**Lecture 3.**

**Plan.**



## 1. Green's functions in complex time and euclidean QFT.

*1.1. FEYNMAN PROPAGATOR, ANALYTIC PROPERTIES OF GREEN'S FUNCTIONS IN COMPLEX TIME.*

In the previous lecture we found that vacuum expectations of KG fields can be represented as certain limits of a single analytic function $D_c(\vec{x}, t)$ of a complex time variable $t$. As an example we found that Feynman propagator $D_F(\vec{x}, t)$ is the function $D_c(\vec{x}, t)$ when the time is going along the countour $C_F$ which goes **above** the real $t$-axis in the region $Ret < -|\vec{x}|$, crosses the segment $-|\vec{x}| < t < -|\vec{x}|$ and goes **below** the real $t$-axis in the region $Ret > |\vec{x}|$.

One can define a new function which is called **euclidean, or imaginary-time** expectation value. Another name for this quantity is **Correlation Function**. Consider the function $D_c(\vec{x}, t)$ for the time running along contour $C_E$ which is the imaginary axis in comlex $t$ plane going from $+\infty$ to $-\infty$. The imaginary time gives euclidean interval

$$t = \imath x_4 \Rightarrow -d\tau^2 \rightarrow ds^2 = dx_4^2 + d\vec{x}^2. \tag{1}$$

Thus, instead of Lorentz group we get $O(4)$ group of rotations in euclidean space $(\vec{x}, x_4)$.

Let us introduce the notation

$$D(x_E) = D(\vec{x}, x_4) = D_c(\vec{x}, -\imath x_4) \tag{2}$$

and call this function as **euclidean correlation function**.

It is clear that

$$(m^2 - \frac{\partial^2}{\partial x_4^2} - (\nabla)^2) D(x_E) = \delta(x_4)\delta(\vec{x}) \Leftrightarrow$$
$$(m^2 - \Delta_E) D(x_E) = \delta^4(x_E) \tag{3}$$

The solution of this equation is given by

$$D(x_E) = \int \frac{d^4 p_E}{(2\pi)^4} \frac{\exp{(\imath p_E x_E)}}{p_E^2 + m^2} \tag{4}$$

where $p_E = (\vec{p}, p_4)$ and $p_E^2 = \vec{p}^2 + p_4^2$. (In this euclidean version of Feynman propagator the poles are at $\pm\imath\omega_{\vec{p}}$, so the contour over $p_4$ can be closed from above if $x_4 > 0$ or it is closed from below if $x_4 < 0$.)

The form (4) is similar to covariant representation for the Feynman propagator. Indeed, if one introduces real variable $\omega$ as $p_4 = \imath\omega$, then this integral takes the form

$$-\imath \int_{\tilde{C}_E} \frac{d\omega}{2\pi} \int \frac{d^3p_E}{(2\pi)^3} \frac{\exp\left(-\omega x_4 + \imath\vec{p}\vec{x}\right)}{-\omega^2 + \vec{p}^2 + m^2} \tag{5}$$

where the $\omega$-integration is going along the imaginary $p_4$-axis (contour $\tilde{C}_E$) from $-\infty$ to $+\infty$. Thus, it relates to the Fourier transform of $D_F$ by $90°$ rotation of $\tilde{C}_F$.

Another form for this solution is given by introducing auxilliary integration:

$$D(x_E) = \int_0^\infty d\tau \int \frac{d^4p_E}{(2\pi)^4} \exp\left(-\tau(p_E^2 + m^2)\right) \exp\left(\imath p_E x_E\right) \tag{6}$$

This form is known as **Schwinger's proper time representation**.

### *1.2. REAL TIME QFT/ IMAGINARY TIME QFT RELATION.*

That was an **example of relation between euclidean (or imaginary time) correlation function, and real time (Minkowski) Green's function**. In other words, the analytic function $D_c(\vec{x}, t)$ is an analytic continuation of $D(x_E)$ for the complex values $x_4 = \imath t$. We have seen that all interesting characteristics of our QKG in real time, such as $D_F$ and commutator of fields, are expressed in terms of appropriate limiting values of $D_c$.

### *1.3. QFT/STATISTICAL MECHANICS RELATION.*

In a more general case the relation between the euclidean QFT (QFT in imaginary time) and real-time QFT is the following: **the time-ordered**

**expectation value in Minkowski space are obtained by analytic continuation from the euclidean space correlation functions**.

The advantages of euclidean point of view:

• one can apply the path integral method to quantize FT and the path integral (functional integral) is easier to define and to handle in imaginary time formulation.

• in the euclidean formulation QFT shows deep connection to Statistical Machanics.

## 2. Path integral in quantum and statistical mechanics.

### *2.1. TRANSITION AMPLITUDE IN QUANTUM MECHANICS.*

In quantum mechnics we are interested in calculation of **matrix element of evolution operator between two states**:

$$< f|\exp{(-\frac{\imath}{\hbar}\hat{H}t)}|i >= \sum_n < f|n >< n|i > \exp{(-\frac{\imath}{\hbar}E_n t)} \qquad (7)$$

where $|n >$ and $E_n$ are stationary states and associated energies. Due to our main postulate that $\hat{H}$ **is bounded from below** and allowing $t$ to be complex we see that **the above sum defines an analytic function in the lower half-plane** $Imt < 0$. Then the **real-time matrix elements are limiting values of this analytic function**.

### *2.2. IMAGINARY TIME TRANSITION AMPLITUDE.*

It make sense therefore to take $t = -\imath\tau$ and evaluate imaginary-time transition amplitude

$$< f|\exp{(-\frac{1}{\hbar}\hat{H}\tau)}|i >, \ \tau > 0 \qquad (8)$$

and then analytically continue the result to real-time value.

### *2.3. PATH INTEGRAL CALCULATION OF IMAGINARY TIME*

*TRANSITION AMPLITUDE.*

Consider a quantum mechanical system described by the Hamiltonian $\hat{H}(\hat{p}, \hat{q})$. By the definition, the (imaginary-time) evolution operator $\hat{U}(\tau, \tau_0)$ satisfies the equation

$$-\frac{\partial}{\partial \tau}\hat{U}(\tau, \tau_0) = \hat{H}\hat{U}(\tau, \tau_0), \; \hat{U}(\tau_0, \tau_0) = 1. \tag{9}$$

**Notice that $\hat{U}(\tau, \tau_0)$ does not exists for $\tau < \tau_0$ because the energy spectrum is bounded from below but not from above**.

It is easy to check that

$$\hat{U}(\tau, \tau_1)\hat{U}(\tau_1, \tau_0) = \hat{U}(\tau, \tau_0), \; \tau \geq \tau_1 \geq \tau_0 \tag{10}$$

Therefore

$$\hat{U}(\tau, \tau_0) = \hat{U}(\tau_n, \tau_{n-1})\hat{U}(\tau_{n-1}, \tau_{n-2})...\hat{U}(\tau_1, \tau_0) \tag{11}$$

where $\tau_k = \tau_0 + k\Delta$, $\Delta = \frac{\tau - \tau_0}{n}$ and $\tau_n = \tau$.

Suppose $\hat{H}$ **is local in the basis of states which diagonalize** $\hat{q}$ (this means that $< q_f|\hat{H}|q_i >$ have a support at $q_f = q_i$) the composition property above allows to construct a **path integral representation** for $< q_f|\hat{U}(\tau, \tau_0)|q_i >$:

$$< q_f|\hat{U}(\tau, \tau_0)|q_i > = \int \prod_{k=1}^{n-1} dq_k \prod_{k=1}^{n} < q_k|U(\tau_k, \tau_{k-1})|q_{k-1} >, \tag{12}$$

where $q_0 = q_i$, $q_n = q_f$. Taking the large $n$, small $\Delta$ limit the problem reduces to

$$< q|\hat{U}(\tau + \Delta, \tau)|q' > \tag{13}$$

calculation. Because of the locality of $\hat{H}$ only the matrix elements with small $|q - q'|$ will contribute in this limit.

*2.4. HEAT EQUATION AND ITS SOLUTION.*

Let us consider the following local Hamiltonian

$$\hat{H} = \frac{p^2}{2} + V(q). \tag{14}$$

Then the equation (9) takes the form of heat equation

$$-\frac{\partial}{\partial\tau} < q|\hat{U}(\tau,\tau_0)|q' >= (-\frac{1}{2}\frac{\partial^2}{\partial q^2} + V(q)) < q|\hat{U}(\tau,\tau_0)|q' >,$$
$$where \ < q|\hat{U}(\tau_0,\tau_0)|q' >= \delta(q-q'). \tag{15}$$

Indeed

$$\frac{\partial}{\partial\tau} < q_f|\hat{U}(\tau,\tau_0)|q_i >=$$
$$lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}[\int dq(\tau) < q_f|\hat{U}(\tau+\Delta\tau,\tau)|q(\tau) >< q(\tau)|\hat{U}(\tau,\tau_0)|q_i > -$$
$$< q_f|\hat{U}(\tau,\tau_0)|q_i >] =$$
$$lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\int dq(\tau)[< q_f|(1-\Delta\tau\hat{H})|q(\tau) >< q(\tau)|\hat{U}(\tau,\tau_0)|q_i > -$$
$$< q_f|\hat{U}(\tau,\tau_0)|q_i >] =$$
$$-lim_{\Delta\tau\to 0}\int dq(\tau) < q_f|(\frac{p^2}{2} + V(q))|q(\tau) >< q(\tau)|\hat{U}(\tau,\tau_0)|q_i >=$$
$$-\int dq(\tau)(-\frac{1}{2}\frac{\partial^2}{\partial q(\tau)^2} + V(q(\tau))) < q_f|q(\tau) >< q(\tau)|\hat{U}(\tau,\tau_0)|q_i >=$$
$$(\frac{1}{2}\frac{\partial^2}{\partial q_f^2} - V(q_f)) < q_f|\hat{U}(\tau,\tau_0)|q_i > . \tag{16}$$

Let us find first $V = 0$ **solution** for (15):

$$< q|\hat{U}(\tau_0+\Delta,\tau_0)|q' >= \frac{1}{\sqrt{2\pi\Delta}}\exp(-\frac{(q-q')^2}{2\Delta}). \tag{17}$$

For $V \neq 0$ we can take the solution in the form

$$\frac{1}{\sqrt{2\pi\Delta}}\exp(-\frac{(q-q')^2}{2\Delta} - \Delta\sigma(q,q') + O(\Delta^2)). \tag{18}$$

Substituting this into (15) we find

$$\sigma(q,q') = \frac{1}{q-q'}\int_{q'}^{q} V(y)dy. \tag{19}$$

We see that leading term does not depend on $V$, while $\sigma$ is given by the potential overaged over the interval $[q', q]$. One can easy to see also that $< q|\hat{U}(\tau_0,\tau_0)|q' >= \delta(q-q')$.

The function $< q|\hat{U}(\tau_0+\Delta,\tau_0)|q' >$ is sharply peaked around $|q-q'| = 0$, with the width $|q-q'| \approx \Delta^{\frac{1}{2}} = |\tau-\tau_0|^{\frac{1}{2}}$ which is typical for the **Brownian motion**.

*2.5. IMAGINARY TIME ACTION.*

If the potential is differentiable function we can write

$$\sigma(q,q') = \frac{1}{2}(V(q)+V(q')) + .... \tag{20}$$

Therefore (12) takes the form

$$< q_f|\hat{U}(\tau_f,\tau_i)|q_i >= \lim_{n\to\infty}(\frac{1}{2\pi\Delta})^{\frac{n}{2}}\int(\prod_{k=1}^{n-1}dq_k)\exp{(-A(q_k))}$$
$$where$$
$$A(q_k) = \sum_{k=1}[\frac{(q_k-q_{k-1})^2}{2\Delta} + \Delta\frac{V(q_k)+V(q_{k-1})}{2}],$$
$$\Delta = \frac{\tau_f-\tau_i}{n},\ \tau_k = \tau_i + k\Delta,\ q_0 = q_i,\ q_n = q_f. \tag{21}$$

**The integration here can be understood as going over the piece-wise linear trajectories $q(\tau)$ going from $q_i$ to $q_f$.** In the limit $\Delta \to 0$, $n \to \infty$ one can write the function in the exponential as

$$A[q(\tau)] = \int_{\tau_i}^{\tau_f} d\tau[\frac{1}{2}(\frac{dq}{d\tau})^2 + V(q)]. \tag{22}$$

That is $A[q(\tau)]$ is an **imaginary-time action**. It is also possible in this limit to rewrite the $n$-fold integral as

$$< q_f|\hat{U}(\tau_f,\tau_i)|q_i >= \int_{q(\tau_i)}^{q(\tau_f)} [Dq(\tau)] \exp\left(-A[q(\tau)]\right) \tag{23}$$

and call it **imaginary-time path integral**, where

$$\exp\left(\frac{\imath}{\hbar} S[q(t)]\right) \to \exp\left(-A[q(\tau)]\right) \tag{24}$$

because in r.h.s. we integrate over all continous paths going from $q_i$ to $q_f$.

### *2.6. PATH INTEGRAL AS A SUPERPOSITION PRINCIPLE.*

The path integral representation is most explicit expression for the superposition principle in quantum mechanics, which states that **the transition amplitude is a superposition of transition amplitudes associated to each possible way to go from initial state to final state**.

### *2.7. FRACTAL PATHS GIVE MAIN CONTRIBUTION.*

One can see that in $n \to \infty$ limit the absolute majority of paths entering the integration are not smooth curves. This is in agreement with the fact that for $|\tau_1 - \tau_2| \to 0$, $|q(\tau_1) - q(\tau_2)| \approx \sqrt{|\tau_1 - \tau_2|}$, not $|\tau_1 - \tau_2|$ as one would expect for differential curves. So the typical path entering the integral is an example of geometrical object know as **fractal**.

From this point of view the 2 terms in euclidean action play different role. The kinetic term selects the class of path entering the integral, namely **it selects those path for which** $\frac{(q(\tau+\Delta)-q(\tau))^2}{\Delta}$ **remains finite** as $\Delta \to 0$. Hence the factor

$$\exp\left(-\int (\frac{dq}{d\tau})^2 d\tau\right) \tag{25}$$

shoud be considered as a part of functional mesure. The factor

$$\exp\,(-\int V(q(\tau))d\tau) \tag{26}$$

**weights the path according to the average potential energy**.

*2.8. VACUUM ENERGY AND PATH INTEGRAL PREFACTOR.*

The symbol

$$[Dq(\tau)] = \lim_{n\to\infty}(\frac{1}{2\pi\Delta})^{\frac{n}{2}}\prod_{k=1}^{n-1} dq(\tau_k) \tag{27}$$

entering the path integral (23) contains an infinite factor

$$(\frac{1}{2\pi\Delta})^{\frac{n}{2}} \approx \exp\,(-(\frac{\log 2\pi\Delta}{2\Delta})T) \tag{28}$$

where $T = \tau_f - \tau_i = n\Delta$ is a volume of the system in imaginary time. **This is similar to the infinity we had for the vacuum energy in the KG theory**. Thus absorbing it into $[Dq]$ is analogous to subtracting $E_0$ from $\hat{H}$. One can get rid of this factor considering the ratio of the given path integral to some reference path integral (when $V = 0$ for example).

*2.9. PATH INTEGRAL IN REAL TIME AND LAGRANGIAN APPROACH TO QM.*

One can repeat the above calculations considering real time $t$ instead of imaginary time $\tau$. This way we arrive at

$$< q_f|\exp\,(-\frac{\imath}{\hbar}\hat{H}T)|q_i >= \int_{q_i}^{q_f}[Dq(t)]\exp\,(\frac{\imath}{\hbar}S[q(t)]). \tag{29}$$

Because of the action functional is given by integral of a Lagrangian we come back thereby to the **Lagrangian approach to quantum mechanics**.

*2.10. MANY DEGREES OF FREEDOM GENERALIZATION.*

In case we have system with many degrees of freedom $q_\mu$ and the Hamiltonian

$$H = \sum_\mu (\frac{1}{2} p_\mu^2 + V(q_\mu)) \tag{30}$$

the path integral expression can be generalized starightforwardly

$$\int [Dq_\mu] \exp\left(-A[q_\mu(\tau)]\right) \tag{31}$$

Also the path integral in the phase space may appear for more complicated Hamiltonians.

**QFT can be treated as QM with infinite number of degrees of freedom**. Thus, in QFT the path integral becomes functional integral where we integrate over all possible field configurations in space-time (or in euclidean spase) weighted by the action. From this point of view **the functional integral can be considered as a Lagrangian approach to QFT**.

## 3. Feynman propagator as a path integral.

Now we represent the Feynman propagator $D_F$ in terms of path integral. More precisely, we want to represent the function $D_F(x_f - x_i)$ as a **relativistic particle transtion amplitude from the point $x_i$ to the point $x_f$** using the path integral.

*3.1. RELATIVISTIC PARTICLE ACTION.*

In classical mechanics the relativistic particle action going from the point $x_i$ to the point $x_f$ is

$$
\begin{aligned}
S &= -m\int_{x_i}^{x_f}\sqrt{dx_\mu dx^\mu}, \\
dx_\mu dx^\mu &= dt^2 - d\vec{x}^2 = (1-(\frac{d\vec{x}}{dt})^2)dt^2 \Rightarrow \\
S &= -m\int_{t_i}^{t_f}\sqrt{(1-(\frac{d\vec{x}}{dt})^2)}dt \qquad (32)
\end{aligned}
$$

The transition amplitude is given by

$$\int_{path(i\to f)}[D\vec{x}(t)]\exp{(\imath\hbar S)} \qquad (33)$$

**The problem 1** whith this expression as that the $dx_\mu dx^\mu$ in $S$ can be positive or negative depending on whether $dx^\mu$ is time like or space like. This causes the problem of choosing the right branch of the square root. But we can not exclude the paths with space like $dx^\mu$ because $D_F$ does not vanish outside the light cone, thought it is exponentially small there.

**The problem 2** is that the relativistic invariance demands admitting also the paths going backward in time once we admit space like $dx^\mu$ (as one can check). For such paths the $\vec{x}(t)$ is not a function of $t$, so the integral over $[D\vec{x}(t)]$ does not make sense.

*3.2. EUCLIDEAN FORMULATION.*

These **problems can be solved going to euclidean (imaginary time) picture**:

$$
\begin{aligned}
t &\to -\imath x_4,\ x \to x_E \\
-\frac{\imath}{\hbar}S &\to A = m\int\sqrt{dx_E^2} \\
dx_E^2 &= dx_4^2 + d\vec{x}^2 \geq 0. \qquad (34)
\end{aligned}
$$

So one needs to compute

$$\int_{path(i \to f)} [Dx_E] \exp(-A) \tag{35}$$

and then analytically continue the result to the real time. Note that

$$A[path] = m_0 L[path], \tag{36}$$

where $L[path]$ **is the euclidean length of given path and we replaced** $m$ **by** $m_0$.

*3.3. EUCLIDEAN PATH INTEGRAL CALCULATION.*

**We define the paths integral as a limit of finite dimensional integral**. Namely, for a given large integer $N$ we consider a piecewise-linear paths from the point 0 to the point $x_E$, with each linear piece having the lenght $\Delta$. The lenght of such path is

$$L_N = N\Delta. \tag{37}$$

Thus, we use discrete approximation considering a lattice with the spacing $\Delta$ instead of continuous euclidean space.

Let $n_k$ be the unit vector in the direction of k-th piece. The discret approximation of the path integral is given by

$$\sum_{N=0}^{\infty} \exp(-m_0 N\Delta) \int (\prod_{k=1}^{N} dn_k) \delta^4(\Delta \sum_k n_k - x_E), \tag{38}$$

where $dn_k$ is the standard mesure on $S^3$ and $\delta$-function extracts the vectors $\Delta n_k$ to add up to $x_E$. We also have taken into account the paths with arbitrary lengths by the summation over $N$. **In this discrete approximation of the path integral the continuous limit is achieved when** $\Delta \to 0,\ N \to \infty$**. Our aim is to determine this limit**

First of all we have to look what kind of paths give the main contribution to the integral in the continuous limit. We have

$$\delta^4(\Delta \sum_k n_k - x_E) = \int \frac{d^4P}{(2\pi)^4} \exp\left(\imath P \cdot (x_E - \Delta \sum_k n_k)\right). \tag{39}$$

Then

$$\prod_{k=1}^{N} \int dn_k \exp\left(-\imath \Delta n_k \cdot P\right) = (I(P\Delta))^N. \tag{40}$$

The integral $I$ can be evaluated by the Bessel functions but its explicit form does not matter. Instead, its behaviour at large $N$ and small $\Delta$ matter

$$I \approx 4\pi^2 (1 - \frac{1}{8} P^2 \Delta^2 + ...),$$
$$(I)^N \approx (4\pi^2)^N \exp\left(-\frac{\Delta^2 N}{8} P^2\right). \tag{41}$$

Now we can calculate the (Gaussian) integral over $P$ for given $N$

$$\int \frac{d^4P}{(2\pi)^4} \exp\left(\imath P x_E - \frac{\Delta^2 N}{8} P^2\right) \approx \exp\left(-\frac{2x_E^2}{\Delta^2 N}\right). \tag{42}$$

Hence, the result for (38) is

$$\sum_{N=0}^{\infty} (4\pi^2)^N \exp\left(-m_0 N \Delta\right) \exp\left(-\frac{2x_E^2}{\Delta^2 N}\right). \tag{43}$$

**Notice here that $\Delta$ enters in the combination $\Delta^2 N$ not as $\Delta N = L$,** as one would expect. By this reason taking the limit

$$\Delta \to 0, \; N \to \infty, \; \Delta N = finite \tag{44}$$

we find that **integral (42) is equal zero for any finite** $x_E$.

*3.4. THE CORRECT LIMIT AND FRACTAL GEOMETRY OF PATHS.*

The above calculation is nearly identical to the probability calculation for Brownian particle. We know that Brownian particle with typical microscopic velocity $v$ travels the distance

$$\bar{x} = v\sqrt{T\Delta t} \tag{45}$$

in time $T$, not $vT$, where $\Delta t$ is typical intercollision time. In this analogy

$$\Delta \approx v\Delta t,\ N \approx \frac{T}{\Delta t} \Rightarrow \bar{x} = \frac{\Delta}{\Delta t}\sqrt{N(\Delta t)^2} = \Delta\sqrt{N}. \tag{46}$$

Therefore, the correct continous limit is achieved by taking

$$\Delta \to 0,\ N \to \infty,\ s \equiv \frac{1}{8}\Delta^2 N = finite \tag{47}$$

so that $s$ **is the parameter characterizing geometry of typical path**. The microsopic length $L = \Delta N$ becomes infinite in this limit so the typical path dominating in this limit becomes **fractal** so that

$$s \approx [length = \Delta\sqrt{N}]^2, \tag{48}$$

which means that **typical path has a fractal dimension 2**.

*3.5. MASS RENORMALIZATION.*

There is also a factor

$$(4\pi^2)^N \exp\left(-m_0 N\Delta\right) = \exp\left(-N(m_0\Delta - \log(4\pi^2))\right). \tag{49}$$

It makes sense to consider **the parameter $m_0$ as depending on $\Delta$**:

$$m_0 = m_0(\Delta) \tag{50}$$

so that it can be adjusted as

$$\Delta m_0(\Delta) - \log(4\pi^2) \to \frac{\Delta^2}{8} m^2, \; as \; \Delta \to 0 \tag{51}$$

Then the above 2 factors become

$$(4\pi^2)^N \exp\left(-m_0 N\Delta\right) = \exp\left(-m^2 \frac{\Delta^2 N}{8}\right) \to \exp\left(-m^2 s\right). \tag{52}$$

*3.6. FEYNMAN PROPAGATOR.*

In the limit $N \to \infty$ we can replace the sum over $N$ by the integral over $s$ and obtain

$$\int_0^\infty ds \exp\left(-m^2 s\right) \int \frac{d^4 P}{(2\pi)^4} \exp\left(-sP^2\right) \exp\left(\imath P x_E\right), \tag{53}$$

which **is exactly the correlation function** $D(x_E)$ from (6). Taking the analytic continuation to the real time we obtain path integral representation for Feynman propagator $D_F$.