0|a^r_{\vec{p}}(a^s_{\vec{p}})^\dagger|0 >=< 0|a^r_{\vec{p}}(a^s_{\vec{p}})^\dagger \exp{(\imath\vec{P}\vec{x})}|0 >= \\ \exp{(\imath(\vec{p}-\vec{q}\vec{x})} < 0| \exp{(\imath\vec{P}\vec{x})} a^r_{\vec{p}}(a^s_{\vec{p}})^\dagger|0 >= \\ \exp{(\imath(\vec{p}-\vec{q}\vec{x})} < 0|a^r_{\vec{p}}(a^s_{\vec{p}})^\dagger|0 > . \tag{29}$$

Therefore

$$\vec{p}-\vec{q} = 0 \ iff \ < 0|a^r_{\vec{p}}(a^s_{\vec{p}})^\dagger|0 >\neq 0. \tag{30}$$

The next step is to use Lorentz invariance of the vacuum. It gives $r = s$. Thus we find

$$< 0|a^r_{\vec{p}}(a^s_{\vec{q}})^\dagger|0 >= (2\pi)^3\delta^{sr}\delta(\vec{p}-\vec{q})A(\vec{p}), \tag{31}$$

where $A(\vec{p}) = A(p^2) = A(m^2)$ **is a positive constant** because of the norm of any state $(< 0|a^r_{\vec{p}}(a^s_{\vec{q}})^\dagger|0 >\sim ||(a^s_{\vec{q}})^\dagger|0 > |^2)$ must be positive. Now we can write

$$\begin{aligned} < 0|\psi(x)\bar{\psi}(y)|0 >= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_r u^r(p)\bar{u}^r(p)A(m^2)\exp\left(-\imath p(x-y)\right) = \\ \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(p_\mu\gamma^\mu + m)A(m^2)\exp\left(-\imath p(x-y)\right) = \\ (\imath\gamma^\mu\partial_\mu + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\exp\left(-\imath p(x-y)\right)A(m^2), \end{aligned} \tag{32}$$

where the realation

$$\sum_r u^r(p)\bar{u}^r(p) = p_\mu\gamma^\mu + m \tag{33}$$

has been used.

One can find similarly

$$< 0|\bar{\psi}(y)\psi(x)|0 >= -(\imath\gamma^\mu\partial_\mu + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\exp\left(\imath p(x-y)\right)B(m^2), \tag{34}$$

where we have used

$$\sum_r v^r(p)\bar{v}^r(p) = p_\mu\gamma^\mu - m, \tag{35}$$

the annihilation condition from (22) and

$$< 0|b_{\vec{p}}^{s\dagger} b_{\vec{q}}^{r}|0 >= (2\pi)^3 \delta^{sr} \delta(\vec{p} - \vec{q}) B(m^2). \tag{36}$$

Because of A, B are positive the vacuum expectation

$$< 0|[\psi(x), \bar{\psi}(y)[|0 >=< 0|\psi(x)\bar{\psi}(y)|0 > - < 0|\bar{\psi}(y), \psi(x)[|0 >\neq 0 \tag{37}$$

outside the light-cone which is in contradiction with causality principle! (Recal that outside the light cone we can use Lorentz transform to get $-(x - y)$ instead of $(x - y)$.)

3. The correct quantization.

3.1. THE ANTI-COMMUTATORS RECOVER THE CAUSALITY.

The solution of the causality problem is to take $A = B$ but demand

$$< 0|\psi(x)\bar{\psi}(y)|0 >= - < 0|\bar{\psi}(y)\psi(x)|0 > \ \textit{outside the light} - \textit{cone}. \tag{38}$$

It means that **outside the light-cone the fields** $\psi_a(x)$ **and** $\bar{\psi}_b(y)$ **must anti-commute**. In this case the arbitrary operators $O_1(x)$ and $O_2(y)$ composed of the even number of $\psi, \bar{\psi}$ will commute outside the light-cone:

$$[O_1(x), O_2(y)] = 0\ ,\ (x - y)^2 < 0. \tag{39}$$

Therefore the correct canonical variables relations are anti-

commutation relations:

$$\{\psi_a(\vec{x},t), \psi_b^\dagger(\vec{y}),t)\} = \delta_{ab}\delta(\vec{x}-\vec{y}),$$
$$\{\psi_a(\vec{x},t), \psi_b(\vec{y},t)\} = \{\psi_a^\dagger(\vec{x},t), \psi_b^\dagger(\vec{y},t)\} = 0 \Leftrightarrow$$
$$\{a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta_{rs}\delta(\vec{p}-\vec{q}),$$
$$\{a^r_{\vec{p}}, a^s_{\vec{q}}\} = \{b^r_{\vec{p}}, b^s_{\vec{q}}\} = 0. \tag{40}$$

3.2. VACUUM STATE, CREATION-ANNIHILATION OPERATORS AND THE SPECTRUM OF THE HAMILTONIAN.

Let us look at the Hamiltonian once more

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} E_{\vec{p}}((a^s_{\vec{p}})^\dagger a^s_{\vec{p}} - (b^s_{\vec{p}})^\dagger b^s_{\vec{p}}). \tag{41}$$

Using the anti-commutators one can cure the problem with negative energies.

Indeed, let us do the following redefinitions

$$b^s_{\vec{p}} \to d^{s\dagger}_{\vec{p}} \ , \ b^{s\dagger}_{\vec{p}} \to d^s_{\vec{p}}, \tag{42}$$

use the anti-commutation relations for this new operators and use the following definition of vacuum state

$$a^s_{\vec{p}}|0 >= d^s_{\vec{p}}|0 >= 0. \tag{43}$$

Then (41) takes the form of Hamiltonian whose spectrum of energies is restricted from below

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} E_{\vec{p}}(a^{\dagger s}_{\vec{p}} a^s_{\vec{p}} + d^{\dagger s}_{\vec{p}} d^s_{\vec{p}}). \tag{44}$$

3.3. SPIN-STATISTICS RELATION THEOREM.

The space of states now is obtained by applying creation operators $a^{s\dagger}_{\vec{p}}$, $d^{s\dagger}_{\vec{p}}$. Notice however

$$(a^{\dagger s}_{\vec{p}})^2|v>=(d^{\dagger s}_{\vec{p}})^2|v>=0 \tag{45}$$

due to anti-commutation relations. Moreover, any multiparticle state is now anti-symmetric. For example

$$...a^{\dagger r}_{\vec{q}}a^{\dagger s}_{\vec{k}}...|0>=-...a^{\dagger s}_{\vec{k}}a^{\dagger r}_{\vec{q}}...|0>. \tag{46}$$

It means that the corresponding particles obey **Fermi-Dirac statistics**.

Thus, we have shown that, having simultaneously demanded

1) Poincare invariance,

2) boundedness of the spectrum of the Hamiltonian from below,

3) causality

leads to the Fermi-Dirac statistics for the Dirac field.

This is an example of general

Theorem (Pauli):

Demanding simultaneously Poincare invariance, positivity of energy and causality, requires that integer spin particles (bosons) obey Bose-Einstein statistics, while the half-integer spin particles (fermions) obey Fermi-Dirac statistics.

4. Dirac's propagator.

4.1. EXPECTATION VALUES OF DIRAC'S FIELD OPERATORS.

Now one can clculate the vacuum expectation values of the Dirac's field

Heisenberg operators using the correct quantization:

$$< 0|\psi_a(x)\bar{\psi}_b(y)|0 >= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s u_a^s(\vec{p})\bar{u}_b^s(\vec{p}) \exp{(-\imath p(x-y))} =$$
$$(\imath\gamma^\mu\partial_\mu + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} \exp{(-\imath p(x-y))}$$
$$< 0|\bar{\psi}_b(y)\psi_a(x)|0 >= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s v_a^s(\vec{p})\bar{v}_b^s(\vec{p}) \exp{(-\imath p(x-y))} =$$
$$-(\imath\gamma^\mu\partial_\mu + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} \exp{(-\imath p(y-x))}, \tag{47}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

4.2. DIRAC PROPAGATOR.

Using definition of Dirac propagator and calculations (47) we find

$$S_F(x-y) =< 0|T(\psi_a(x)\bar{\psi}_b(y))|0 >=$$
$$\Theta(x^0-y^0) < 0|\psi_a(x)\bar{\psi}_b(y)|0 > -\Theta(y^0-x^0) < 0|\bar{\psi}_b(y)\psi_a(x)|0 >=$$
$$\int \frac{d^4p}{(2\pi)^4}\frac{\imath(p_\mu\gamma^\mu + m)}{p^2-m^2+\imath\epsilon} \exp{(-\imath p(x-y))}, \tag{48}$$

Similar to the KG case, the shift $\imath\epsilon$ determines the contour when we integrate over p^0.

The Fourier transform of the propagator above is given by

$$\tilde{S}_F(p) = \frac{\imath(p_\mu\gamma^\mu + m)}{p^2-m^2}. \tag{49}$$

5. Discret symmetries of Dirac field.

In addition to continuous Lorentz group transformations there are discret symmetries leaving unchanged the Minkowski distance $t^2 - \vec{x}^2$:

$$\begin{aligned} parity\ P &: (\vec{x}, t) \to (-\vec{x}, t), \\ time\ reversing\ T &: (\vec{x}, t) \to (\vec{x}, -t) \end{aligned} \tag{50}$$

According to this symmetries the total Lorentz group can be decomposed into 4 disconnected parts:

$$L_+^\uparrow -\ proper\ orthochronous\ Lorentz\ group,\ L_-^\uparrow = PL_+^\uparrow, \\ L_+^\downarrow = TL_+^\uparrow,\ L_-^\downarrow = PTL_+^\uparrow. \tag{51}$$

The trnsformations $L_+^\uparrow \cup L_-^\uparrow$ are called **orthochronous**. The transformation $L_+^\downarrow \cup L_-^\downarrow$ are called **nonorthochronous**.

5.1. P AND T DISCRET SYMMETRIES.

The action of P on the Dirac field must change the direction of momentum of the particle and must not change its spin. Indeed, the 3-dimensional orbital momentum operator is invariant:

$$P : x_i p_j - x_j p_i \to (-x_i)(-p_j) - (-x_j)(-p_i). \tag{52}$$

Hence, the spin operator Σ_{ij} must be invariant also. Therefore

$$P a^s_{\vec{p}} P = \eta_a a^s_{-\vec{p}},\ P b^s_{\vec{p}} P = \eta_b b^s_{-\vec{p}}, \tag{53}$$

where $\eta_{a,b}$ are phase factors. Because of $P^2 = 1$ we conclude that

$$\eta^2_{a,b} = \pm 1. \tag{54}$$

Applying P transformation to the field $\psi(x)$ we should obtain

$$P\psi(\vec{x},t)P =$$
$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s(\eta_a a^s_{-\vec{p}}u^s(\vec{p})\exp{(-\imath px)} + \bar{\eta}_b b^{s\dagger}_{-\vec{p}}v^s(\vec{p})\exp{(\imath px)}) =$$
$$M\psi(-\vec{x},t), \tag{55}$$

where M is a constant matrix.

Making the change of variables $p \to \tilde{p} = (p^0, -\vec{p})$ and taking into account that $px = \tilde{p}(-\vec{x},t)$ and

$$u(p) = \gamma^0 u(\tilde{p}),\ v(p) = -\gamma^0 v(\tilde{p}), \tag{56}$$

we find that

$$\bar{\eta}_b = -\eta_a. \tag{57}$$

Hence

$$\eta_a\eta_b = -\eta_a\bar{\eta}_a = -1. \tag{58}$$

It gives

$$P\psi(\vec{x},t)P = -\eta_a\gamma^0\psi(-\vec{x},t). \tag{59}$$

Let us consider P transformation of $\bar{\psi}\psi$:

$$P\bar{\psi}(\vec{x},t)\psi(\vec{x},t)P = \bar{\eta}_a\eta_a\bar{\psi}(-\vec{x},t)\psi(-\vec{x},t) = \bar{\psi}(-\vec{x},t)\psi(-\vec{x},t). \tag{60}$$

Because of P transformation of fermion-bilinear expressions does not depend on η_a we can take

$$\eta_a = -\eta_b = 1. \tag{61}$$

Now we consider T action on the Dirac field. Under this transformation the 3-dim. momentum operator M_{0i} must change its direction: $p_i \to -p_i$. Hence, the 3-dimensional orbital momentum operator must change its direction also

$$T : x_i p_j - x_j p_i \to x_i(-p_j) - x_j(-p_i). \tag{62}$$

The spin operator must transform by the same way

$$T : \Sigma_{ij} \to -\Sigma_{ij}. \tag{63}$$

It can be considered as an operator transforming $\psi(\vec{x}, t)$ to $\tilde{M}\psi(\vec{x}, -t)$ where $\tilde{M}$ is a constant matrix. It means that

$$T a^s_{\vec{p}} T = a^{-s}_{-\vec{p}},\ T b^s_{\vec{p}} T = -b^{-s}_{-\vec{p}}. \tag{64}$$

Moreover since

$$\psi(t, \vec{x}) = \exp{(\imath H t)} \psi(0, \vec{x}) \exp{(-\imath H t)} \tag{65}$$

and the Hamiltonian H is diagonalized in the basis $a^s_{\vec{p}}, a^{s\dagger}_{\vec{p}}, b^s_{\vec{p}}, b^{s\dagger}_{\vec{p}}$ and must commute with the operator T we conclude that

$$T c T = c^* \tag{66}$$

for any complex number c. **In other words T is antilinear operator**. Then we find

$$T\psi(\vec{x},t)T =$$
$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s (a_{-\vec{p}}^{-s}(u_{\vec{p}}^s)^* \exp(\imath px) + b_{-\vec{p}}^{-s\dagger}(v_{\vec{p}}^s)^* \exp(-\imath px)) =$$
$$-(\gamma^1\gamma^3)\psi(\vec{x},-t), \tag{67}$$

where the explicit expressions for the basic vectors $u_{\vec{p}}^s$, $v_{\vec{p}}^s$ have been used.

5.2. INTERACTION WITH EM FIELD AND CHARGE CONJUGATION.

The Lagrangian density (10) can be generalized to the case of interaction with EM field

$$\mathcal{L}_{EM} = \bar{\psi}(\imath\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi. \tag{68}$$

We interprate the term $e\bar{\psi}\gamma^\mu\psi A_\mu$ as describing the interraction of electrically charged particle with EM field like $j^\mu A_\mu$ where j^μ is a current of the particle and e is a charge. It is Lorentz invariant again and gives the following equations of motion

$$(\imath\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0. \tag{69}$$

If we claim that ψ describes electrons and positrons we should find a symmetry

$$\psi(x) \to \psi^c(x) \tag{70}$$

of the Lagrangian changing the charge of the particle in (69):

$$(\imath\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi^c = 0. \tag{71}$$

This symmetry must be local and involutive

$$(\psi^c)^c = \psi. \tag{72}$$

To find this transformation let us consider the complex conjugated equation

$$(-\imath(\gamma^\mu)^*\partial_\mu - e(\gamma^\mu)^* A_\mu - m)\psi^* = -(\imath(\gamma^\mu)^*\partial_\mu + e(\gamma^\mu)^* A_\mu + m)\psi^* = 0. \tag{73}$$

We can try to find a new basis in ψ^* where the linear equation (73) takes the form of (71)

$$\psi^* \to \psi^c \equiv C^{-1}\psi^*$$
$$(\imath\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\psi^c = C^{-1}C(\imath\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)C^{-1}\psi^* = 0. \tag{74}$$

It gives the equation for the matrix C:

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^* \ , \ C^2 = 1. \tag{75}$$

We can take

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{76}$$

Then the solution of (75) in this representation is

$$C = \imath\gamma^2. \tag{77}$$

Let us find the Lorentz transformation of $\psi^c(x)$

$$C^{-1}(\exp(-\frac{\imath}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\psi)^* = C^{-1}\exp(+\frac{\imath}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})^*)\psi^* =$$
$$C^{-1}C\exp(-\frac{\imath}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})C^{-1}\psi^* = \exp(-\frac{\imath}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\psi^c. \tag{78}$$

We see that ψ^c spinor transforms by the same rule that ψ itself, so there is no contradiction with Lorentz invariance.

For the Heisenberg field $\psi(x)$ we then obtain

$$Ca^s_{\vec{p}}C = b^s_{\vec{p}},\ Cb^s_{\vec{p}}C = a^s_{\vec{p}}. \tag{79}$$