Lecture 4.

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1. Path integral in QM and T-ordered expectation values.

1.0. D_c, D_F, D_E AS A FUNCTIONS IN COMPLEX TIME (reminder).

Function D_c.

First of all we considered the function

$$[\hat{\phi}(x), \hat{\phi}(y)] = D_-(x-y) - D_+(x-y) \tag{1}$$

and found that there is a single analytic function of complex variable t:

$$D_c(\vec{x},t) = \int_{p^2=m^2} d\mu(p)\exp\left(-\imath\omega_{\vec{p}}t + \imath\vec{p}\vec{x}\right),\ Imt < 0$$

$$D_c(\vec{x},t) = \int_{p^2=m^2} d\mu(p)\exp\left(\imath\omega_{\vec{p}}t - \imath\vec{p}\vec{x}\right),\ Imt > 0$$

$$D_-(\vec{x},t) = D_c(\vec{x},t-\imath 0),\ Imt = 0$$

$$D_+(\vec{x},t) = D_c(\vec{x},t+\imath 0),\ Imt = 0 \tag{2}$$

such that for x is timelike and real t, the last limiting values do not coincide, so $D_c(\vec{x},t)$ has a branch cuts along the real t which extend from $t = |\vec{x}|$ to ∞ and from $t = -|\vec{x}|$ to ∞. And these cuts are due to the causality principle for KG theory.

Function D_F.

This function which is known as **Feynmam propagator is the first example of time-ordered expectation value of Heisenberg's operators**:

$$D_F(x-y) =< 0|T(\hat{\phi}(\vec{x},t_x)\hat{\phi}(\vec{y},t_y))|0> . \tag{3}$$

We found that Feynman propagator $D_F(\vec{x},t)$ is the limiting value of the function $D_c(\vec{x},t)$ when the time is going along the countour C_F which goes slightly **above** the real t-axis in the region $Ret < -|\vec{x}|$, cross the segment $-|\vec{x}| < t < |\vec{x}|$ and goes slightly **below** the real t-axis in the region $Ret > |\vec{x}|$. This representation allowed us to write

$$D_F(x) = \int_{p^2=m^2} d\mu(p)\exp\left(-\imath\omega_{\vec{p}}t + \imath\vec{p}\vec{x}\right),\ t > 0,$$

$$D_F(x) = \int_{p^2=m^2} d\mu(p)\exp\left(\imath\omega_{\vec{p}}t - \imath\vec{p}\vec{x}\right),\ t < 0,$$

$$\Leftrightarrow$$

$$D_F(x) = \int_{\tilde{C}_F}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\imath\exp\left(-\imath\omega t + \imath\vec{p}\vec{x}\right)}{\omega^2 - \vec{p}^2 - m^2} \Leftrightarrow$$

$$D_F(x) = \int\frac{d^4\vec{p}}{(2\pi)^4}\frac{\imath\exp\left(-\imath p_\mu x^\mu\right)}{p^2 - m^2 + \imath 0} \tag{4}$$

where $d\mu(\vec{p}) = \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}$. We saw from the last expression that $D_F(x)$ is a Green's function for the KG equation

$$(\partial_\mu\partial^\mu + m^2)D_F(x) = \imath\delta(x). \tag{5}$$

Function $D(x_E)$.

It has been determined also the euclidean, or imaginary-time expectation value (or correlation function) as the limiting value of the function $D_c(\vec{x}, t)$ for the time running along contour C_E which is the imaginary axis in comlex t-plane going from $+\infty$ to $-\infty$:

$$D(x_E) \equiv D(\vec{x}, x_4) = D_c(\vec{x}, -\imath x_4). \tag{6}$$

We used Schwinger's proper time representation:

$$D(x_E) = \int_0^\infty d\tau \int \frac{d^4p_E}{(2\pi)^4} \exp\left(-\tau(p_E^2 + m^2) + \imath p_E x_E\right) \tag{7}$$

to express this function by the path integral in euclidean space:

$$D(x_E) = \int_{path(i\to f)} [Dx_E] \exp(-A) =$$

$$\int_0^\infty ds \exp(-m^2 s) \int \frac{d^4P}{(2\pi)^4} \exp(-sP^2) \exp(\imath P x_E). \tag{8}$$

And we got path integral representation for Feynman propagator D_F taking the analytic continuation of $D(x_E)$ to the real time, in such a way that to get

$$\exp\left(\frac{\imath}{\hbar} S[q(t)]\right) \to \exp\left(-A[q(\tau)]\right). \tag{9}$$

1.1. PRELIMINARIES FROM STATISTICAL MECHANICS.

The connection between the vacuum average of the T-ordered product of Heisenberg operators and the Euclidean expectation, which holds for the cases $D_F(x)$ and $D(x_E)$, can be generalized.

Let us consider some dynamical system with the canonical coordinates in phase-space $\{Q_i, P_i\}$ and Hamiltonian function $H(Q_i, P_i)$. In thermal equilibrium state with some temeperature $\mathcal{T}$, the probability distribution of microscopic states is given by Hibbs formula

$$\rho(Q_i, P_i) \prod_i dQ_i dP_i = \frac{1}{Z} \exp\left(-\beta H(Q_i, P_i)\right) \frac{1}{N!} \prod_i dQ_i dP_i, \tag{10}$$

where $\beta = \frac{1}{k\mathcal{T}}$, and

$$Z = \int \frac{\prod_i dQ_i dP_i}{N!} \exp\left(-\beta H(Q_i, P_i)\right). \tag{11}$$

We assume also that typical form of H is

$$H = \sum_i \frac{1}{2} P_i^2 + W(Q_i). \tag{12}$$

Then one can integrate out the momenta P_i using Gaussian integral formula

$$\int dy \exp(-by^2) = \sqrt{\frac{\pi}{b}} \tag{13}$$

so that one can write

$$Z = \frac{1}{N!} \left(\frac{2\pi}{\beta}\right)^{\frac{N}{2}} Z_{conf}, \tag{14}$$

where

$$Z_{conf} = \int \prod_i dQ_i \exp\left(-\beta W(Q_i)\right) \tag{15}$$

is the **configuration-space integral** which contains most of interesting physics.

1.2. STATISTICAL MECHANICS OF ELASTIC STRING IN POTENTIAL.

Let us consider a particular case of statistical system which is elastic string whose configuration is given by a function $q(\tau)$, $\tau \in (0, T)$ and which lies in a potential valley $V(q)$. So the potential energy of the string is

$$W[q(\tau)] = \int_0^T d\tau (\frac{1}{2}(\frac{dq}{d\tau})^2 + V(q)), \tag{16}$$

where the first term accounts for the elastic energy of the string (whose tension is equal 1).

Calculating configuration part of the partition function we have to integrate over the functions $q(\tau)$ with the boundary conditions $q(0) = q_i$, $q(T) = q_f$ so that the configuration-space integral is given by

$$Z_{conf} = \int_{q_i}^{q_f} [Dq(\tau)] \exp(-\beta W[q(\tau)]). \tag{17}$$

This expression is identical to the path integral for the imaginary-time quantum mechanical transition amplitude

$$Z_{conf} =< q_f|U(T,0)|q_i >= \int_{q_i}^{q_f} [Dq(\tau)] \exp(-A[q(\tau)]),\ q_f = q(T),\ q_i = q(0) \tag{18}$$

provided we made also the identification

$$\beta = \frac{1}{\hbar},\ W[q(\tau)] = A[q(\tau)]. \tag{19}$$

Thus, we conclude that **quantum mechanics with Hamiltonian operator**

$$\hat{H} = \frac{1}{2}p^2 + V(q) \tag{20}$$

is related in this way to the classical Statistical Mechanics of a system with continously many degrees of freedom $q(\tau)$.

1.3. CORRELATION FUNCTIONS.

Besides the partition function one may be interested in **correlation functions**. In our string case this correlation function is given by the integral

$$< q(\tau_1), ...q(\tau_N) >= \frac{1}{Z} \int [Dq(\tau)] q(\tau_1), ...q(\tau_N) \exp\left(-A[q(\tau)]\right) \tag{21}$$

because $[Dq(\tau)] \exp\left(-A[q(\tau)]\right)$ is a statistical weight of the microstate $q(\tau)$.

1.4. PATH INTEGRAL REPRESENTATION OF τ-ORDERED EXPECTATION VALUES OF HEISENBERG OPERATORS IN QM.

Now we consider the interpretation of this quantity from the point of view of QM.

Without loss of generality we can take

$$-\frac{T}{2} \le \tau_1 \le \tau_2 \le ... \le \tau_N \le \frac{T}{2}. \tag{22}$$

One can perform this integral in 2 steps: at the first step we fix the values of $q(\tau)$ at the points τ_1,...τ_N to be

$$q(\tau_1) = q_1, \ ..., \ q(\tau_N) = q_N \tag{23}$$

and integrate over the $q(\tau)$ with these constraints. At the second step we integrate over the values q_1,..., q_N.

Step 1: Using the path integral representation for the transition apmlitude in imaginary time

$$< q_{i+1} | U(\tau_{i+1}, \tau_i) | q_i >= \int_{q_i}^{q_{i+1}} [Dq(\tau)] \exp\left(-A[q(\tau)]\right) \tag{24}$$

we can write

$$< q(\tau_1)...q(\tau_N) >=$$
$$\frac{1}{Z}\int\prod_{k=1}^{N} dq_k < q_f|U(\frac{T}{2},\tau_N)|q_N > q_N < q_N|U(\tau_N,\tau_{N-1})|q_{N-1} > q_{N-1}...$$
$$...q_1 < q_1|U(\tau_1,-\frac{T}{2})|q_i > . \tag{25}$$

Here we have not yet integrated over the positions of q_k, but only over the intermediate configurations that led to the insertions (24).

Step 2: Recall that for $\tau_i \geq \tau_j$ we can write

$$U(\tau_i,\tau_j) = \exp\left(-(\tau_i-\tau_j)\hat{H}\right), \tag{26}$$

because of the spectrum of $\hat{H}$ is bounded from below.

Using

$$\hat{q}|q >= q|q > \tag{27}$$

we see that

$$\int\prod_{k=1}^{N} dq_k < q_f|U(\frac{T}{2},\tau_N)|q_N > q_N < q_N|U(\tau_N,\tau_{N-1})|q_{N-1} > ...$$
$$q_1 < q_1|U(\tau_1,-\frac{T}{2})|q_i >=$$
$$< q_f|\exp\left(-(\frac{T}{2}-\tau_N)\hat{H}\right)\hat{q}\exp\left(-(\tau_N-\tau_{N-1})\hat{H}\right)\hat{q}...$$
$$\hat{q}\exp\left(-(\tau_1+\frac{T}{2})\hat{H}\right)|q_i >=$$
$$< q_f|\exp\left(-\frac{T}{2}\hat{H}\right)\hat{q}_E(\tau_N)\hat{q}_E(\tau_{N-1})...\hat{q}_E(\tau_1)\exp\left(-\frac{T}{2}\hat{H}\right)|q_i >, \tag{28}$$

where

$$\hat{q}_E(\tau) = \exp{(\tau \hat{H})} \hat{q} \exp{(-\tau \hat{H})} \tag{29}$$

is the **imaginary-time Heisenberg operator**. It is related to the real-time Heisenberg operator $\hat{q}(t)$ by continuation

$$t = -\imath \tau. \tag{30}$$

1.5. $T \to \infty$ LIMIT.

One can simplify our relation between N-point correlation function and τ-ordered Heisenberg operators expectation value considering the limit $T \to \infty$, thus getting to the correlation function of the infinite string.

Notice that the states $|q_{f,i}>$ can be expanded in stationary states $|n>$ of $\hat{H}$:

$$|q_{f,i}> = \sum_n |n>< n|q_{f,i}> = \sum_n \Psi_n^*(q_{f,i}). \tag{31}$$

We then observe that in the limit $T \to \infty$ only the ground state survives, so that we obtain

$$\begin{aligned} &< q(\tau_1)...q(\tau_N) > = \\ \frac{< q_f|0 >< 0|q_i > \exp{(-E_0 T)}}{Z} &< 0|\hat{q}_E(\tau_N)\hat{q}_E(\tau_{N-1})... \\ &\hat{q}_E(\tau_1)|0 > . \end{aligned} \tag{32}$$

Taking into account that (see (18))

$$Z = < q_f| \exp{(-T\hat{H})}|q_i > \tag{33}$$

we see that the first factor in (32) is equal 1 so that we find **the relation between the N-point correlation function and expectation value of τ-ordered Heisenberg operators** $\hat{q}_E(\tau_k)$.

It is important to note that by construction, **the operators** $\hat{q}_E(\tau)$ **are nessesserily placed in the order of τ increasing from the right to the left**. Thus, **the expectation values of imaginary-time Heisenberg operators make sense only if they are τ-ordered**, for other wise the sums over the intermediate states are diverge.

1.6. IMAGINARY-TIME/REAL-TIME EXPECTATION VALUES RELATION.

Actually, **the expectation value above makes sense even for complex values of** τ_k **if**

$$Re(\tau_1) \leq Re(\tau_2) \leq ... \leq Re(\tau_N) \tag{34}$$

so that the correlation function is analytic (because the serieses are converge).

By this reason **one can relate naturally the imaginary-time expectation values to the real-time ones setting**

$$\tau_k = \exp{(\imath\alpha)} t_k,\ t_k \in \mathbb{R},\ 0 \leq \alpha < \frac{\pi}{2}. \tag{35}$$

Therefore, taking

$$t_1 \leq t_2 \leq ... \leq t_N \tag{36}$$

we satisfy the inequalities (34). Then, sending $\alpha_k \to \frac{\pi}{2} - 0$ we obtain the relation:

$$< q(\tau_1)...q(\tau_N) >=< 0|T(\hat{q}(t_1)...\hat{q}(t_N))|0 >,\ where\ \tau_k = (\imath - 0)t_k, t_k \in \mathbb{R} \tag{37}$$

and $\hat{q}(t)$ is usual **real-time Heisenberg operators**. The relation between Feynman propagator D_F and the correlation function $D(x_E)$ previously discussed is a particular example of (37).

Taking into account the expression (21) for the correlation function:

$$< q(\tau_1), ...q(\tau_N) >= \frac{1}{Z} \int [Dq(\tau)] q(\tau_1), ...q(\tau_N) \exp\left(-A[q(\tau)]\right) \tag{38}$$

as well as the relation between imaginary-time action and real-time action

$$\exp\left(\frac{\imath}{\hbar} S[q(t)]\right) \rightarrow \exp\left(-A[q(\tau)]\right) \tag{39}$$

the relation (37) can be represented in the form

$$< 0|T(\hat{q}(t_1)...\hat{q}(t_N))|0 >= \lim_{T \to \infty} \frac{\int [Dq(t)] q(t_1), ...q(t_N) \exp\left[\frac{\imath}{\hbar} \int_{-T}^{T} L(q(t)) dt\right]}{\int [Dq(t)] \exp\left[\frac{\imath}{\hbar} \int_{-T}^{T} L(q(t)) dt\right]}. \tag{40}$$

2. Functional integral in KG theory.

We would like to generalize the formula (40) for the case of QFT. For simplicity we consider KG theory, though the discussion to be presented is true for any physically acceptable model of QFT.

2.1. DEFINITION OF FUNCTIONAL INTEGRAL BY AN EUCLIDEAN LATTICE.

We start from the definition of the euclidean path integral (functional integral) for the KG field.

First of all we replace the coordinate q by a field

$$q(\tau) \rightarrow \phi(\vec{x}, \tau) = \phi(\vec{x}, x_4) = \phi(x). \tag{41}$$

The euclidean action is obtained by the substitution $t \to -\imath x_4$:

$$A = \int dx_4 d\vec{x} \frac{1}{2}((\partial_4 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2) =$$
$$\int d^4 x \frac{1}{2}((\partial_a \phi)^2 + m^2 \phi^2) = \int d^4 x \mathcal{L}(\phi(x), \partial\phi(x)), \qquad (42)$$

where $a = 1, ..., 4$. We are interested in the path integral

$$Z = \int [D\phi(x)] \exp\left(-A[\phi(x)]\right). \qquad (43)$$

To explain this expression we define it first as an integral over the finite number of variables introducing instead of 4-dimensional euclidean space 4-dimensional (euclidean) lattice with spacing Δ:

$$x^a = n^a \Delta, \;\; n^a \in \mathbb{Z}. \qquad (44)$$

Then

$$\begin{aligned} \phi(x) &\to \phi(n), \;\; n = (n^1, ..., n^4) \in \mathbb{Z}^4, \\ \partial_a \phi(x) &\to \frac{1}{\Delta}(\phi(n + e^a \Delta) - \phi(n)), \\ \int d^4 x &\to \Delta^4 \sum_{n \in \mathbb{Z}^4}, \\ A = \int d^4 x \mathcal{L}(\phi(x), \partial\phi(x)) &\to \Delta^4 \sum_{n \in \mathbb{Z}^4} \mathcal{L}(n), \\ \int [D\phi(x)] &\to \prod_n d\phi(n), \end{aligned} \qquad (45)$$

so that we have an integral

$$\int \prod_n d\phi(n) \exp\left(-A[\phi(n)]\right). \qquad (46)$$

The right hand side of (43) can be understood as a limit $\Delta \to 0$ of the lattice integral above

$$Z = \int [D\phi(x)] \exp\left(-A[\phi(x)]\right) = \lim_{\Delta \to 0} \int \prod_n d\phi(n) \exp\left(-A[\phi(n)]\right). \quad (47)$$

2.2. CORRELATION FUNCTIONS.

We are interested also in the correlations functions

$$< \phi(x_N)...\phi(x_1) >= \frac{1}{Z} \int [D\phi]\phi(x_N)...\phi(x_1) \exp\left(-A[\phi]\right). \quad (48)$$

The points x_i are in euclidean space and $[D\phi] \exp\left(-A[\phi]\right)$ **can be considered as a statistical weight of microstate** $\phi(\vec{x}, x_4)$. Therefore, this expression can also be interprated from the Statistical Mechanics point of view.

We would like to continue this expression into the complex values of $(x_i)_4$ **and relate it to the real-time ordered expectation values of Heisenberg operators**

$$< 0|T(\hat{\phi}_H(\vec{x}_N, t_N)...\hat{\phi}_H(\vec{x}_1, t_1))|0 >, \quad (49)$$

where $(x_i)_4 = \imath t_i$.

2.3. PATH INTEGRAL REPRESENTATION OF τ-ORDERED EXPECTATION VALUES OF HEISENBERG OPERATORS IN QFT.

Without loss of generality we can take

$$-\frac{T}{2} \leq \tau_1 \leq \tau_2 \leq ... \leq \tau_N \leq \frac{T}{2} \quad (50)$$

and perform the functional integral on the r.h.s of (48) by 2 steps. First of all we fix the configurations of $\phi(x)$ at the imaginary time moments

τ_1,...,τ_N as the following

$$\phi(\vec{x},-\frac{T}{2})=\phi_i(\vec{x}),\ \phi(\vec{x},\tau_1)=\phi_1(\vec{x}),\ ...,\ \phi(\vec{x},\tau_N)=\phi_N(\vec{x}),\ \phi(\vec{x},\frac{T}{2})=\phi_f(\vec{x}). \tag{51}$$

Then we integrate over the $\phi(x)$ with these constraints. At the second step we integrate over the values $\phi_1(\vec{x})$,...,$\phi_N(\vec{x})$.

Step 1: At the first step we use the path integral representation for the transition amplitude in imaginary time

$$< \phi_{k+1}|U(\tau_{k+1},\tau_k)|\phi_k >= \int_{\phi_k}^{\phi_{k+1}} [D\phi(\vec{x},\tau)] \exp\left(-A[\phi]\right)$$
$$where\ U(\tau_{k+1},\tau_k)=\exp\left(-(\tau_{k+1}-\tau_k)\hat{H}\right) \tag{52}$$

and we integrate over the all configurations $\phi(x)$ such that $\phi(\vec{x},\tau_k)=\phi_k(\vec{x})$, $\phi(\vec{x},\tau_{k+1})=\phi_{k+1}(\vec{x})$.

This expression can be obtained similar to the QM derivation. **The main assumption is used to derive this formula is that the spectrum of the Hamiltonian is bounded from below so that r.h.s of this expression is an analytic function of real values** $\tau_{k+1}-\tau_k$ when

$$\tau_{k+1}\geq\tau_k. \tag{53}$$

Therefore we can write

$$< \phi(x_N)...\phi(x_1) >=$$
$$\frac{1}{Z}\int\prod_{k=1}^{N}[D\phi_k] < \phi_f|U(\frac{T}{2},\tau_N)|\phi_N > \phi_N(\vec{x}_N) < \phi_N|U(\tau_N,\tau_{N-1})|\phi_{N-1} >$$
$$...\phi_2(\vec{x}_2) < \phi_2|U(\tau_2,\tau_1)|\phi_1 > \phi_1(\vec{x}_1) < \phi_1|U(\tau_1,-\frac{T}{2})|\phi_i > . \tag{54}$$

Thus, we have integrated over the intermediate configurations $\phi(\vec{x},\tau)$ such that for the moments of (imaginary) time τ_k the field configurations are fixed by $\phi_k(\vec{x})$. Moreover, the initial configuration $\phi_i(\vec{x})$ at the moment $-\frac{T}{2}$ and the finite configuration $\phi_f(\vec{x})$ at the moment $\frac{T}{2}$ are also fixed.

Step 2: Now we replace the functions $\phi_k(\vec{x})$ with the Schrödinger operators which are defined by

$$\hat{\phi}_S(\vec{x})|\phi_k >= \phi_k(\vec{x})|\phi_k > \tag{55}$$

and use the relation

$$\int [D\phi_k]|\phi_k >< \phi_k| = 1 \tag{56}$$

at the second step. We obtain thereby

$$\int \prod_{k=1}^{N}[D\phi_k] < \phi_f|U(\frac{T}{2},\tau_N)|\phi_N > \phi_N(\vec{x}_N) < \phi_N|U(\tau_N,\tau_{N-1})|\phi_{N-1} > ...$$
$$...\phi_1(\vec{x}_1) < \phi_1|U(\tau_1,-\frac{T}{2})|\phi_i >=$$
$$< \phi_f|U(\frac{T}{2},\tau_N)\hat{\phi}_S(\vec{x}_N)U(\tau_N,\tau_{N-1})...U(\tau_2,\tau_1)\hat{\phi}_S(\vec{x}_1)U(\tau_1,-\frac{T}{2})|\phi_i >, \tag{57}$$

where the amplitudes $< \phi_k|U(\tau_k,\tau_{k-1})|\phi_{k-1} >$ are given by the integrals (52).

The Schrödinger operators can be replaced by **imaginary-time Heisenberg operators** according to the definition

$$\exp{(\tau\hat{H})}\hat{\phi}_S(\vec{x})\exp{(-\tau\hat{H})} = \hat{\phi}_H(\vec{x},\tau) \tag{58}$$

and hence

$$< \phi_f|U(\frac{T}{2},\tau_N)\hat{\phi}_S(\vec{x}_N)U(\tau_N,\tau_{N-1})...U(\tau_2,\tau_1)\hat{\phi}_S(\vec{x}_1)U(\tau_1,-\frac{T}{2})|\phi_i >= \\ < \phi_f|\exp{(-\frac{T}{2}\hat{H})}\hat{\phi}_H(\vec{x}_N,\tau_N)...\hat{\phi}_H(\vec{x}_1,\tau_1)\exp{(\frac{T}{2}\hat{H})}|\phi_i > . \tag{59}$$

2.4. $T \to \infty$ LIMIT.

Now we take the limit $T \to \infty$. Similar to the QM case we obtain in this limit

$$< \phi(\vec{x}_N,\tau_N)...\phi(\vec{x}_1,\tau_1) >= \\ \frac{< \phi_f|0 >< 0|\phi_i > \exp{(-E_0T)}}{Z} < 0|\hat{\phi}_H(\vec{x}_N,\tau_N)...\hat{\phi}_H(\vec{x}_1,\tau_1)|0 > . \tag{60}$$

But the first factor is equal 1 if we take into account that

$$Z = \int_{\phi_i(\vec{x})}^{\phi_f(\vec{x})}[D\phi(x)]\exp{(-A[\phi(x)])}. \tag{61}$$

2.5. IMAGINARY-TIME/REAL-TIME EXPECTATION VALUES RELATION.

The expression (60) have been obtained in euclidean space. But it can be defined for the complex values of x_i^4 similarly to the QM case.

When continued to the complex values of $x_i^4 = \tau_i$ and specialized to $x_i^4 = \imath t_i$, where t_i are real, the correlation functions give the time-ordered expectation values of Heisenberg operators. Thus, the rule is the following:

$$< \phi(\vec{x}_N,\tau_N)...\phi(\vec{x}_1,\tau_1) >=< 0|T(\hat{\phi}(\vec{x}_N,t_N)...\hat{\phi}(\vec{x}_1,t_1)|0 >,\ \tau_k = (\imath - 0)t_k, \tag{62}$$

where $\hat{\phi}(\vec{x}, t)$ is the **real-time Heisenberg field operator**. Taking into account (48):

$$< \phi(x_N)...\phi(x_1) >= \frac{1}{Z} \int [D\phi]\phi(x_N)...\phi(x_1) \exp\left(-A[\phi]\right) \tag{63}$$

and euclidean action definition

$$\exp\left(\frac{\imath}{\hbar} S[\phi(\vec{x}, t)]\right) \rightarrow \exp\left(-A[\phi(\vec{x}, \tau)]\right) \tag{64}$$

we obtain functional integral representation for the expectation value of **real time-ordered Heisenberg operators**

$$< 0|T(\hat{\phi}(\vec{x}_N, t_N)...\hat{\phi}(\vec{x}_1, t_1))|0 >= \lim_{T \rightarrow \infty} \frac{\int [D\phi]\phi(\vec{x}_1, t_1)...\phi(\vec{x}_N, t_N) \exp\left[\frac{\imath}{\hbar} \int_{-T}^{T} dt d^3x \mathcal{L}(\phi, \partial\phi)\right]}{\int [D\phi] \exp\left[\frac{\imath}{\hbar} \int_{-T}^{T} dt d^3x \mathcal{L}(\phi, \partial\phi)\right]}. \tag{65}$$