Lecture 5.

Plan.

1. Lorentz invariant wave equations.

1.1 LORENTZ INVARIANCE IN KG THEORY.

An arbitrary Lorentz transformation in Minkowski space-time with metric tensor $g^{\mu\lambda}$ can be represented as follows

$$x^\mu \to \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu, \ where \ \Lambda^\mu_\nu \Lambda^\lambda_\rho g^{\nu\rho} = g^{\mu\lambda}. \tag{1}$$

It means that the point x with coordinates x^μ maps to the point $\tilde{x}$ with coordinates $\Lambda^\mu_\nu x^\nu$.

By the definition, the scalar fields $\phi(x)$ transforms by the rule

$$\phi(x) \to \tilde{\phi}(x) = \phi(\Lambda x). \tag{2}$$

Suppose the scalar field $\phi(x)$ obeys KG equation. Let us check that the transformed field also obeys KG equation:

$$\frac{\partial}{\partial x^\mu}\tilde{\phi}(x) = \frac{\partial}{\partial x^\mu}\phi(\Lambda x) = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\phi(y) = (\Lambda)^\nu_\mu \frac{\partial}{\partial y^\nu}\phi(y), \ where \ y = \Lambda x \tag{3}$$

Then

$$g^{\mu\rho}\partial_\mu\partial_\rho\tilde{\phi}(x) = g^{\mu\rho}(\Lambda)^\nu_\mu \frac{\partial}{\partial y^\nu}(\Lambda)^\lambda_\rho \frac{\partial}{\partial y^\lambda}\phi(y) = g^{\nu\lambda}\frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y^\lambda}\phi(y). \tag{4}$$

Therefore

$$(g^{\mu\rho}\partial_\mu\partial_\rho\tilde{\phi}(x) + m^2\tilde{\phi}(x)) = g^{\nu\lambda}\frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y^\lambda}\phi(y) + m^2\phi(y) = 0. \tag{5}$$

Thus, any solution of KG equation still satisfy this equation after the Lorentz transformation. **It means that the KG equations of motion are Lorentz invariant.**

One can also check that the Lagrangian density and hence, the action of KG theory is Lorentz invariant.

1.2 LORENTZ INVARIANCE OF EM FIELD.

The EM field is described by the vector potential $A_\mu(x)$. By the definition, the vector potential transforms under the Lorentz transformation (1) by the rule

$$A_\mu(x) \to \tilde{A}_\mu(x) = (\Lambda)^\nu_\mu A_\nu(\Lambda x). \tag{6}$$

The vector potential $A_\mu(x)$ transforms as a covariant rank-1 tensor.

Let us calculate the derivative of the transformed field

$$\frac{\partial}{\partial x^\nu}\tilde{A}_\mu(x) = (\Lambda)^\rho_\nu \frac{\partial}{\partial y^\rho}(\Lambda)^\lambda_\mu A_\lambda(y) = (\Lambda)^\rho_\nu (\Lambda)^\lambda_\mu \frac{\partial}{\partial y^\rho} A_\lambda(y). \tag{7}$$

Then, for the electro-magnetic field components we obtain

$$\begin{aligned}\tilde{F}_{\nu\mu}(x) = \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x) = \\ (\Lambda)^\rho_\nu (\Lambda)^\lambda_\mu \frac{\partial}{\partial y^\rho} A_\lambda(y) - (\Lambda)^\rho_\mu (\Lambda)^\lambda_\nu \frac{\partial}{\partial y^\rho} A_\lambda(y) = \\ (\Lambda)^\rho_\nu (\Lambda)^\lambda_\mu F_{\rho\lambda}(\Lambda x).\end{aligned} \tag{8}$$

Thus, the EM field stress-tensor $F_{\mu\nu}$ **transforms as a covariant rank-2 tensor.**

Now it is easy to check the Lorentz invariance of Maxwell equations

$$\begin{aligned}\partial^\mu \tilde{F}_{\mu\nu}(x) = g^{\mu\rho}\partial_\rho \tilde{F}_{\mu\nu}(x) = \\ g^{\mu\rho}(\Lambda)^\tau_\rho \frac{\partial}{\partial y^\tau}(\Lambda)^\epsilon_\mu (\Lambda)^\lambda_\nu F_{\epsilon\lambda}(y) = \\ g^{\epsilon\tau}(\Lambda)^\lambda_\nu \frac{\partial}{\partial y^\tau} F_{\epsilon\lambda}(y) = (\Lambda)^\lambda_\nu \partial^\epsilon F_{\epsilon\lambda}(y) = 0.\end{aligned} \tag{9}$$

1.3. TENSOR REPRESENTATIONS OF LORENTZ GROUP.

One can write out the Lorentz transformation rule for the general p-covariant and q-contravariant tensor field:

$$A^{\nu_1...\nu_q}_{\mu_1...\mu_p}(x) \rightarrow \tilde{A}^{\nu_1...\nu_q}_{\mu_1...\mu_p}(x) = (\Lambda)^{\sigma_1}_{\mu_1}...(\Lambda)^{\sigma_p}_{\mu_p}(\Lambda^{-1})^{\nu_1}_{\tau_1}...(\Lambda^{-1})^{\nu_q}_{\tau_q} A^{\tau_1...\tau_q}_{\sigma_1...\sigma_p}(\Lambda x). \quad (10)$$

In general one can imagine a tensor field ϕ_a with multi index $a = 1, ..., n$ which transforms under the Lorentz transformation (1) by the rule

$$\tilde{\phi}_a(x) = M^b_a(\Lambda)\phi_b(\Lambda x). \quad (11)$$

where $M^b_a(\Lambda)$ is a $n \times n$ matrix depending on Λ. The matrix M must satisfy the following equations: for a pair of Lorentz transformations Λ_1 and Λ_2

$$M^a_b(\Lambda_1)M^b_c(\Lambda_2) = M^a_c(\Lambda_1\Lambda_2), \ and \ M^a_b(1) = \delta^a_b. \quad (12)$$

in such a case we say that Lorentz group representation is given on the tensor fields $\phi_a(x)$.

1.4. LINEAR REPRESENTATION OF A GROUP.

It makes sense to consider a group G and a vector space V. If for an arbitrary element $g \in G$ there is a linear transformation $M(g)$ of V such that for a pair of elements $g_{1,2} \in G$ and an arbitrary vector $v \in V$

$$M(g_1)M(g_2)v = M(g_1g_2)v, \ and \ M(1)v = v, \quad (13)$$

where $1 \in G$ is a unity of the group, one says that on V is given **linear representation of** G.

An example of representation is a group of orthogonal rotations in $\mathbb{R}^3$.

1.5. LIE ALGEBRA OF A LIE GROUP AND ITS REPRESENTATIONS.

Definition.

A differential manifold G is called a group Lie (continuous group) if it is a differentil manifold and the group structure mappings

$$s : G \to G \ , \ s(g) = g^{-1}$$
$$m : G \times G \to G \ , \ m(g,h) = gh \tag{14}$$

are differential mappings.

If the group G is a Lie group (continuous group) and a representation of this group in vector space V is given, one can consider a set of transformations of V by the elements from G which are close to the unit of the group:

$$g \approx 1 + \epsilon_i J^i ... \tag{15}$$

The set of matrices J^i constitute a basis of some vector space $\mathbf{g}$ (because G is a manifold). Moreover, because of G is a Lie group, these matrices satisfy some important property: the commutator of basic matrices J^i, J^j is a linear combination of the basic matrices J^k

$$[J^i, J^j] = f^{ij}_k J^k, \ f^{ij}_k = -f^{ji}_k. \tag{16}$$

It allows to define so called **Lie algebra structure on the vector space g** introducing a bilinear skew-symmetric operation (commutator) by the rule: for a pair of elements $\epsilon_i J^i \in \mathbf{g}$, $\eta_j J^j \in \mathbf{g}$ the commutator is given by

$$[\epsilon_i J^i, \eta_j J^j] = \epsilon_i \eta_j f^{ij}_k J^k \in \mathbf{g}. \tag{17}$$

Definition of a Lie algebra.

The vector space $\mathbf{g}$ is endowed with the bilinear skew- symmetric operation called the commutator:

$$[X,Y] = -[Y,X] \in \mathbf{g} \tag{18}$$

is called a Lie algebra if for any triple of vectors $X, Y, Z \in \mathbf{g}$ the **Jacobi identity** is true

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]. \tag{19}$$

In this situation, the basic vectors J^i of $\mathbf{g}$ are called **the generators of the Lie algebra**, the coefficients f_k^{ij} are called **the structure constants** of Lie algebra.

Example:

For the group of orthogonal rotations in $\mathbb{R}^3$ the corresponding Lie algebra is determined by

$$[J^i, J^j] = \epsilon^{ijk} J^k, \;\; i,j,k = 1,2,3 \tag{20}$$

Having a representation of a Lie group on vector space V one can obtain a representation of its Lie algebra on V considering a transformations of V by the elements of G from the vicinity of the group unit.

1.6. LIE ALGEBRA OF THE LORENTZ GROUP.

The Lorentz group can be determined as a set of 4×4 real matrices Λ leaving the Lorentz metric g invariant:

$$\Lambda g \Lambda^T = g \leftrightarrow \Lambda^\mu_\lambda \Lambda^\nu_\sigma g_{\mu\nu} = g_{\lambda\sigma}. \tag{21}$$

As a vector space, the Lie algebra of the Lorentz group (Lorentz algebra) is spanned over the generators $J^{\mu\nu} = -J^{\nu\mu}$, $\mu, \nu = 0, ..., 3$, which satisfy the following commutators

$$[J^{\mu\nu}, J^{\rho\sigma}] = \imath(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{22}$$

2. Spinor representations and Clifford algebras.

2.1.CLIFFORD ALGEBRAS.

We have seen that various tensor spaces give the examples of the Lorentz group representations. But this set **does not exaust all possible linear representations of the Lorentz group**. There are so-called **spinor representations**. The construction of spinor representations is closely related to the **Clifford algebras**.

Definition.

A Clifford algebra $Cl(1,3)$ is an associative algebra (that is the vector space endowed with an assotiative multiplican low which respects the vector space structure) with unit over the complex numbers, which is generated by the elements b_μ, $\mu = 0, ..., 3$ obeying the relations

$$\{b_\mu, b_\nu\} = 2g_{\mu\nu}. \tag{23}$$

Notice that Clifford algebra can be determined for an arbitrary number of generators. If the number of generators is d we get $Cl(1, d-1)$ algebra. As a (complex) vector space it is spanned by the elements

$$1,\ b_\mu,\ b_\mu b_\nu,\ b_\mu b_\nu b_\lambda, ..., b_0 ... b_{d-1} \tag{24}$$

and has a dimension 2^d.

2.2. EXAMPLES OF CLIFFORD ALGEBRAS.

In $d = 2$ dimensions one can take

$$b_0 = \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; b_1 = \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{25}$$

Then one can check

$$\gamma_0^2 = -\gamma_1^2 = 1, \; \{\gamma_0, \gamma_1\} = 0 \Leftrightarrow \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{26}$$

When $d = 4$ the Clifford algebra generators are given by

$$\begin{gathered} b_\mu = \gamma_\mu^{(4)} = \gamma_\mu^{(2)} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\gamma_\mu^{(2)} & 0 \\ 0 & \gamma_\mu^{(2)} \end{pmatrix} \; \mu = 0,1 \\ b_2 = \gamma_2^{(4)} = 1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; b_3 = \gamma_3^{(4)} = 1 \otimes \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}. \end{gathered} \tag{27}$$

For an arbitrary d the construction goes by iterations

$$\begin{gathered} \gamma_\mu^{(d)} = \gamma_\mu^{(d-2)} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; \mu = 0, 1, ..., d-3 \\ \gamma_{d-2}^d = 1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \gamma_{d-1}^d = 1 \otimes \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}. \end{gathered} \tag{28}$$

Notice that we have constructed thereby **a Clifford algebra representations** because the generators are realized as the matrices (linear transformations in a complex vector space).

2.3. CONSTRUCTION OF A CLIFFORD ALGEBRA REPRESENTATION.

To construct a representation of Clifford algebra $Cl(1, d-1)$ one needs to realize the generators b_μ by a matrices $\gamma_\mu = M(b_\mu)$ acting in some (finite dimensional) vector space (over $\mathbb{C}$) V such that

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}1. \tag{29}$$

The constructions of Clifford algebras above were at the same time the examples of Clifford algebras representations.

For any representation of the Clifford algebra (if d is even) tere is some canonical construction of representation. Let us introduce the new basis for generators of the algebra:

$$\gamma_0^{\pm} = \frac{1}{2}(\pm\gamma_0 + \gamma_1),$$
$$\gamma_a^{\pm} = \frac{1}{2}(\gamma_{2a} \pm \imath\gamma_{2a+1}),\ a = 1, ..., k = \frac{d-2}{2}. \tag{30}$$

Then

$$\{\gamma_a^+, \gamma_b^-\} = \delta_{ab}\ ,\ \{\gamma_a^{\pm}, \gamma_b^{\pm}\} = 0 \tag{31}$$

It is clearly that one can find a vector $v_0 \in V$ whith the properties

$$\gamma_a^- v_0 = 0. \tag{32}$$

Then all the vectors from V can be generated from v_0:

$$\gamma_a^+ v_0,\ \gamma_a^+\gamma_b^+ v_0,\ ... \tag{33}$$

Proposition.

The vector v_0 exits.

Indeed, let us take an arbitrary vector $v \in V$ such that $\gamma_0^- v \neq 0$. Then consider the vector

$$w = \gamma_0^- v \Rightarrow \gamma_0^- w = 0. \tag{34}$$

If $\gamma_1^- w \neq 0$ consider the vector

$$u = \gamma_1^- w \Rightarrow \gamma_1^- u = 0\ ,\ \gamma_0^- u = 0. \tag{35}$$

Going by this way we construct at the end the vector v_0 satisfying (32).

One can describe the basic vectors of V by the tuples $s = (s_0, ..., s_k)$, where $s_a = \pm\frac{1}{2}$:

$$v_s = (\gamma_k^+)^{s_k+\frac{1}{2}} ... (\gamma_0^+)^{s_0+\frac{1}{2}} v_0, \;\; v_0 = v_{-\frac{1}{2},...,-\frac{1}{2}} \tag{36}$$

and find explicitly the matrix elements of generators:

$$\gamma_\mu v_s = \sum_{s'} (\gamma_\mu)_{s,s'} v_{s'} \tag{37}$$

Thus, we have constructed the representation of $Cl(1, d-1)$ in $dimV = 2^{k+1} = 2^{\frac{d}{2}}$ dimensional (complex) vector space V. **The space V is called the space of Dirac's pinors**.

2.4. CLIFFORD ALGEBRA AUTOMORPHYSMS GROUP.

A remarkable property of Clifford algebra: **automorphisms group of Clifford algebra $Cl(1, d-1)$ is the Lorentz group $O(1, d-1)$**:

$$b_\mu \to \Lambda_\mu^\nu b_\nu$$

$$\{b_\mu, b_\rho\} \to \{\Lambda_\mu^\nu b_\nu, \Lambda_\rho^\sigma b_\sigma\} = \Lambda_\mu^\nu \Lambda_\rho^\sigma \{b_\nu, b_\sigma\} = 2\Lambda_\mu^\nu \Lambda_\rho^\sigma g_{\nu\sigma} = 2g_{\mu\rho}. \tag{38}$$

(In other words, the automorphysm preserve the structure of vector space and multiplication structure of the algebra.)

This property allows to construct Lorentz group representation on the space of Dirac's spinors.

Indeed, suppose we have a representation of $Cl(1, d-1)$ in a vector space V. If we could find for each element Λ of the Lorentz group a linear transformation of V which is given by a matrix $S(\Lambda)$ such that

$$S^{-1}(\Lambda)\gamma_\mu S(\Lambda) = \Lambda_\mu^\nu \gamma_\nu \tag{39}$$

we would get the representation of Lorent group in V.

It is easy to construct this representation for the subgroup $SO(1, d-1) \subset O(1, d-1)$ which is given by the Lorentz group elements with unit determinant.

2.5. DIRAC'S SPINORS AND GROUP $Spin(1, d-1)$.

Suppose we have already constructed the representation of the Lorentz group in the space of spinors V. Let us consider then the representation of the Lie algebra of the Lorentz group in V. For infinitesimal Lorentz transformation we must get

$$S(1 + \omega^{\mu\nu} J_{\mu\nu} + ...) = 1 + \imath\omega^{\mu\nu}\Sigma_{\mu\nu} + ..., \qquad where\ \Sigma_{\mu\nu} \in Cl(1, d-1) \tag{40}$$

and $\omega^{\mu\nu}$ are the parameters of the Lorentz group transformation.

Hence, in order to get the relation (39) we must find the generators $\Sigma_{\mu\nu}$ such that

$$\exp[-\frac{\imath}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}]\gamma_\rho \exp[\frac{\imath}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}] \approx \gamma_\rho - \frac{\imath}{2}\omega^{\mu\nu}[\Sigma_{\mu\nu}, \gamma_\rho] = (1 + \frac{1}{2}\omega^{\mu\nu}J_{\mu\nu})^\lambda_\rho \gamma_\lambda. \tag{41}$$

The matrix elements $(J_{\mu\nu})^\lambda_\rho$ can be calculated from the commutators

$$[J_{\mu\nu}, \frac{\partial}{\partial x^\rho}] = [x_\mu\partial_\nu - x_\nu\partial_\mu, \partial_\rho] = -g_{\mu\rho}\partial_\nu + g_{\nu\rho}\partial_\mu \equiv (J_{\mu\nu})^\lambda_\rho \partial_\lambda \tag{42}$$

because γ_ρ must transforms as a co-vector $\frac{\partial}{\partial x^\rho}$. Hence we must obtain

$$-\frac{\imath}{2}\omega^{\mu\nu}[\Sigma_{\mu\nu}, \gamma_\rho] = -\frac{\omega^{\mu\nu}}{2}(g_{\mu\rho}\gamma_\nu - g_{\nu\rho}\gamma_\mu). \tag{43}$$

The answer is

$$\Sigma_{\mu\nu} = \frac{\imath}{4}[\gamma_\mu, \gamma_\nu]. \tag{44}$$

It is easy to check that Lorentz Lie algebra generators are realized indeed by the Clifford algebra elements $\Sigma_{\mu\nu}$:

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = \imath(g_{\nu\rho}\Sigma_{\mu\sigma} - g_{\mu\rho}\Sigma_{\nu\sigma} - g_{\nu\sigma}\Sigma_{\mu\rho} + g_{\mu\sigma}\Sigma_{\nu\rho}). \tag{45}$$

Hence, the Lorentz algebra representation in V is given by $\Sigma_{\mu\nu}$. One can obtain the representation of the group $SO(1, d-1)$ considering the exponentials of the form

$$S = S(\exp{(\frac{\imath}{2}\omega^{\mu\nu}J_{\mu\nu})}) = \exp{(\frac{\imath}{2}\omega^{\mu\nu}\Sigma_{\mu\nu})}. \tag{46}$$

The representation of the group $SO(1, d-1)$ we just have constructed in V using the matrices (44), (46) is called **spinor representation**.

Notice that the transformation $S(\Lambda)$ and $-S(\Lambda)$ give the same Lorentz transformation on the elemnts of Clifford algebra (as well as on the vectors of Minkowski space) **but correspond to different transformations in the space of Dirac's spinors** V. It means in particular that the set of elements (46) constitute the group which is not $SO(1, d-1)$ group but **a double cover of the group** $SO(1, d-1)$:

$$\phi : Spin(1, d-1) \to SO(1, d-1) \ , \ \phi(S) = \phi(-S) \in SO(1, d-1). \tag{47}$$

This group is called $Spin(1, d-1)$.

2.6. WEYL'S SPINORS.

The spinor representation of the Lorentz group we have constructed in the space of Dirac's spinors V is **reducible**. It means that under the

$Spin(1, d-1)$ transformations the space decomposes into the direct sum of invariant subspaces

$$V = V_+ \oplus V_-. \tag{48}$$

The subspaces $V_\pm$ are $\pm$ eigenspaces of the operator $\Gamma_{d+1} = \gamma_0\gamma_1 ... \gamma_{d-1}$:

$$\Gamma_{d+1} V_\pm = \pm V_\pm. \tag{49}$$

This statement follows from the simple calculations:

$$\Gamma_{d+1}^2 = 1 \ , \Gamma_{d+1}\gamma_\mu = -\gamma_\mu \Gamma_{d+1}. \tag{50}$$

The vectors from $V_\pm$ are called **Weyl's spinors** of positive and negative chiralities correspondingly.

3. Dirac equation and Weyl equation.

3.1. DIRAC EQUATION.

Dirac equation is an equation of motion of Dirac field $\psi(x)$:

$$(\imath\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{51}$$

$\psi(x)$ is a function taking values in the space of Dirac's spinors V we considered above. Dirac equation is Lorentz invariant if we demand the following rule for the Dirac field transformation

$$\psi(x) \to \tilde{\psi}(x) = S(\Lambda)\psi(\Lambda x). \tag{52}$$

Indeed

$$
\begin{aligned}
(\imath\gamma^\mu\partial_\mu - m)\tilde{\psi}(x) = (\imath\gamma^\mu(\Lambda)^\nu_\mu\frac{\partial}{\partial y^\nu} - m)S(\Lambda)\psi(y) = \\
S(\Lambda)S^{-1}(\Lambda)(\imath\gamma^\mu(\Lambda)^\nu_\mu\frac{\partial}{\partial y^\nu} - m)S(\Lambda)\psi(y) = \\
S(\Lambda)(\imath S^{-1}(\Lambda)\gamma^\mu S(\Lambda)(\Lambda)^\nu_\mu\frac{\partial}{\partial y^\nu} - m)\psi(y) = \\
S(\Lambda)(\imath(\Lambda^{-1})^\mu_\rho\gamma^\rho(\Lambda)^\nu_\mu\frac{\partial}{\partial y^\nu} - m)\psi(y) = 0,
\end{aligned}
\tag{53}
$$

where $y = \Lambda x$.

To construct the Lagrangian leading to the Dirac equation one needs to construct first a scalar from spinors. One can check that

$$
\bar{\psi} \equiv \psi^\dagger\gamma^0 \tag{54}
$$

thransforms by

$$
\bar{\psi} \to \bar{\psi}S^{-1}(\Lambda) \tag{55}
$$

so that $\bar{\psi}(x)\psi(x)$ is a scalar field. Then one can construct the Dirac field Lorentz invariant Lagrangian density as

$$
\mathcal{L} = \bar{\psi}(x)(\imath\gamma^\mu\partial_\mu - m)\psi(x). \tag{56}
$$

3.2. WEYL EQUATION.

Because of the space of Dirac's spinors is reducible representation of the Lorentz group ($V = V_+ \oplus V_-$) one can try to construct Lorentz invariant equation for the Weyl's spinors.

Let us project the Dirac spinor field on its V_+ component by the projection operator $\frac{1}{2}(1+\Gamma_5)$ and find Lorentz invariant equation it may obey. Acting by the Dirac operator we find

$$(\imath\gamma^\mu\partial_\mu - m)\frac{1}{2}(1+\Gamma_5)\psi(x) = \frac{1}{2}(1-\Gamma_5)\imath\gamma^\mu\partial_\mu\psi - \frac{1}{2}(1+\Gamma_5)m\psi. \tag{57}$$

It means that Dirac equation is not consistent with the projection on chiral subspace V_+ due to the mass term.

We see that one would obtain the Lorentz invariant equation for Weyl spinors if we put $m=0$:

$$\imath\gamma^\mu\partial_\mu\frac{1}{2}(1\pm\Gamma_5)\psi(x) = 0. \tag{58}$$

Thus, Weyl's spinors of positive chirality $\psi_L(x)\in V_+$ obey the equation

$$\imath(\partial_0 - \vec{\nabla}\vec{\sigma})\psi_L(x) = 0, \tag{59}$$

while the Weyl spinors of negative chirality $\psi_R(x)\in V_-$ obey the equation

$$\imath(\partial_0 + \vec{\nabla}\vec{\sigma})\psi_R(x) = 0 \tag{60}$$

where we have used the following realization of gamma-matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; \gamma_k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \Rightarrow \Gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{61}$$

while the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{62}$$

The neutrino are described by the Weyl's spinors and obey the equations (59), (60) (breaking the parity P).

3.3. PLANE WAVE SOLUTIONS OF DIRAC EQUATION.

It can be seen that any solution to the Dirac equation is also a solution to the KG equation, hence one can represent Dirac equation solution in the form

$$\psi(x) = u(p) \exp\left(-\imath px\right) \ , \ p^2 = m^2. \tag{63}$$

Let us consider first the solution with $p^0 = \sqrt{\vec{p}^2 + m^2}$ ($p^0 > 0$). Dirac equation then reduces to the linear equation for the complex 4-vector $u(p)$

$$(\gamma^\mu p_\mu - m)u(p) = 0. \tag{64}$$

It can be analyzed in the rest frame of the particle: $p = (m, 0)$:

$$(m\gamma^0 - m)u(p) = m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p^0) = 0 \Rightarrow \\ u(p^0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}. \tag{65}$$

Under the space rotations the vector ξ transforms as a Weyl spinor and hence determines the spin orientation of a particle. It will also be convenient to normalize ξ as

$$\xi^\dagger \xi = 1. \tag{66}$$

One can apply a boost to generate the solution in an arbitrary frame. We aplly the boost along the third axis x^3. The boost can be parametrized by a rapidity η. The boost action on the momentum vector is given by

$$\left(\begin{array}{c} E \\ p^3 \end{array}\right) = \exp{(\eta\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right))}\left(\begin{array}{c} m \\ 0 \end{array}\right) = \left(\begin{array}{c} m\cosh{(\eta)} \\ m\sinh{(\eta)} \end{array}\right). \tag{67}$$

Now we apply the boost to the Dirac spinor $u(m,0)$. We have

$$\Sigma_{03} = \frac{\imath}{4}[\gamma_0,\gamma_3] = \frac{\imath}{2}\left(\begin{array}{cc} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{array}\right), \tag{68}$$

$$\begin{array}{r} \exp{(\frac{\imath}{2}\eta\Sigma_{03})}\sqrt{m}\left(\begin{array}{c} \xi \\ \xi \end{array}\right) = \\ \frac{\sqrt{m}}{2}\left(\begin{array}{c} (\exp{(\frac{\eta}{2})}(1-\sigma^3)+\exp{(-\frac{\eta}{2})}(1+\sigma^3))\xi \\ (\exp{(\frac{\eta}{2})}(1+\sigma^3)+\exp{(-\frac{\eta}{2})}(1-\sigma^3))\xi \end{array}\right) = \\ \left(\begin{array}{c} (\sqrt{E+p^3}\frac{1-\sigma^3}{2}+\sqrt{E-p^3}\frac{1+\sigma^3}{2})\xi \\ (\sqrt{E+p^3}\frac{1+\sigma^3}{2}+\sqrt{E-p^3}\frac{1-\sigma^3}{2})\xi \end{array}\right) = u(p), \end{array} \tag{69}$$

where $E+p^3 = m\exp(\eta)$, $E-p^3 = m\exp(-\eta)$ have been taken into account. The last expression can be rewritten in the form

$$u(p) = \left(\begin{array}{c} \sqrt{p\sigma}\xi \\ \sqrt{p\bar{\sigma}}\xi \end{array}\right) \tag{70}$$

where $\sigma = (1,\vec{\sigma})$, $\bar{\sigma} = (1,-\vec{\sigma})$.

The solutions for $p^0 = -\sqrt{\vec{p}^2+m^2}$ can be found similarly. But we use the ansatz with reversed p^0

$$\psi(x) = v(p)\exp{(\imath px)},$$
$$v(p) = \begin{pmatrix} \sqrt{p\sigma}\chi \\ -\sqrt{p\bar{\sigma}}\chi \end{pmatrix}. \tag{71}$$

Lorentz invariant norm of the solutions (70), (71) is given by

$$\bar{u}(p)u(p) = 2m\xi^\dagger\xi = 2m\ ,\ \bar{v}(p)v(p) = -2m\chi^\dagger\chi = -2m. \tag{72}$$

As a result we obtaine the folowing basis of Dirac equation solutions

$$\psi(x)^s = u(p)^s\exp{(-\imath px)}\ ,\ u^s(p) = \begin{pmatrix} \sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s \end{pmatrix}$$
$$\psi(x) = v^s(p)\exp{(\imath px)}\ ,\ v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\chi^s \\ -\sqrt{p\bar{\sigma}}\chi^s \end{pmatrix}, \tag{73}$$

where ξ^s, $s = 1, 2$ is a pair of basic Weyl spinors, χ^s, $s = 1, 2$ is another pair of basic Weyl spinors. The orthogonality conditions for the basic solutions have the form

$$\bar{u}^s(p)u^r(p) = 2m\delta^{sr}\ ,\ \bar{v}^s(p)v^r(p) = -2m\delta^{sr}$$
$$\bar{u}^s(p)v^r(p) = \bar{v}^s(p)u^r(p) = 0. \tag{74}$$

3.4. SPIN SUMMS.

$$\sum_s (u^s(\vec{p})\bar{u}^s(\vec{p})) = \begin{pmatrix} \sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s \end{pmatrix} (\xi^{\dagger s}\sqrt{p\bar{\sigma}}, \xi^{\dagger s}\sqrt{p\sigma}) =$$
$$\begin{pmatrix} \sqrt{p\sigma}\sqrt{p\bar{\sigma}} & \sqrt{p\sigma}\sqrt{p\sigma} \\ \sqrt{p\bar{\sigma}}\sqrt{p\sigma} & \sqrt{p\sigma}\sqrt{p\bar{\sigma}} \end{pmatrix} = \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} = \gamma^\mu p_\mu + m. \tag{75}$$

Similarly one can obtain

$$\sum_s (v^s(\vec{p})\bar{v}^s(\vec{p})) = \gamma^\mu p_\mu - m. \tag{76}$$

Appendix.

The differential geometry definition of covariant rank-p tensor:

suppose we have a map of the point $x = (x^0, ..., x^N)$ to the point $\tilde{x} = (\tilde{x}^0(x), ..., \tilde{x}^N(x))$. Then the image $\tilde{A}_{\mu_1,...,\mu_p}(x)$ of covariant rank-p tensor $A_{\mu_1,...,\mu_p}$at the point x is given by

$$\tilde{A}_{\mu_1,...,\mu_p}(x) = \frac{\partial \tilde{x}^{\nu_1}}{\partial x^{\mu_1}} ... \frac{\partial \tilde{x}^{\nu_p}}{\partial x^{\mu_p}} A_{\nu_1,...\nu_p}(\tilde{x}(x)). \tag{77}$$

As an exmple let us consider the transformation of covariant rank-1 tensor under the Lorentz transformation, which is given by the map

$$x \to \tilde{x} = \Lambda x \Leftrightarrow \tilde{x}^\nu = \Lambda^\nu_\mu x^\mu. \tag{78}$$

We have from the definition above

$$\tilde{A}_\mu(x) = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} A_\nu(\Lambda x) = \Lambda^\nu_\mu A_\nu(\Lambda x). \tag{79}$$

Similarly, contravariant rank-p tensor transforms as

$$\tilde{A}^{\mu_1,...,\mu_p}(x) = \frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\nu_1}} ... \frac{\partial x^{\mu_p}}{\partial \tilde{x}^{\nu_p}} A^{\nu_1,...\nu_p}(\tilde{x}(x)). \tag{80}$$