Lecture 14. LSZ theorem and S-matrix.

Plan.

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1. S-matrix.

1.1. ASYMPTOTIC STATES AND S-MATRIX.

On the last lecture we introduced the asymtotic in-states

$$\begin{aligned} \hat{A}^{in}_{\vec{p}} &= \lim_{\epsilon\to 0}\lim_{t\to -\infty} \hat{A}_{\vec{p},\epsilon}(\tau + \imath t), \\ \hat{A}^{\dagger in}_{\vec{p}} &= \lim_{\epsilon\to 0}\lim_{t\to -\infty} \hat{A}^{\dagger}_{\vec{p},\epsilon}(\tau + \imath t), \end{aligned} \tag{1}$$

where

$$\begin{aligned} \hat{A}_{\vec{p},\epsilon}(\tau) &= \frac{1}{\sqrt{Z}}\int d^3x(\partial_\tau\hat{\phi} f_{\vec{p},\epsilon} - \hat{\phi}\partial_\tau f_{\vec{p},\epsilon}) \\ \hat{A}^{\dagger}_{\vec{p},\epsilon}(\tau) &= \frac{1}{\sqrt{Z}}\int d^3x(\partial_\tau\hat{\phi} \bar{f}_{\vec{p},\epsilon} - \hat{\phi}\partial_\tau \bar{f}_{\vec{p},\epsilon}) \end{aligned} \tag{2}$$

and $f_{\vec{p},\epsilon}$, $\bar{f}_{\vec{p},\epsilon}$ are the wave packets solutions of KG equation in euclidean space:

$$\begin{aligned} f_{\vec{p},\epsilon}(\vec{x},\tau) &= \exp{(\omega_p\tau - \imath\vec{p}\vec{x})}\psi_\epsilon(\vec{x} - \imath\vec{v}\tau) \\ \bar{f}_{\vec{p},\epsilon}(\vec{x},\tau) &= \exp{(\omega_p\tau + \imath\vec{p}\vec{x})}\psi_\epsilon(\vec{x} + \imath\vec{v}\tau), \\ \psi_\epsilon(\vec{y}) &= \exp{(-\frac{\epsilon}{2}\vec{y}^2)} \end{aligned} \tag{3}$$

Similarly, we introduced the asymptotic out-operators taking another limit

$$\hat{A}^{out}_{\vec{p}} = \lim_{\epsilon \to 0} \lim_{t \to \infty} \hat{A}_{\vec{p},\epsilon}(\tau + \imath t),$$
$$\hat{A}^{\dagger out}_{\vec{p}} = \lim_{\epsilon \to 0} \lim_{t \to \infty} \hat{A}^{\dagger}_{\vec{p},\epsilon}(\tau + \imath t). \tag{4}$$

It was shown then that the asymptotic in-operators and out-operators satisfy canonical commutation relations of creation-annihilation operators (Heisenberg algebra) and the asymptotic in-states

$$|\vec{p}_1, ..., \vec{p}_n >_{in} \equiv A^{\dagger in}_{\vec{p}_1} ... A^{\dagger in}_{\vec{p}_n} |\Omega > \tag{5}$$

and out-states

$$|\vec{q}_1, ..., \vec{q}_n >_{out} \equiv A^{\dagger out}_{\vec{q}_1} ... A^{\dagger out}_{\vec{q}_n} |\Omega > \tag{6}$$

constitute an in- and out- basises in Hilbert space $\mathcal{H}$ of the theory.

The spaces of in-states (5) and out-states (6) do not coincide in the theory with interaction but are related by a unitary transformation

$$|\vec{p}_1, ..., \vec{p}_N >_{in} = \sum_M \int \hat{S}(\vec{p}_1, ..., \vec{p}_N | \vec{q}_1, ..., \vec{q}_M) |\vec{q}_1, ..., \vec{q}_N >_{out} d\mu(\vec{q}_1) ... d\mu(\vec{q}_M), \tag{7}$$

where $\partial\mu(\vec{q})$ is a Lorentz invariant measure. The unitary operator S is called S-**matrix** and

$$\hat{S}(\vec{p}_1, ..., \vec{p}_N | \vec{q}_1, ..., \vec{q}_M) \approx \delta^4(\sum_{i=1}^{N} p_i - \sum_{j=1}^{M} q_j). \tag{8}$$

Thus, matrix elements of $\hat{S}$

$${}_{out} < \vec{q}_1, ..., \vec{q}_M | \vec{p}_1, ..., \vec{p}_N >_{in} \equiv S(\vec{p}_1, ..., \vec{p}_N | \vec{q}_1, ..., \vec{q}_M) \tag{9}$$

describe the overlapping between the in and out states.

1.2. T-OPERATOR.

Let us consider for example $2 \to 2$ amplitude

$$S(\vec{p}_1, \vec{p}_2 | \vec{q}_1, \vec{q}_2) = {}_{out}< \vec{q}_1, \vec{q}_2 | \vec{p}_1, \vec{p}_2 >_{in} \tag{10}$$

Besides the scattering, this amplitude contains the probability of the passage of the particles without collision. Therefore

$$\begin{aligned} S(\vec{p}_1, \vec{p}_2 | \vec{q}_1, \vec{q}_2) = \\ (2\pi)^4 2\omega_1\omega_2 (\delta(\vec{p}_1 - \vec{q}_1)\delta(\vec{p}_2 - \vec{q}_2) + \delta(\vec{p}_1 - \vec{q}_2)\delta(\vec{p}_2 - \vec{q}_1)) + \\ (2\pi)^4 \delta(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2) T(\vec{p}_1, \vec{p}_2 | \vec{q}_1, \vec{q}_2) \end{aligned} \tag{11}$$

The term $T(\vec{p}_1, \vec{p}_2 | \vec{q}_1, \vec{q}_2)$ describes the collision. By the definition, so called **differential crossection** is determined by the formula

$$\begin{aligned} d\sigma_{2\to 2} = \\ (2\pi)^4 \delta(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2) \frac{|T(\vec{p}_1, \vec{p}_2 | \vec{q}_1, \vec{q}_2)|^2}{\sqrt{s(s - 4m^2)}} d\mu(p_1) d\mu(p_2) \end{aligned} \tag{12}$$

where $s = (p_1 + p_2)^2$. **It describes the probability of collision in per unit time in unite volume for unite flow of particles**.

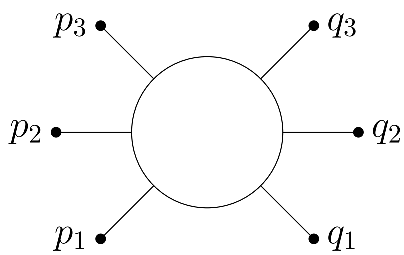
In general case we can write

$$\hat{S} = I + \imath \hat{T} \tag{13}$$

where I is an identity operator which corresponds to the scattering without collisions while $\hat{T}$ **describes the process with at least one pair of particles collide**.

Example: $3 \to 3$ amplitude.

$$_{out} < 3|3 >_{in} =$$



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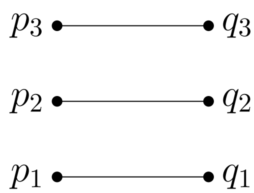
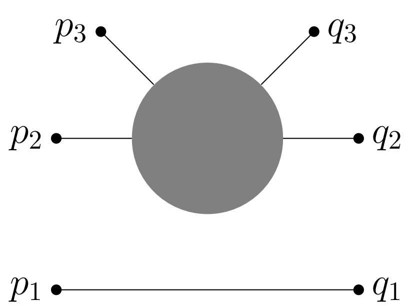
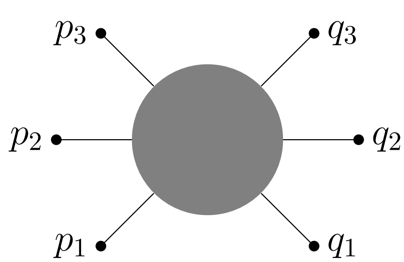


$$+ \ permutations \ + \qquad (14)$$



$$+ \ permutations \ +$$



$$(15)$$

2. LSZ theorem.

LSZ theorem allows to express the S-matrix elements in terms of the Green's functions.

2.1. RELATION BETWEEN IN AND OUT CREATION-ANNIHILATION OPERATO

We have by definition

$$|\vec{p}_1,...,\vec{p}_N>_{in}\equiv A^{\dagger in}_{\vec{p}_1}...A^{\dagger in}_{\vec{p}_N}|\Omega>,$$
$$_{out}<\vec{q}_1,...,\vec{q}_M| =<\Omega|A^{out}_{\vec{q}_1}...A^{out}_{\vec{q}_M},$$
$$S(\vec{p}_1,...,\vec{p}_N|\vec{q}_1,...,\vec{q}_M) =_{out}<\vec{q}_1,...,\vec{q}_M|\vec{p}_1,...,\vec{p}_N>_{in}=$$
$$<\Omega|A^{out}_{\vec{q}_1}...A^{out}_{\vec{q}_M}|A^{\dagger in}_{\vec{p}_1}...A^{\dagger in}_{\vec{p}_N}|\Omega> . \tag{16}$$

The operators $A^{\dagger in}_{\vec{p}}$, $A^{in}_{\vec{k}}$ satisfy Heisenberg algebra of creation-annihilation operators. The same is true for the operators $A^{\dagger out}_{\vec{p}}$, $A^{out}_{\vec{k}}$. But it is **not true for the commutators between in and out operators in interacting theory**.

Let us consider the operators $A_{\vec{p}}(\tau)$, $A^{\dagger}_{\vec{p}}(\tau)$ determined in (2) in real time:

$$a_{\vec{p}}(t) = \imath\int d^3x(\hat{\phi}(\vec{x},t)\partial_t f_{\vec{p}}(\vec{x},t) - \partial_t\hat{\phi}(\vec{x},t)f_{\vec{p}}(\vec{x},t))$$
$$a^{\dagger}_{\vec{p}}(t) = -\imath\int d^3x(\hat{\phi}(\vec{x},t)\partial_t f^*_{\vec{p}}(\vec{x},t) - \partial_t\hat{\phi}(\vec{x},t)f^*_{\vec{p}}(\vec{x},t)) \tag{17}$$

where $f_{\vec{p}}(\vec{x},t)$, $f^*_{\vec{p}}(\vec{x},t)$ are the solutions of KG equation in Minkowski space, which in the limit $\epsilon\to 0$ go to the plane waves solutions and

$$\hat{\phi}(\vec{x},t) = \exp(\imath\hat{H}t)\hat{\phi}(\vec{x},0)\exp(-\imath\hat{H}t) \tag{18}$$

is Heisenberg field operator.

Let us calculate the difference between these operators in different moments of time $t_1 > t_2$:

$$
\begin{aligned}
a^\dagger_{\vec{p}}(t_1) - a^\dagger_{\vec{p}}(t_2) = \\
-\imath \int_{\Sigma_1} d^3x(\hat{\phi}(\vec{x},t)\partial_t f^*_{\vec{p}}(\vec{x},t) - \partial_t\hat{\phi}(\vec{x},t) f^*_{\vec{p}}(\vec{x},t)) + \\
\imath \int_{\Sigma_2} d^3x(\hat{\phi}(\vec{x},t)\partial_t f^*_{\vec{p}}(\vec{x},t) - \partial_t\hat{\phi}(\vec{x},t) f^*_{\vec{p}}(\vec{x},t)) = \\
-\imath \int_{t_2}^{t_1} dt \int d^3x \partial^\mu (f^*_{\vec{p}}\partial_\mu\hat{\phi} - \partial_\mu f^*_{\vec{p}}\hat{\phi}) = \\
-\imath \int_{t_2}^{t_1} dt \int d^3x f^*_{\vec{p}}(\partial^\mu\partial_\mu + m^2)\hat{\phi}
\end{aligned}
\tag{19}
$$

where we have applied KG equation for $f^*_{\vec{p}}$ and Gauss theorem assuming that $\hat{\phi} \to 0$ as $|\vec{x}| \to \infty$.

Taking the limit $t_1 \to \infty$, $t_2 \to -\infty$ we obtain

$$
a^{\dagger in}_{\vec{p}} = a^{\dagger out}_{\vec{p}} + \imath \int d^4x f^*(\partial^\mu\partial_\mu + m^2)\hat{\phi}
\tag{20}
$$

2.2. LSZ REDUCTION FORMULA.

The relation (20) between in and out operators can be used to calculate the S-matrix element

$$
\begin{aligned}
S(\vec{p}_1,...,\vec{p}_N|\vec{q}_1,...,\vec{q}_M) =< \Omega|a^{out}_{\vec{q}_1}...a^{out}_{\vec{q}_M} a^{\dagger in}_{\vec{p}_1}...a^{\dagger in}_{\vec{p}_N}|\Omega >= \\
< \Omega|a^{out}_{\vec{q}_1}...a^{out}_{\vec{q}_M}(a^{\dagger out}_{\vec{p}_1} + \imath \int d^4x f^*_{\vec{p}_1}(x)(\partial_x^2 + m^2)\hat{\phi}) a^{\dagger in}_{\vec{p}_2}...a^{\dagger in}_{\vec{p}_N}|\Omega >= \\
< \Omega|a^{out}_{\vec{q}_1}...a^{out}_{\vec{q}_M} a^{\dagger out}_{\vec{p}_1} a^{\dagger in}_{\vec{p}_2}...a^{\dagger in}_{\vec{p}_N}|\Omega > + \\
\imath \int d^4x f^*_{\vec{p}_1}(x)(\partial_x^2 + m^2)_{out} < \vec{q}_1,...,\vec{q}_M|\hat{\phi}(x)|\vec{p}_2,...,\vec{p}_N >_{in} .
\end{aligned}
\tag{21}
$$

In the first term of this expression we can move $a^{\dagger out}_{\vec{p}_1}$ to the left and use creation-annihilation commutation relations as well as

$$
< \Omega|a^{\dagger out}_{\vec{p}_1} = 0
\tag{22}
$$

It gives zero or **unconnected contribution**

$$\sum_{i=1}^{M} 2\omega_{p_1}\delta^3(\vec{p}_1-\vec{q}_i)_{out} < \vec{q}_1,...,\check{\vec{q}_i},...,\vec{q}_M|\vec{p}_2,...,\vec{p}_N >_{in} \tag{23}$$

to the matrix element.

Let us look at the **connected contribution**:

$$\begin{aligned} {}_{out} < \vec{q}_1,...,\vec{q}_{M-1}|a^{out}_{\vec{q}_M}\hat{\phi}(x)|\vec{p}_2,...,\vec{p}_N >_{in} = \\ \lim_{t'\to\infty} \imath \int_{\Sigma_\infty} d^3x_{out} < \vec{q}_1,...,\vec{q}_{M-1}|(\hat{\phi}(\vec{x},t')\frac{\partial}{\partial t'}f_{\vec{p}}(\vec{x},t') - \frac{\partial}{\partial t'}\hat{\phi}(\vec{x},t')f_{\vec{p}}(\vec{x},t'))\hat{\phi}(x) \\ |\vec{p}_2,...,\vec{p}_N >_{in} \end{aligned} \tag{24}$$

We can write the integral as

$$\begin{aligned} \lim_{t'\to\infty} \imath \int_{\Sigma_\infty} d^3x[\frac{\partial f_{\vec{p}}(x')}{\partial t'}{}_{out} < \vec{q}_1,...,\vec{q}_{M-1}|T\hat{\phi}(x')\hat{\phi}(x)|\vec{p}_2,...,\vec{p}_N >_{in} - \\ f_{\vec{p}}(x')\frac{\partial}{\partial t'}{}_{out} < \vec{q}_1,...,\vec{q}_{M-1}|T\hat{\phi}(x')\hat{\phi}(x)|\vec{p}_2,...,\vec{p}_N >_{in}] \end{aligned} \tag{25}$$

which is correct because $t' \to \infty$. Now one can use again the arguments similar to (19) and Gauss formula to get

$$\begin{aligned} {}_{out} < \vec{q}_1,...,\vec{q}_{M-1}|\hat{\phi}(x)a^{in}_{\vec{q}_M}|\vec{p}_2,...,\vec{p}_N >_{in} + \\ \imath \int d^4x' f_{\vec{q}_{M-1}}(x')(\partial^2_{x'} + m^2)_{out} < \vec{q}_1,...,\vec{q}_{M-1}|T\hat{\phi}(x')\hat{\phi}(x)|\vec{p}_2,...,\vec{p}_N >_{in} . \end{aligned} \tag{26}$$

Using the creation-annihilation commutation relations in the first term we obtain again unconnected contribution or zero. The second term gives connected diagram contribution to the amplitude.

Going by this way we arrive at the end the **LSZ-reduction formula**

$$S(\vec{p}_1,...,\vec{p}_N|\vec{q}_1,...,\vec{q}_M) =$$
$$sum\ of\ unconnected\ parts\ +$$
$$\imath^{N+M}\int d^4x_1...d^4x_N d^4y_1...d^4y_M f^*_{\vec{p}_1}(x_1)...f^*_{\vec{p}_N}(x_N) f_{\vec{q}_1}(y_1)...f_{\vec{q}_M}(y_M)$$
$$(\partial^2_{x_1}+m^2)...(\partial^2_{x_N}+m^2)(\partial^2_{y_1}+m^2)...(\partial^2_{y_M}+m^2)$$
$$< \Omega|T(\hat{\phi}(x_1)...\hat{\phi}(x_N)\hat{\phi}(y_1)...\hat{\phi}(y_M))|\Omega > \quad (27)$$

2.3. LSZ REDUCTION FORMULA IN MOMENTA SPACE.

The above formula looks more simpler for its Fourier image.

First of all we consider Fourier transformation for the Green's function

$$G_{N+M}(k_1,...,k_{N+M}) =$$
$$\int d^4x_1...d^4x_{N+M}\exp{(\imath k_1 x_1)}...\exp{(\imath k_{N+M}x_{N+M})} < \Omega|T(\hat{\phi}(x_1)...\hat{\phi}(x_{N+M})|\Omega > \quad (28)$$

Then, we form two sets

$$k_i = (-\omega_{\vec{p}_i}, -\vec{p}_i)\ ,\ i=1,...,N$$
$$k_i = (\omega_{\vec{q}_i}, \vec{q}_i)\ ,\ i=N+1,...,N+M \quad (29)$$

Let us take also into account that

$$\int d^4x\exp{(\imath kx)}(\partial^2_x+m^2)f(x) = (-k^\mu k_\mu + m^2)f(k), \quad (30)$$

where $k^\mu k_\mu = (k^0)^2-(\vec{k})^2$. As a result we obtain

$$S(\vec{p}_1,...,\vec{p}_N|\vec{q}_1,...,\vec{q}_M) =$$
$$sum\ of\ unconnected\ parts\ +$$
$$\imath^{N+M}\prod_{i=1}^{N+M}(m^2-k_i^2)G_{N+M}(k_1,...,k_{N+M}) \quad (31)$$

where the Green's function $G_{N+M}(k_1, ..., k_{N+M})$ is obtained from the corresponding correlation function in euclidean space after the substitution $k_i^0 = -\imath k_i^4$.

Notice that factors $(m^2 - k_i^2)$ vanish on-shell. On the other hand $G_{N+M}(k_1, ..., k_{N+M})$ has a pole when $k_i^2 - m^2 = 0$ because

$$G_{N+M}(k_1, ..., k_{N+M}) = \prod_i G_2(k_i^2) G_{N+M,amp}(k_1, ..., k_{N+M}), \tag{32}$$

where

$$G_2(k^2) = \frac{Z}{k^2 - m^2} + \int_{4m^2}^{\infty} \frac{d\mu^2}{(2\pi)} \frac{\rho(\mu^2)}{k^2 + \mu^2} \tag{33}$$

due to KL representation (see the previous lecture).

We see the pole in $G_2(k^2)$ which cancels the $k^2 - m^2 = 0$ factor in (31). As a result we can write

$$S_c(\vec{p}_1, ..., \vec{p}_N | \vec{q}_1, ..., \vec{q}_M) = Z^{\frac{N+M}{2}} G_{N+M,amp}(k_1, ..., k_{N+M}) \tag{34}$$

where

$$\begin{aligned} k_i = -p_i \ , \ i = 1, ..., N \\ k_i = q_i \ , \ i = N+1, ..., N+M \end{aligned} \tag{35}$$

Therefore, **in order to calculate the element of S-matrix we must calculate the corresponding connected diagrams and cut off the external legs**.