Lecture 2.

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1. KG field quantization in Hamiltonian formalism (reminder).

1.1. Conserved charges and algebra of creation-annihilation operators.

The action of KG field is invariant under some special symmetry:

$$\phi(x) \to \tilde{\phi}(x) = \phi(x) + f(x) \tag{1}$$

where the function $f(x)$ is an arbitrary solution of KG equation. Due to Noether theorem this symmetry allows to conclude that

$$\partial_\mu J^\mu_f = 0,\ J^\mu_f = f\partial^\mu\phi - \phi\partial^\mu f \tag{2}$$

So the corresponding conserved charges (integrals of motion) are given by

$$A_f = \int d^3x(\dot{\phi} f - \dot{f}\phi) = \int d^3x(\pi f - \dot{f}\phi) \tag{3}$$

From the last formula we calculate the Poisson brackets

$$\{A_f, A_g\} = \int d^3x(\dot{f} g - f\dot{g}) \tag{4}$$

Using the plane waves basis of solutions of KG equations

$$\begin{aligned} A_{\vec{p}} \equiv A_{f_{\vec{p}}},\ \ f_{\vec{p}} &= \exp\left(\imath(\omega_{\vec{p}} t - \vec{p}\vec{x})\right), \\ A^*_{\vec{p}} \equiv A_{f^*_{\vec{p}}},\ \ f^*_{\vec{p}} &= \exp\left(-\imath(\omega_{\vec{p}} t - \vec{p}\vec{x})\right), \\ \omega_{\vec{p}} &= \sqrt{\vec{p}^2 + m^2} \end{aligned} \tag{5}$$

we find the following algebra

$$\begin{aligned} \{A_{\vec{p}}, A^*_{\vec{p}'}\} &= \imath(2\pi)^3 2\omega_{\vec{p}}\delta(\vec{p} - \vec{p}'), \\ \{A_{\vec{p}}, A_{\vec{p}'}\} &= \{A^*_{\vec{p}}, A^*_{\vec{p}'}\} = 0 \end{aligned} \tag{6}$$

Thus, in quantum theory we must postulate

$$\begin{aligned} A_{\vec{p}} \to \hat{A}_{\vec{p}},\ \ A^\dagger_{\vec{p}} &\to \hat{A}^\dagger_{\vec{p}} \\ [\hat{A}_{\vec{p}}, \hat{A}^\dagger_{\vec{p}'}] &= (2\pi)^3 2\omega_{\vec{p}}\delta(\vec{p} - \vec{p}'), \\ [\hat{A}_{\vec{p}}, \hat{A}_{\vec{p}'}] &= [\hat{A}^\dagger_{\vec{p}}, \hat{A}^\dagger_{\vec{p}'}] = 0 \end{aligned} \tag{7}$$

Renormalizing the operators above:

$$\hat{A}_{\vec{p}} = \sqrt{2\omega_{\vec{p}}} a_{\vec{p}},\ \ \hat{A}^\dagger_{\vec{p}} = \sqrt{2\omega_{\vec{p}}} a^\dagger_{\vec{p}}. \tag{8}$$

we find the Heisenberg algebra they satisfy

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3\delta(\vec{p} - \vec{p}') \tag{9}$$

1.2. Hamiltonian and momentum operators.

Then the KG Hamiltonian takes the form

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\vec{p}}}{2} (a^\dagger_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} a^\dagger_{\vec{p}}). \tag{10}$$

The KG field momentum operator is given by

$$\hat{\vec{P}} = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{2} (a^\dagger_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} a^\dagger_{\vec{p}}) \tag{11}$$

The operators $a_{\vec{p}}, a^\dagger_{\vec{p}}$ diagonalize the KG Hamiltonian:

$$[\hat{H}, a^\dagger_{\vec{p}}] = \omega_{\vec{p}} a^\dagger_{\vec{p}},\ \ [\hat{H}, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}} \tag{12}$$

and we have also

$$[\hat{P}, a_{\vec{p}}] = -\vec{p} a_{\vec{p}},\ \ [\hat{P}, a^*_{\vec{p}}] = \vec{p} a^*_{\vec{p}}. \tag{13}$$

1.3. Ground state postulate and space of states.

We constructed the space of states of KG theory demanding that **the energy spectrum is bounded from below**: it was postulated that there is a state $|0>$ with minimal energy E_0 such that

$$a_{\vec{p}}|0>= 0\ for\ all\ \vec{p} \tag{14}$$

Then the space of states of KG theory $\mathcal{H}$ is spanned by the vectors

$$\begin{aligned} |\vec{p}_1, ..., \vec{p}_N> &= a^\dagger_{\vec{p}_1} ... a^\dagger_{\vec{p}_N}|0>, \\ \hat{H}|\vec{p}_1, ..., \vec{p}_N> &= (\omega_{\vec{p}_1} + ... + \omega_{\vec{p}_N} + E_0)|\vec{p}_1, ..., \vec{p}_N> \\ \hat{P}|\vec{p}_1, ..., \vec{p}_N> &= (\vec{p}_1 + ... + \vec{p}_N + \vec{P}_0)|\vec{p}_1, ..., \vec{p}_N> . \end{aligned} \tag{15}$$

1.4. Particle interpretation.

We noticed that infinite value of vacuum energy could be ignored as long as the difference between the energy of a given state and vacuum energy matters. Therefore we redefined $\hat{H}$ by subtracting E_0:

$$: \hat{H} := \hat{H} - E_0 = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}} \tag{16}$$

and do similar subtraction for the momentum operator (though the $P_0 = 0$)

$$: \hat{\vec{P}} := \int \frac{d^3p}{(2\pi)^3} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}} \tag{17}$$

It allowed us to interprate the operator $a^\dagger_{\vec{p}}$ as **creating a particle** with the energy $\omega_{\vec{p}}$ and momentum $\vec{p}$ so that the vector $|\vec{p}_1, ..., \vec{p}_N >= a^\dagger_{\vec{p}_1} ... a^\dagger_{\vec{p}_N} |0>$ is an N-particles state with the momenta $\vec{p}_1, ..., \vec{p}_N$ because of the correct relation $\omega_{\vec{p}_i} = \sqrt{\vec{p}_i^2 + m^2}$ between the momentum and energy of each particle.

2. Heisenberg picture in KG theory.

One can rewrite the relations (5), (8) in opposit direction:

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_{\vec{p}} + a^\dagger_{-\vec{p}}) \exp(\imath \vec{p}\vec{x})$$
$$\hat{\pi}(\vec{x}) = -\imath \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} (a_{\vec{p}} - a^\dagger_{-\vec{p}}) \exp(\imath \vec{p}\vec{x}) \tag{18}$$

and deduce the following:

$$\frac{\partial \hat{\phi}}{\partial \vec{x}} = -\imath [\hat{\vec{P}}, \hat{\phi}], \ \frac{\partial \hat{\pi}}{\partial \vec{x}} = -\imath [\hat{\vec{P}}, \hat{\pi}] \tag{19}$$

These relations are the quantization of the qlassical statements that **the momentum operator generate translations in space**. Thus we get translation invariance statement for the KG theory along the space directions.

In Heisenberg picture **the operators evaluate in time** while the wave functions do not depend on time. The time dependent Heisenberg operators are related to the time independent operators in Schr$\ddot{o}$dinger picture as

$$\hat{O}(t) = \exp(\imath \hat{H} t)\hat{O}\exp(-\imath \hat{H} t) \tag{20}$$

Hence, the Heisenberg field and canonical momentum operators are given by

$$\begin{aligned}\hat{\phi}(\vec{x},t) &= \exp(\imath \hat{H} t)\hat{\phi}(\vec{x})\exp(-\imath \hat{H} t)\\ \hat{\pi}(\vec{x},t) &= \exp(\imath \hat{H} t)\hat{\pi}(\vec{x})\exp(-\imath \hat{H} t)\end{aligned} \tag{21}$$

where $\hat{H}$ is KG Hamiltonian (10). Using the explicit form of $\hat{H}$ we find from (18)

$$\begin{aligned}\hat{\phi}(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_{\vec{p}}\exp(\imath\vec{p}\vec{x} - \imath\omega_{\vec{p}}t) + a^{\dagger}_{\vec{p}}\exp(-\imath\vec{p}\vec{x} + \imath\omega_{\vec{p}}t)))\\ \hat{\pi}(\vec{x},t) &= -\imath\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(\omega_{\vec{p}}a_{\vec{p}}\exp(\imath\vec{p}\vec{x} - \imath\omega_{\vec{p}}t) - \omega_{\vec{p}}a^{\dagger}_{\vec{p}}\exp(-\imath\vec{p}\vec{x} + \imath\omega_{\vec{p}}t))\end{aligned} \tag{22}$$

2.1. Heisenberg equations of motion for KG field.

The Heisenberg operators satisfy Heisenberg equations of motion

$$\imath\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}] \tag{23}$$

In the case of KG field we find

$$\frac{\partial}{\partial t}\hat{\phi} = \hat{\pi},\ \frac{\partial}{\partial t}\hat{\pi} = (\nabla^2 - m^2)\hat{\phi} \tag{24}$$

that is the Heisenberg field $\hat{\phi}(\vec{x},t)$ satisfy the KG equation. This is a linear equation so that the general solution is given by the first line from (22). The creation-anihilation operators appear in this solution as the integration constants.

2.2. Relativistics invariance.

Similar to

$$\frac{\partial \hat{\phi}}{\partial t} = \imath[\hat{H}, \hat{\phi}] \tag{25}$$

one can check also

$$\frac{\partial \hat{\phi}}{\partial \vec{x}} = -\imath[\hat{\vec{P}}, \hat{\phi}] \tag{26}$$

They can be combined into the covariant form

$$\frac{\partial \hat{\phi}}{\partial x^\mu} = \imath[\hat{P}^\mu, \hat{\phi}], \ \mu = 0, ..., 3 \tag{27}$$

where $\hat{P}^0 = \hat{H}$ and $\hat{P}^i = -\vec{P}_i$, $i = 1, ..., 3$. The eq. (27) express translational invariance of the theory. Similar relations for the KG field orbital momentum operator $\hat{M}^{\mu\nu}$ (which can be found from Noether theorem)

$$\hat{M}^{\mu\nu} = \int d^3x (x^\mu \hat{T}^{0\nu} - x^\nu \hat{T}^{0\mu}) \tag{28}$$

expresses the Lorentz invariance of the theory:

$$x^\mu \frac{\partial \hat{\phi}}{\partial x_\nu} - x^\nu \frac{\partial \hat{\phi}}{\partial x_\mu} = \imath[\hat{M}^{\mu\nu}, \hat{\phi}(x)] \tag{29}$$

As an exercise one can check the Poincare algebra of commutators

$$\begin{gathered} [\hat{P}^\mu, \hat{P}^\nu] = 0, [\hat{M}^{\mu\nu}, \hat{P}^\lambda] = \imath(g^{\lambda\mu}\hat{P}^\nu - g^{\lambda\nu}\hat{P}^\mu), \\ [\hat{M}^{\mu\nu}, \hat{M}^{\lambda\rho}] = \imath(g^{\mu\lambda}\hat{M}^{\nu\rho} - g^{\nu\lambda}\hat{M}^{\mu\rho} + g^{\nu\rho}\hat{M}^{\mu\lambda} - g^{\mu\rho}\hat{M}^{\nu\lambda}) \end{gathered} \tag{30}$$

The relations (27), (29), (30) all together expresses the relativistics invariance of the theory.

3. KG in space-time, analitical properties of Green's functions, causality in QFT, Feynman propagator.

3.1. $[\phi(x), \phi(y)]$ calculation and $D_\pm$ functions.

Let us calculate the commutator

$$[\hat{\phi}(x), \hat{\phi}(y)] \tag{31}$$

using the expression (22). To do that it is helpfull first to use the decomposition

$$\begin{aligned} \hat{\phi}(x) &= \hat{\phi}_-(x) + \hat{\phi}_+(x),\ \ where \\ \hat{\phi}_-(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}} a_{\vec{p}} \exp\left(\imath\vec{p}\vec{x} - \imath\omega_{\vec{p}}t\right) \\ \hat{\phi}_+(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}} a^\dagger_{\vec{p}} \exp\left(-\imath\vec{p}\vec{x} + \imath\omega_{\vec{p}}t))\right) \end{aligned} \tag{32}$$

$\hat{\phi}_+(x)$ **can be interpreted as an operator creating a particle at a space-time point** x**, while** $\hat{\phi}_-(x)$ **absorbs a particle at that point**.

We can write

$$\begin{aligned} [\hat{\phi}(x), \hat{\phi}(y)] &= D_-(x-y) - D_+(x-y) \\ & \qquad\qquad\qquad\qquad where \\ D_-(x-y) &= [\hat{\phi}_-(x), \hat{\phi}_+(y)], \\ D_+(x-y) &= [\hat{\phi}_-(y), \hat{\phi}_+(x)] = D_-(y-x) \end{aligned} \tag{33}$$

Explicitly, one finds

$$\begin{aligned} D_-(x) = D_-(\vec{x}, t) &= \int d\mu(\vec{p}) \exp\left(-\imath\omega_p t + \imath\vec{p}\vec{x}\right) \\ D_+(x) = D_+(\vec{x}, t) &= \int d\mu(\vec{p}) \exp\left(\imath\omega_p t - \imath\vec{p}\vec{x}\right) \end{aligned} \tag{34}$$

where the notation

$$d\mu(\vec{p}) = \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p} \tag{35}$$

has been used for the invariant volume form $\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}$.

3.2. Properties of $D_{\pm}(x)$.

1. First af all $D_{\pm}(x)$ are Lorentz invariant functions:

$$D_{\pm}(Rx) = D_{\pm}(x) \tag{36}$$

where $(Rx)^{\mu} = R^{\mu}_{\nu} x^{\nu}$ and R is a Lorentz group element that can be continously connected to the identity element.

Let us consider for example

$$D_{-}(x) = \int_{p^2=m^2} d\mu(p) \exp \imath p_{\mu} x^{\mu} \tag{37}$$

The transformation $x^{\mu} \rightarrow R^{\mu}_{\nu} x^{\nu}$ is compensated by the transformation $p_{\mu} \rightarrow R^{\lambda}_{\mu} p_{\lambda}$ which leaves the mass shell condition $p^2 = m^2$ and the mesure $d\mu(\vec{p})$ invariant.

It means in particular that

$$D_{+}(x) = D_{-}(x) \; if \; x \; is \; spacelike: \; x^2 = t^2 - \vec{x}^2 < 0 \tag{38}$$

because in this case $-x$ can be continously transformed to x. (One needs first to chose the frame where $t = 0$ and transform then $(0, -\vec{x})$ to $(0, \vec{x})$ by a rotation.)

2. Consider $D_{\vec{x},t}$ as a function of **complex** variable t. From the definition

$$D_{-}(x) = \int_{p^2=m^2} d\mu(p) \exp\left(-\imath \omega_{\vec{p}} t + \imath \vec{p}\vec{x}\right) \tag{39}$$

we see that integral **converges absolutely as long as** $Imt < 0$ (because $\omega_{\vec{p}} > 0$). Therefore this integral defines a function $D_c(\vec{x}, t)$ of complex variable t which is analytic in the lower half-plane. For real t the value of $D_{-}(\vec{x}, t)$ is obtained as the limiting value of this analytic function from $Imt < 0$:

$$D_{-}(\vec{x}, t) = D_c(\vec{x}, t - \imath 0) \tag{40}$$

Analogously, the integral for $D_+(x)$:

$$D_+(x) = \int_{p^2=m^2} d\mu(p) \exp\left(\imath\omega_{\vec{p}} t - \imath\vec{p}\vec{x}\right) \tag{41}$$

converges absolutely as long as $Imt > 0$ and defines a function $\tilde{D}_c(\vec{x}, t)$ which is analytic in the upper half-plane, such that for real t $D_+(\vec{x}, t)$ is understood as a limiting value of this analytic function from $Imt > 0$:

$$D_+(\vec{x}, t) = \tilde{D}_c(\vec{x}, t + \imath 0) \tag{42}$$

3. The segment $-|\vec{x}| < t < |\vec{x}|$ of the real axis corresponds to the real spacelike $x = (\vec{x}, t)$ and due to **1** we have

$$D_+(\vec{x}, t) = D_-(\vec{x}, t), \ when \ -|\vec{x}| < t < |\vec{x}| \tag{43}$$

This means that $D_c(\vec{x}, t)$ can be analytically continued to the upper half plane across this segment so that **we obtain a single analytic function**:

$$\begin{aligned} D_c(\vec{x}, t) &= \int_{p^2=m^2} d\mu(p) \exp\left(-\imath\omega_{\vec{p}} t + \imath\vec{p}\vec{x}\right), \ Imt < 0 \\ D_c(\vec{x}, t) &= \int_{p^2=m^2} d\mu(p) \exp\left(\imath\omega_{\vec{p}} t - \imath\vec{p}\vec{x}\right), \ Imt > 0 \\ D_-(\vec{x}, t) &= D_c(\vec{x}, t - \imath 0), \ Imt = 0 \\ D_+(\vec{x}, t) &= D_c(\vec{x}, t + \imath 0), \ Imt = 0 \end{aligned} \tag{44}$$

For x is timelike and real t, these limiting values do not coinside, so $D_c(\vec{x}, t)$ has a branch cuts along the real t which extends from $t = |\vec{x}|$ to ∞ and from $t = -|\vec{x}|$ to ∞.

3.3. Causality in quantum KG theory.

Due to the arguments above one can write

$$[\hat{\phi}(\vec{x}, t), \hat{\phi}(0, 0)] = D_c(\vec{x}, t - \imath 0) - D_c(\vec{x}, t + \imath 0) \tag{45}$$

Because of there is no discontinuity at $-|\vec{x}| < t < |\vec{x}|$ we can make important conclusion

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0, \ when \ (x-y)^2 < 0 \tag{46}$$

To understand what does it mean remember that in quantum mechanics a commutator of two operators says how a disturbance introduced by one operator affects the mesurement of observable described by another operator. The result (46) says that a perturbation caused by a field operator $\hat{\phi}$ at the point x can not affect a mesurement of the field at y if the points are separated by a spacelike interval. It expresses **causality** of the theory (which is also known as a local commutativity). Causality is a general requirement in QFT.

We can conclude also that commutator of field operators is nonzero for the timelike $x-y$ (inside the light-cone) because of discontinuity is nonzero at the branch cuts, that is at $t_x - t_y > |\vec{x} - \vec{y}|$.

3.4. Feynman propagator.

The functions $D_\pm$ can be related also to the vacuum expectation values of the Heisenberg field operators

$$< 0|\hat{\phi}(x)\hat{\phi}(y)|0 >=< 0|\hat{\phi}_-(x)\hat{\phi}_+(y)|0 >=$$
$$< 0|[\hat{\phi}_-(x), \hat{\phi}_+(y)]|0 >= D_-(x-y) = D_+(y-x) \tag{47}$$

and is obtained from the analytic function of the complex variable t $D_c(\vec{x}, t)$ as $D_c(\vec{x}, t - \imath 0)$. One can use this function to define another important quantity.

Consider the value of $D_c(x)$ as t approaches the real t-axis along the contour C_F which goes above the real t-axis in the region $Ret < -|\vec{x}|$, crosses the segment $-|\vec{x}| < t < -|\vec{x}|$ and goes below the real t-axis in the region $Ret > |\vec{x}|$.

We can see that this limiting function is

$$D_- =< 0|\hat{\phi}(\vec{x},t),\hat{\phi}(0,0)|0>,\ \ Ret > |\vec{x}| \tag{48}$$

while this limiting function is given by

$$D_+(\vec{x},t) = D_-(-\vec{x},-t) =< 0|\hat{\phi}(0,0),\hat{\phi}(\vec{x},t)|0>,\ \ Ret < -|\vec{x}| \tag{49}$$

Hence, this new limiting function can be written as the **time − ordered** expectation value

$$D_F(x-y) =< 0|T(\hat{\phi}(\vec{x},t_x)\hat{\phi}(\vec{y},t_y)|0> \tag{50}$$

where time-ordering symbol T means

$$\begin{aligned} T(\hat{\phi}(\vec{x},t_x)\hat{\phi}(\vec{y},t_y)) &= \hat{\phi}(\vec{x},t_x)\hat{\phi}(\vec{y},t_y),\ \ when\ t_x > t_y \\ T(\hat{\phi}(\vec{x},t_x)\hat{\phi}(\vec{y},t_y)) &= \hat{\phi}(\vec{y},t_y)\hat{\phi}(\vec{x},t_x),\ \ when\ t_y > t_x \end{aligned} \tag{51}$$

Equivalently

$$\begin{aligned} D_F(x) &= \int_{p^2=m^2} d\mu(p)\exp\left(-\imath\omega_{\vec{p}}t + \imath\vec{p}\vec{x}\right),\ \ t > 0 \\ D_F(x) &= \int_{p^2=m^2} d\mu(p)\exp\left(\imath\omega_{\vec{p}}t - \imath\vec{p}\vec{x}\right),\ \ t < 0 \end{aligned} \tag{52}$$

This function, known as Feynmam propagator can also be represented as an $4d$ integral

$$D_F(x) = \int_{\tilde{C}_F}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\imath\exp\left(-\imath\omega_{\vec{p}}t + \imath\vec{p}\vec{x}\right)}{\omega^2-\vec{p}^2-m^2} \tag{53}$$

As a function of ω, the integrand has the poles at $\omega = \pm\omega_{\vec{p}}$ and contour **goes along the real t axis from $t = -\infty$ to $t = \infty$ axis and bypasses the pole $-\omega_{\vec{p}}$ from below, while it bypasses the pole $\omega_{\vec{p}}$ from above**.

Let us show that this integral reproduces the formula (52). Indeed, if $t > 0$, the integrand exponentially decays in the lower ω-half-plane ($Im\omega <$

0) and the contour can be closed on the pole $\omega_{\vec{p}}$ so we obtain $D_-(\vec{x},t)$. Similarly for $t<0$ the contour can be closed on $-\omega_{\vec{p}}$, yelding $D_+(\vec{x},t)$. More convenient way to write D_F is

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{d^3\vec{p}}{(2\pi)^3}\frac{\imath\exp\left(-\imath\omega_{\vec{p}}t+\imath\vec{p}\vec{x}\right)}{\omega^2-\vec{p}^2-m^2+\imath 0} \tag{54}$$

It is equivalent formula, since $\imath 0$ term in the denominator shifts the poles $\omega_{\vec{p}}\to\omega_{\vec{p}}-\imath 0, -\omega_{\vec{p}}+\imath 0$. It can be rewritten also in the covariant form

$$D_F(x)=\int\frac{d^4\vec{p}}{(2\pi)^4}\frac{\imath\exp\left(-\imath p_\mu x^\mu\right)}{p^2-m^2+\imath 0} \tag{55}$$

In other words it is a Fourier transform of the function

$$D_F(p)=\frac{\imath}{p^2-m^2+\imath 0} \tag{56}$$

3.5. Feynman propagator as a Green function of KG equation.

The last formula shows that $D_F(x)$ is a Green's function for the KG equation

$$(\partial_\mu\partial^\mu+m^2)D_F(x)=\imath\delta(x) \tag{57}$$

At the same time, the $D_\pm(x)$ are the solutions of homogeneous KG equation:

$$(\partial_\mu\partial^\mu+m^2)D_\pm(x)=0 \tag{58}$$