Lecture 13.

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1. LSZ theory.

As we have seen the QFT has two mutually complementary points of view: one can consider the theory in euclidean space (statistical mechanics) or one can consider it in the Minkowski space (quantum mechanics). The important ussumption is boundness of spectrum of H from below.

For the Lorentz invariant theory it make sense to analyze the spectrum $\mathcal{P}$ of the momentum operator P^μ, basing on the Lorentz invariance property. Boundnes from below means that there is the unique vacuum state $|\Omega>$ (one can also cosider the theories with degenerate vacuum state), which is determined by

$$P^\nu|\Omega>=0,\ \nu=0,...,3 \tag{1}$$

and for any other state $|\alpha>$ of the theory

$$H|\alpha>=E_\alpha|\alpha>,\ E_\alpha>0\ iff\ \alpha\neq\Omega \tag{2}$$

Note also that for each $\hat{P}^\nu$-eigenstate $|\alpha>$ with eigenvalues P^ν_α its Lorentz group orbit is also in the spectrum of the theory. This orbit $\mathcal{P}_{\mu^2} \subset \mathcal{P}$ is the surface $P^\nu P_\nu = \mu^2$ and $\mathcal{P} = \bigcup_{\mu^2} \mathcal{P}_{\mu^2}$.

1.1. CLASTER PROPERTY.

The ussumptions (1), (2) allows to prove so called **claster property** of the correlation functions

$$\lim_{R\to\infty} < O_1(\vec{x}_1,\tau_1+R)...O_N(\vec{x}_N,\tau_N+R)O_{N+1}(\vec{x}_{N+1},\tau_{N+1}...O_{N+M}(\vec{x}_M,\tau_{N+M}) >= \\ < O_1(\vec{x}_1,\tau_1)...O_N(\vec{x}_N,\tau_N) >< O_{N+1}(\vec{x}_{N+1},\tau_{N+1}...O_{N+M}(\vec{x}_M,\tau_{N+M}) > . \tag{3}$$

Proof.

If $\tau_1 \geq \tau_2 \geq ...\tau_N$

$$< O_1(\vec{x}_1,\tau_1)...O_N(\vec{x}_N,\tau_N) >=< \Omega|\hat{O}_1(\vec{x}_1,\tau_1)...\hat{O}_N(\vec{x}_N,\tau_N)|\Omega >, \tag{4}$$

where

$$\hat{O}(\vec{x},\tau) = \exp{(\hat{H}\tau)}O(\vec{x},0)\exp{(-\hat{H}\tau)} \ (\tau = \imath t). \tag{5}$$

Hence

$$< \Omega|\hat{O}_1(\vec{x}_1,\tau_1+R)...\hat{O}_N(\vec{x}_N,\tau_N+R)\hat{O}_{N+1}(\vec{x}_{N+1},\tau_{N+1})... \\ ...\hat{O}_{N+M}(\vec{x}_{N+M},\tau_{N+M})|\Omega >= \\ < \Omega|\hat{O}_1(\vec{x}_1,\tau_1)...\hat{O}_N(\vec{x}_N,\tau_N)\exp{(-\hat{H}R)}\hat{O}_{N+1}(\vec{x}_{N+1},\tau_{N+1}) \\ ...\hat{O}_{N+M}(\vec{x}_{N+M},\tau_{N+M})|\Omega > . \tag{6}$$

Now we insert the identity operator $\sum_\alpha |\alpha><\alpha|$ and use that $|\alpha>$ is a Hamiltonian eigenstate:

$$\begin{aligned}\sum_\alpha <\Omega|\hat{O}_1(\vec{x}_1,\tau_1)...\hat{O}_N(\vec{x}_N,\tau_N)\exp{(-\hat{H}R)}|\alpha> \\ <\alpha|\hat{O}_{N+1}(\vec{x}_{N+1},\tau_{N+1}) \\ ...\hat{O}_{N+M}(\vec{x}_{N+M},\tau_{N+M})|\Omega>= \\ \exp{(-E_\alpha R)}\sum_\alpha <\Omega|\hat{O}_1(\vec{x}_1,\tau_1)...\hat{O}_N(\vec{x}_N,\tau_N)|\alpha> \\ <\alpha|\hat{O}_{N+1}(\vec{x}_{N+1},\tau_{N+1}) \\ ...\hat{O}_{N+M}(\vec{x}_{N+M},\tau_{N+M})|\Omega>\Rightarrow \\ \lim_{R\to\infty} <\Omega|\hat{O}_1(\vec{x}_1,\tau_1+R)...\hat{O}_N(\vec{x}_N,\tau_N+R)\hat{O}_{N+1}(\vec{x}_{N+1},\tau_{N+1}) \\ ...\hat{O}_{N+M}(\vec{x}_{N+M},\tau_{N+M})|\Omega>= \\ <\Omega|\hat{O}_1(\vec{x}_1,\tau_1)...\hat{O}_N(\vec{x}_N,\tau_N)|\Omega> \\ <\Omega|\hat{O}_{N+1}(\vec{x}_{N+1},\tau_{N+1})...\hat{O}_{N+M}(\vec{x}_M,\tau_{N+M})|\Omega>\end{aligned} \tag{7}$$

Thus, **the claster property follows from boundedness of the sperctrum from below**.

1.2. LOCALITY.

Let us consider $<O_1(\vec{x},\tau)O_2(\vec{0},0)>$. If $\tau>0$

$$\begin{aligned}<O_1(\vec{x},\tau)O_2(\vec{0},0)>=<\Omega|O_1(\vec{x},\tau)O_2(\vec{0},0)|\Omega>= \\ <\Omega|O_1(\vec{x},0)\exp{(-\hat{H}\tau)}O_2(\vec{0},0)|\Omega>= \\ \sum_\alpha \exp{(-E_\alpha\tau)} <\Omega|O_1(\vec{x},0)|\alpha><\alpha|O_2(\vec{0},0)|\Omega>\,.\end{aligned} \tag{8}$$

The eigenvalues $E_\alpha \geq 0$ and we assume that the series is convergent (so that 2-point fuction exists). **It then defines analytic function** $G_+(\vec{x},\tau)$

for $Re(\tau) > 0$.

Let us analogously calculate the correlation function with opposit order,

$$< O_2(\vec{0},0)O_1(\vec{x},\tau) >=< \Omega|O_2(\vec{0},0)O_1(\vec{x},\tau)|\Omega >=$$
$$< \Omega|O_2(\vec{0},0)\exp{(\hat{H}\tau)}O_1(\vec{x},0)|\Omega >=$$
$$\sum_\alpha < \Omega|O_1(\vec{x},0)|\alpha >< \alpha|O_2(\vec{0},0)|\Omega > \exp{(E_\alpha \tau)} \tag{9}$$

This correlator determines analytic function $G_-(\vec{x},\tau)$ for $Re(\tau) < 0$. **The imaginary $\tau = \imath t$ axis corresponds to the theory in Minkowski space**.

When $-|\vec{x}| < t < |\vec{x}|$ the vector $(\vec{x},t)$ is space-like. Due to Lorentz invariance we conclude that

$$G_-(\vec{x},\imath t - 0) = G_+(\vec{x},\imath t + 0) \ when \ -|\vec{x}| < t < |\vec{x}| \tag{10}$$

(recall similar arguments for the KG theory where we rotated the 3-vector $\vec{x}$ to the vector $-\vec{x}$ in the framework $t = 0$). Thus, $G_-(\vec{x},\tau)$ is analitic continuation of $G_+(\vec{x},\tau)$ into the half-plane $Re(\tau) < 0$. But in the region $t > \pm|\vec{x}|$ the limiting values of $G_\pm(\vec{x},\tau)$ may not coinside. In other words, similar to KG theory

$$G_+(\vec{x},\imath t + 0) - G_-(\vec{x},\imath t - 0) \equiv< \Omega|[\hat{O}_1(\vec{x},t),\hat{O}_2(\vec{0},0)]|\Omega > \tag{11}$$

and it vanishes for the space-like vectors $(\vec{x},t)$. This is **locality property of the Green's functions in the theory with interaction**.

1.3. ADDITIVITY OF THE SPECTRUM.

Let us consider the correlation function

$$< O(\vec{x},\tau)O(\vec{x},0)O(\vec{0},\tau)O(\vec{0},0) > \tag{12}$$

In euclidean space the correaltion functions are invariant w.r.t. $O(4)$ (it is the Lorentz group in imaginary time). Hence, one can take any linear combination of components of P^ν as a Hamiltonian and use the claster property above to show that

$$\lim_{|\vec{x}|\to\infty} < O(\vec{x},\tau)O(\vec{x},0)O(\vec{0},\tau)O(\vec{0},0) >=< O(\vec{0},\tau)O(\vec{0},0) >^2 . \tag{13}$$

On the other hand, because of the correlation fucntions are related with the vacuum expectation value of the product of time-ordered (in our case, τ-ordered) Heisenberg field operators, one can write

$$\begin{aligned}
< O(\vec{x},\tau)O(\vec{x},0)O(\vec{0},\tau)O(\vec{0},0) >= \\
< \Omega|T(\hat{O}(\vec{x},\tau)\hat{O}(\vec{x},0)\hat{O}(\vec{0},\tau)\hat{O}(\vec{0},0))|\Omega >= \\
< \Omega|\hat{O}(\vec{0},\tau)\hat{O}(\vec{x},\tau)\hat{O}(\vec{x},0)\hat{O}(\vec{0},0)|\Omega >= \\
\sum_\gamma < \Omega|\hat{O}(\vec{0},0)\hat{O}(\vec{x},0)\exp{(-\hat{H}\tau)}|\gamma >< \gamma|\hat{O}(\vec{x},0)\hat{O}(\vec{0},0)|\Omega >= \\
\sum_\gamma \exp{(-E_\gamma\tau)} < \Omega|\hat{O}(\vec{0},0)\hat{O}(\vec{x},0)|\gamma >< \gamma|\hat{O}(\vec{x},0)\hat{O}(\vec{0},0)|\Omega >= \\
\sum_\gamma \exp{(-E_\gamma\tau)}| < \Omega|\hat{O}(\vec{0},0)\hat{O}(\vec{x},0)|\gamma > |^2.
\end{aligned} \tag{14}$$

At the same time

$$\begin{aligned}
< O(\vec{0},\tau)O(\vec{0},0) >^2=< \Omega|\hat{O}(\vec{0},\tau)\hat{O}(\vec{0},0)|\Omega >^2= \\
\sum_{\alpha,\beta} | < \Omega|\hat{O}(\vec{0},0)|\alpha > |^2| < \Omega|\hat{O}(\vec{0},0)|\beta > |^2 \exp{(-(E_\alpha+E_\beta)\tau)}
\end{aligned} \tag{15}$$

which shows that if $E_\alpha, E_\beta \in \mathcal{P}$ **then** $E_\alpha + E_\beta \in \mathcal{P}$. Due to the Lorentz ($O(4)$ invarince in euclidean formulation) invariance **we conclude that if** $P^\mu_\alpha, P^\mu_\beta \in \mathcal{P}$ **then** $P^\mu_\alpha + P^\mu_\beta \in \mathcal{P}$. It is clear also that if $P^\mu_\alpha \in \mathcal{P}$ then $\Lambda^\mu_\nu P^\nu_\alpha \in \mathcal{P}$ for any Lorentz transformation Λ. **If the spectrum** $\mathcal{P}$ **contains at least one** 1**-particle state with the mass** m **then the spectrum contains also a surface of states** $P^2 = m^2$.

1.4. KÄLLEN-LEHMANN SPECTRAL REPRESENTATION.

Let us consider 2-point correlation function with $\tau > 0$

$$\begin{aligned} &< \phi(\vec{x},\tau)\phi(\vec{0},0) >=< \Omega|\hat{\phi}(\vec{x},\tau)\hat{\phi}(\vec{0},0)|\Omega >= \\ &\sum_\alpha \int \frac{d^3q}{(2\pi)^3 2E_{\vec{q}}(\alpha)} < \Omega|\hat{\phi}(\vec{x},\tau)|\alpha_{\vec{q}} >< \alpha_{\vec{q}}|\hat{\phi}(\vec{0},0)|\Omega >= \\ &\sum_\alpha \int \frac{d^3q}{(2\pi)^3 2E_{\vec{q}}(\alpha)} < \Omega|\hat{\phi}(\vec{x},0)\exp(-\hat{H}\tau)|\alpha_{\vec{q}} >< \alpha_{\vec{q}}|\hat{\phi}(\vec{0},0)|\Omega >= \\ &\sum_\alpha \exp(-E_\alpha\tau) \int \frac{d^3q}{(2\pi)^3 2E_{\vec{q}}(\alpha)} < \Omega|\hat{\phi}(\vec{x},0)|\alpha_{\vec{q}} >< \alpha_{\vec{q}}|\hat{\phi}(\vec{0},0)|\Omega > . \end{aligned} \tag{16}$$

Due to translation invariance

$$[\hat{\vec{P}}, \hat{\phi}(\vec{x},0)] = -\imath\nabla\hat{\phi}(\vec{x},0) \tag{17}$$

we can write

$$\hat{\phi}(\vec{x},0) = \exp(\imath\hat{P}_i x^i)\hat{\phi}(0,0)\exp(-\imath\hat{P}_i x^i),\ \ i = 1,2,3. \tag{18}$$

Therefore

$$
\begin{aligned}
&< \Omega|\hat{\phi}(\vec{x},\tau)\hat{\phi}(\vec{0},0)|\Omega >= \\
\sum_\alpha \int \frac{d^3q}{(2\pi)^3 2E_{\vec{q}}(\alpha)} &< \Omega|\hat{\phi}(\vec{0},0)|\alpha >< \alpha|\hat{\phi}(\vec{0},0)|\Omega > \exp\left(-E_{\vec{q}}(\alpha)\tau + \imath\vec{q}\vec{x}\right) = \\
&\sum_\alpha \int \frac{d^3q}{(2\pi)^3 2E_{\vec{q}}(\alpha)} | < \Omega|\hat{\phi}(\vec{0},0)|\alpha > |^2 \exp\left(-E_{\vec{q}}(\alpha)\tau + \imath\vec{q}\vec{x}\right),
\end{aligned}
\tag{19}
$$

where $|\alpha >$ is a momentum $\hat{\vec{P}} = \vec{q}$ state. (Recall also that $\frac{d^3q_\alpha}{(2\pi)^3 2E(\alpha)}$ is Lorentz invariant and 4-vector $q^\mu = (q^0, q^i) = (q^0, \vec{q})$, while $q_\mu = (q^0, -q^i) = (q^0, -\vec{q})$). Now we take into account the Lorentz invariance of the spectrum so that the summation over the states $|\alpha >$ can be decomposed in the integration over the Lorentz orbits $q^2 = \mu^2$:

$$
\begin{aligned}
\int_0^\infty \frac{d\mu^2}{2\pi}(2\pi \sum_\alpha | < \Omega|\hat{\phi}(\vec{0},0)|\alpha > |^2 \delta(\mu^2 - m_\alpha^2)) \int \frac{d^3q}{(2\pi)^3 2E_{\vec{q}}} \exp\left(-E_{\vec{q}}\tau + \imath\vec{q}_\alpha\vec{x}\right) = \\
\int_0^\infty \frac{d\mu^2}{2\pi}\rho(\mu^2) \int \frac{d^3q}{(2\pi)^3 2E_{\vec{q}}} \exp\left(-E_{\vec{q}}\tau + \imath\vec{q}_\alpha\vec{x}\right),
\end{aligned}
\tag{20}
$$

where

$$
\rho(\mu^2) = 2\pi \sum_\alpha | < \Omega|\hat{\phi}(\vec{0},0)|\alpha > |^2 \delta(\mu^2 - m_\alpha^2). \tag{21}
$$

For the case $\tau < 0$ we should change $\exp(-E_{\vec{q}}\tau) \to \exp(E_{\vec{q}}\tau)$. In both cases one can use

$$
\frac{1}{2E_{\vec{q}}} \exp\left(-E_{\vec{q}}|\tau|\right) = \int \frac{dq_4}{(2\pi)} \frac{1}{q_4^2 + \vec{q}^2 + \mu^2} \exp\left(\imath q_4 \tau\right) \tag{22}
$$

It allows to write finally

$$
< \phi(\vec{x},\tau)\phi(\vec{0},0) >= \int_0^\infty \frac{d\mu^2}{2\pi}\rho(\mu^2) D(|x|,\mu^2), \tag{23}
$$

where $|x| = \sqrt{\tau^2 + \vec{x}^2}$ and

$$D(|x|, \mu^2) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left(\imath k_4 \tau + \imath \vec{k}\vec{x}\right)}{k_4^2 + \vec{k}^2 + \mu^2}. \tag{24}$$

The 2-point correlation function expression above is called **Kӓllen-Lehmann spectral representation**.

If the theory contains 1-particle state $|1>$ with mass m the function $\rho(\mu^2)$ must take the form

$$\rho(\mu^2) = 2\pi Z\delta(\mu^2 - m^2) + \tilde{\rho}(\mu^2)\ ,\ \ \tilde{\rho}(\mu^2) = 0\ ,\ \ for\ \mu^2 < 4m^2 \tag{25}$$

(this is true if the spaectrum does not contain bound states whose mass $M^2 < 4m^2$), where

$$Z^{\frac{1}{2}} =< \Omega|\hat{\phi}|1 > . \tag{26}$$

2. Asymptotic states.

We need asymptotic states to formulate the scattering problem.

2.1. CORRELATION FUNCTIONS AT LARGE DISTANCES.

The 2-point function (Kӓllen-Lehmann spectral representation) can be written as

$$< \phi(x)\phi(0) >= ZD(|x|, m^2) + \int_{4m^2}^{\infty} \frac{d\mu^2}{(2\pi)} \rho(\mu^2) D(|x|, \mu^2), \tag{27}$$

where

$$D(|x|,\mu^2) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp{(\imath kx)}}{k^2+\mu^2} \tag{28}$$

is euclidean propagator for the free field with mass μ. Its asymptotic at large distancies is

$$D(|x|,m^2) \approx \frac{m^{\frac{1}{2}}}{|x|^{\frac{3}{2}}} \exp{(-m|x|)}. \tag{29}$$

Hence, **at large distances the contributions from the multiparticle states are less essential because they are more massive**, so we can write

$$< \phi(x)\phi(0) > \approx ZD(|x|,m^2) \tag{30}$$

if $Z \neq 0$. Therefore

$$(m^2 - \partial_\tau^2 - \vec{\nabla}^2) < \phi(x)\phi(0) >= 0 + O(exp(-2m|x|)), \tag{31}$$

(the δ-function in the right hand side is equal zero when $x \neq 0$). It means that at large distances the 2-point function satisfy KG equation with a mass m. Due to the cluster property, the same is true for the N-point correlation function:

$$(m^2 - \partial_\mu^2) < \phi(x)\phi(x_1)...\phi(x_{N-1}) > \approx 0 \ when \ |x-x_i| \to \infty. \tag{32}$$

This fact is important to understand the Hilbert space of states in the theory with interaction. It also allows to introduce the notion of asymptotic states.

In **KG theory** we have introduced creation-annihilation operators

$$\hat{A}_f = \imath \int_\Sigma d^3x (\frac{\partial \hat{\phi}}{\partial \tau} f - \hat{\phi} \frac{\partial f}{\partial \tau}), \tag{33}$$

where the integration is going over the 3-dim. surface Σ fixed by an arbitrary chosen value of τ. The function f satisfy KG equation

$$(m^2 - \partial_\mu^2) f(\vec{x}, \tau) = 0 \tag{34}$$

and $\hat{\phi}(\vec{x}, \tau)$ is a Heisenberg operator

$$\hat{\phi}(\vec{x}, \tau) = \exp{(\hat{H}\tau)} \phi(\vec{x}, 0) \exp{(-\hat{H}\tau)} \ , \ \frac{\partial \phi}{\partial \tau} = [\hat{H}, \hat{\phi}] \tag{35}$$

The operators $\hat{A}_f$ satisfy Heisenberg algebra commutation relations

$$[\hat{A}_f, \hat{A}_g] = \int_\Sigma d^3x (\frac{\partial f}{\partial \tau} g - f \frac{\partial g}{\partial \tau}), \tag{36}$$

where

$$\hat{A}_f = \imath \int_\Sigma d^3x (\frac{\partial \hat{\phi}}{\partial \tau} f - \hat{\phi} \frac{\partial f}{\partial \tau}). \tag{37}$$

To prove (36) notice first that for f fulfilling KG equation, we have the conservation low

$$\partial_\mu J_f^\mu = 0 \ , \ J_f^\mu = \partial^\mu \phi f - \phi \partial^\mu f. \tag{38}$$

Therefore the correlation function

$$< A_f(\tau) \phi(\vec{x}_0, \tau_0) > \tag{39}$$

does not change when we change the integration surface Σ in (39) by its smooth deformation $\tilde{\Sigma}$

$$\imath \int_{\Sigma} d^3x (\frac{\partial \phi}{\partial \tau} f - \phi \frac{\partial f}{\partial \tau}) = \imath \int_{\tilde{\Sigma}} d\Sigma^{\mu} (\partial_{\mu}\phi f - \partial_{\mu} f \phi) \tag{40}$$

We may in particular make a shift Σ along τ direction by a small value $\delta\tau$. Hence, we can write

$$\frac{\partial}{\partial \tau} < A_f(\tau)\phi(\vec{x}_0, \tau_0) >= 0 \ , \ \tau \neq \tau_0. \tag{41}$$

However, the correlation function (39) changes if during the deformation the surface $\tilde{\Sigma}$ crosses the point $(\vec{x}_0, \tau_0)$. Indeed, let us consider two surfaces $\Sigma_1 = \{(\vec{x}, \tau = \tau_1 = \tau_0 - \Delta)\}$ and $\Sigma_2 = \{(\vec{x}, \tau = \tau_2 = \tau_0 + \Delta)\}$. Then

$$\begin{aligned} &< A_f(\tau_2)\phi(\vec{x}_0, \tau_0) > - < A_f(\tau_1)\phi(\vec{x}_0, \tau_0) >= \\ &\qquad \imath \int_{\Sigma_0} d\Sigma_{\mu} < J^{\mu}_f(\vec{x}, \tau)\phi(\vec{x}_0, \tau_0) >, \end{aligned} \tag{42}$$

where the integration is going over the surface Σ_0 surrounding point $(\vec{x}_0, \tau_0)$. Using Gauss theorem we can write

$$\begin{aligned} &< A_f(\tau_2)\phi(\vec{x}_0, \tau_0) > - < A_f(\tau_1)\phi(\vec{x}_0, \tau_0) >= \\ &\qquad \imath \int_{D_0} d^4x \partial_{\mu} < J^{\mu}_f(x)\phi(\vec{x}_0, \tau_0) > . \end{aligned} \tag{43}$$

In KG theory

$$\partial_{\mu} < J^{\mu}_f(x)\phi(\vec{x}_0, \tau_0) >= -\delta^4(x - x_0) f(x) \tag{44}$$

because

$$< \phi(x)\phi(x_0) >= D(x - x_0) \ , \ where \ (m^2 - \partial^2_{\mu}) D(x - x_0) = \delta^4(x - x_0). \tag{45}$$

Therefore for the Heisenberg operators

$$< 0|[\hat{A}_f, \hat{\phi}(\vec{x}, \tau_0)]|0 >= -\imath f(\vec{x}_0, \tau_0), \tag{46}$$

which gives the commutator (36).

2.3. ASYMPTOTIC IN-STATES IN THE THEORY WITH INTERACTION.

Let us now consider similar operators in the theory with interaction

$$\hat{A}_f = \imath \int_\Sigma d^3x (\frac{\partial \hat{\phi}}{\partial \tau} f - \hat{\phi} \frac{\partial f}{\partial \tau}), \tag{47}$$

where the function f is still obeys KG equation (34) but $\hat{\phi}(x)$ is now Heisenberg operator of the theory with interaction.

Because the field $\hat{\phi}$ **does not satisfy KG eq.** the current

$$J_f^\mu = \partial^\mu \hat{\phi} f - \hat{\phi} \partial^\mu f \tag{48}$$

is not a conserved current and hence, the operator $\hat{A}_f$ is not an integral of motion so that $\hat{A}_f$, $\hat{A}_g$ **do not satisfy in general Heisenberg algebra commutation relations** (36).

But one can take the **special limit of these operators**

$$\hat{A}_f^{in} = \lim_{t \to -\infty} \hat{A}_f(\tau + \imath t). \tag{49}$$

Due to claster property it appears then that $\hat{A}_f^{in}$ does not depend on τ and satisfy Heisenberg algebra relations (36).

To show this let us consider the wave packets

$$f_{\vec{p},\epsilon}(\vec{x}, \tau) = \int \frac{d^3q}{(2\pi)^3} \exp{(\omega_q \tau - \imath \vec{q}\vec{x})} \exp{(-\frac{(\vec{q} - \vec{p})^2}{2\epsilon})}$$
$$\bar{f}_{\vec{p},\epsilon}(\vec{x}, \tau) = \int \frac{d^3q}{(2\pi)^3} \exp{(\omega_q \tau + \imath \vec{q}\vec{x})} \exp{(-\frac{(\vec{q} - \vec{p})^2}{2\epsilon})} \tag{50}$$

For the small ϵ the packets have the form of localized packets

$$f_{\vec{p},\epsilon}(\vec{x},\tau) = \exp{(\omega_p \tau - \imath \vec{p}\vec{x})} \psi_\epsilon(\vec{x} - \imath \vec{v}\tau),$$
$$\bar{f}_{\vec{p},\epsilon}(\vec{x},\tau) = \exp{(\omega_p \tau + \imath \vec{p}\vec{x})} \psi_\epsilon(\vec{x} + \imath \vec{v}\tau),$$
$$(51)$$

where

$$\psi_\epsilon(\vec{y}) = \exp{(-\frac{\epsilon}{2}\vec{y}^2)} \tag{52}$$

and $\vec{v} = \frac{\vec{p}}{2\omega_{\vec{p}}}$, $\omega_p = \sqrt{\vec{p}^2 + m^2}$.

In the limit $t \to -\infty$ the packet $f_{\vec{p},\epsilon}(\vec{x}, \tau + \imath t)$ **is centered at** $\vec{x} = \vec{v}t$**. It differs sufficiently from zero in some domain of the size** $\frac{1}{\sqrt{\epsilon}}$. Hence, in this limit the wave packets $f_{\vec{p}_i,\epsilon}(\vec{x}, \tau + \imath t)$ with different $\vec{v}_i = \frac{\vec{p}_i}{2\omega_{\vec{p}_i}}$ are well spaced.

Let us define

$$\hat{A}_{\vec{p},\epsilon} = \imath \int_\Sigma d^3x (\frac{\partial \hat{\phi}(\vec{x},\tau)}{\partial \tau} f_{\vec{p},\epsilon} - \hat{\phi}(\vec{x},\tau) \frac{\partial f_{\vec{p},\epsilon}}{\partial \tau})$$
$$\hat{A}^\dagger_{\vec{p},\epsilon} = \imath \int_\Sigma d^3x (\frac{\partial \hat{\phi}(\vec{x},\tau)}{\partial \tau} \bar{f}_{\vec{p},\epsilon} - \hat{\phi}(\vec{x},\tau) \frac{\partial \bar{f}_{\vec{p},\epsilon}}{\partial \tau}) \tag{53}$$

and consider the correlation function

$$< A^\dagger_{\vec{p}_1,\epsilon}(\tau_1 + \imath t) ... A_{\vec{p}_n,\epsilon}(\tau_n + \imath t) > \tag{54}$$

in the limit $t \to -\infty$. In view of (32), the correlators $< \phi(\vec{x}_1, \tau_1 + \imath t) ... \phi(\vec{x}_n, \tau_n + \imath t) >$ will satisfy in this limit the euclidean KG equation with a good precision because the (4-dimensional) euclidean distances $|x_i - x_j|$ become large. Therefore, the correlation function (54) will not be dependent on τ_i (unless they are still τ-ordered: $\tau_1 < ... < \tau_n$) similar to the KG theory. **This means that the operators**

$$\lim_{t \to -\infty} \hat{A}_{\vec{p},\epsilon}(\tau + \imath t) \equiv \hat{A}^{in}_{\vec{p},\epsilon} \tag{55}$$

do not depend on τ.

Moreover, one can repeat the analysis for KG theory and show that

$$[\hat{A}^{in}_{\vec{p}_1,\epsilon}, \hat{A}^{\dagger in}_{\vec{p}_2,\epsilon}] = \int d^3x (f_{\vec{p}_1,\epsilon} \partial_\tau \bar{f}_{\vec{p}_2,\epsilon} - \bar{f}_{\vec{p}_2,\epsilon} \partial_\tau f_{\vec{p}_1,\epsilon}) \tag{56}$$

In the limit $\epsilon \to 0$ we obtain the Heisenberg algebra in the standard form

$$\begin{aligned} [\hat{A}^{in}_{\vec{p}_1}, \hat{A}^{\dagger in}_{\vec{p}_2}] &= (2\pi)^3 2\omega_{\vec{p}_1} \delta(\vec{p}_1 - \vec{p}_2) \\ [\hat{A}^{in}_{\vec{p}_1}, \hat{A}^{in}_{\vec{p}_2}] &= [\hat{A}^{\dagger in}_{\vec{p}_1}, \hat{A}^{\dagger in}_{\vec{p}_2}] = 0 \end{aligned} \tag{57}$$

The operators $\hat{A}^{in}_{\vec{p}}, \hat{A}^{\dagger in}_{\vec{p}}$ are called **asymptotic in-operators**. One can create the space of asymptotic scatteringstates determining the vacuum state $|\Omega>$ which is determined by the equation

$$\hat{A}^{in}_{\vec{p}} |\Omega >= 0 \tag{58}$$

Then the state

$$|\vec{p}_1, ..., \vec{p}_n >_{in} \equiv A^{\dagger in}_{\vec{p}_1} ... A^{\dagger in}_{\vec{p}_n} |\Omega > \tag{59}$$

is called **asymptotic scattering state, in-state**.

Thus, we have

$$\begin{aligned} \hat{A}^{in}_{\vec{p}} &= \lim_{\epsilon \to 0} \lim_{t \to -\infty} \hat{A}_{\vec{p},\epsilon}(\tau + \imath t) \\ \hat{A}^{\dagger in}_{\vec{p}} &= \lim_{\epsilon \to 0} \lim_{t \to -\infty} \hat{A}^{\dagger}_{\vec{p},\epsilon}(\tau + \imath t) \end{aligned} \tag{60}$$

where

$$\hat{A}_{\vec{p},\epsilon}(\tau) = \frac{1}{\sqrt{Z}} \int d^3x (\partial_\tau \hat{\phi} f_{\vec{p},\epsilon} - \hat{\phi} \partial_\tau f_{\vec{p},\epsilon})$$
$$\hat{A}^\dagger_{\vec{p},\epsilon}(\tau) = \frac{1}{\sqrt{Z}} \int d^3x (\partial_\tau \hat{\phi} \bar{f}_{\vec{p},\epsilon} - \hat{\phi} \partial_\tau \bar{f}_{\vec{p},\epsilon}) \tag{61}$$

and $f_{\vec{p},\epsilon}$, $\bar{f}_{\vec{p},\epsilon}$ are the wave packets solutions of KG equation in euclidean space. **The asymptotic in-operators $\hat{A}^{in}_{\vec{p}}$, $\hat{A}^{\dagger in}_{\vec{p}}$ satisfy canonical commutation relations of creation-annihilation operators (Heisenberg algebra) and the asymptotic in-states**

$$|\vec{p}_1, ..., \vec{p}_n >_{in} \equiv A^{\dagger in}_{\vec{p}_1} ... A^{\dagger in}_{\vec{p}_n} |\Omega > \tag{62}$$

constitute an in- basis in Hilbert space $\mathcal{H}$ of the theory.

2.4. ASYMPTOTIC OUT-STATES IN THE THEORY WITH INTERACTION.

One can introduce another basis of out-states in the Hilbert space $\mathcal{H}$:

$$|\vec{q}_1, ..., \vec{q}_n >_{out} \equiv A^{\dagger out}_{\vec{q}_1} ... A^{\dagger out}_{\vec{q}_n} |\Omega >, \tag{63}$$

where $\hat{A}^{\dagger out}_{\vec{q}}$, $\hat{A}^{out}_{\vec{q}}$ are given by taking another limit

$$\hat{A}^{out}_{\vec{p}} = \lim_{\epsilon \to 0} \lim_{t \to \infty} \hat{A}_{\vec{p},\epsilon}(\tau + \imath t),$$
$$\hat{A}^{\dagger out}_{\vec{p}} = \lim_{\epsilon \to 0} \lim_{t \to \infty} \hat{A}^\dagger_{\vec{p},\epsilon}(\tau + \imath t). \tag{64}$$

One can check completely similar that they do not depend on τ and satisfy **canonical commutation relations of creation-annihilation operators (Heisenberg algebra) and the asymptotic out-states**

$$|\vec{p}_1, ..., \vec{p}_n >_{out} \equiv A^{\dagger out}_{\vec{p}_1} ... A^{\dagger out}_{\vec{p}_n} |\Omega > \tag{65}$$

constitute an out- basis in Hilbert space $\mathcal{H}$ of the theory.