Lecture 12.

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1. Regularization methods.

If the theory is renormalizable one can develop renormalizable perturbation theory. According to this program one has to start from the action in its bare form, as

$$A = \int d^4x(\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4), \tag{1}$$

which is equiped with some regularization or cutoff, i.e. with some modification of the theory at the momenta $\geq \Lambda$ which makes all the perturbative theory integrals convergent. There are many possible implementations of the cutoff. Let us consider some more frequently used ones.

1.1. LATTICE REGULARIZATION.

This regularization has been discussed already in relation with the definition of the functional integral. In this approach continous space d-dimensional (euclidean) space is replaced by d-dimensional, say hypercubic, lattice with some lattice spacing Δ which plays the role of inverse cutoff parameter Λ^{-1}.

$$x \to x_n = \Delta \sum_{a=1}^{d} n^a e_a, \tag{2}$$

where e_a is the unit vector in the direction a and $n^1, ..., n^d$ are integers. The lattice action is obtained by replacing derivatives by finite differences.

$$A_{lat} = \Delta^d \sum_{x\in\Delta Z^d} (\frac{1}{2} \sum_{a=1,...,d} (\frac{\phi_0(x+\Delta e_a) - \phi_0(x)}{\Delta})^2 + \frac{m_0^2}{2}\phi_0(x) + V(\phi_0(x))) \tag{3}$$

and the integration measure in the functional integral is taken as

$$[D\phi_0] \to \prod_{x\in\Delta Z^d} d\phi_0 \tag{4}$$

This regularization makes also sense nonperturbatively which is an advantage of this method.

1.2. PROPER-TIME REGULARIZATION.

We have seen before that the momentum-space propagator admits Schwinger's proper-time representation

$$\tilde{D}(k) = \int_0^\infty d\tau \exp\left(-\tau(k^2+m^2)\right) \tag{5}$$

where τ is interprated as **renormalized lenght** of the path of relativistic particle in the euclidean space. **One can exclude the pathes which are too short by replacing** $\tilde{D}(k) \to \tilde{D}_\Lambda(k)$

$$\tilde{D}_\Lambda(k) = \int_{\frac{1}{\Lambda^2}}^\infty d\tau \exp\left(-\tau(k^2+m^2)\right) = \frac{\exp\left(-\frac{k^2+m^2}{\Lambda^2}\right)}{k^2+m^2}. \tag{6}$$

Notice that this is a particular case of the regularized propagator we considered before

$$\frac{1}{k^2+m^2}\Phi(\frac{k^2}{\Lambda^2}) \tag{7}$$

with $\Phi(x) \approx \exp(-x)$.

1.3. PAULI-VILLARS REGULARIZATION.

This is another version of (7) with

$$\Phi_{PV}(\frac{k^2}{\Lambda^2}) = \frac{\Lambda^2}{k^2+\Lambda^2} \tag{8}$$

1.4. DIMENSIONAL REGULARIZATION.

This is the most technicaly advanced (although perhaps the least physicaly transparent) regularization method. Its advantage is that **it usualy**

preserves main important symmetries (gauge symmetry for example) and simplifies calculations significantly.

The idea is this. We have already observed that lowering the space dimensions generally improves the large-momentum convergence of Feynman diagrams. Suppose that for d sufficiently low the momentum integral associated with given diagram is convergent without any additional cutoff. Suppose in addition that we have managed to calculate this integral as an **analytic functions of** d. Then we can analytically continue the result to the physical value $d = 4$, the 4-dimensional divegences manifest itselves as a singularities (poles) in the variable d at $d = 4$.

Doing so we use the translation invariance of the integrals:

$$\int d^d k F(k+p) = \int d^d k F(k) \tag{9}$$

and the scale invariance

$$\int d^d k F(Ck) = |C|^{-d} \int d^d k F(k) \tag{10}$$

1.5. EXAMPLE.

Consider the integral

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^2}. \tag{11}$$

This integral is convergent for $d < 4$. For any $d < 4$ it can be transformed as follows

$$\int_0^\infty t dt \int \frac{d^d k}{(2\pi)^d} \exp\left(-t(k^2+m^2)\right) = \int_0^\infty t dt (4\pi t)^{-\frac{d}{2}} \exp\left(-m^2 t\right). \tag{12}$$

(Recall Gaussian integral:

$$\int_{-\infty}^{\infty} dx \exp(-ax^2) = (\frac{\pi}{a})^{\frac{1}{2}}. \tag{13}$$

)

By the definition of Euler's Gamma function

$$\Gamma(z) = \int_0^{\infty} dt t^{z-1} \exp(-t) \tag{14}$$

we obtain the integral (11) for $d < 4$ is equal

$$\frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}} \Gamma(2 - \frac{d}{2}). \tag{15}$$

As it is known the Gamma function $\Gamma(z)$ is analytic (meromorphic) function of z with poles at $z = 0, -1, -2, ...$ (and $\Gamma(1) = 1$). In particular, the above expression has pole at $d = 4$, i.e. **exactly where the integral diverges logarithmically**. For $d = 4 - \epsilon$, $\epsilon \to 0$

$$\Gamma(2 - \frac{d}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon) \tag{16}$$

where $\gamma = 0.5772...$ is the Euler constant.

1.6. THE INTERPLAY BETWEEN DIMENSIONAL CONTINUATION AND CUTOFF REGULARIZATIONT.

It is interesting to explain with this example the interplay between dimensional continuation and cutoff regularization. If we regularize the propagators in (11), say by proper-time regularization (which amounts to replacing $\int_0^{\infty} dt \to \int_{\frac{1}{\Lambda^2}}^{\infty} dt$ in (12)) **the integral (11) becomes regular**

function of d for all d. In the neighborhood of $d=4$ one would have instead of (12)

$$\frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\int_{\frac{m^2}{\Lambda^2}}^{\infty} d\tau\tau^{1-\frac{d}{2}}\exp\left(-\tau\right)=$$
$$\frac{(m^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\frac{2}{4-d}(\tau^{2-\frac{d}{2}}\exp\left(-\tau\right)|_{\frac{m^2}{\Lambda^2}}^{\infty}+\int_{\frac{m^2}{\Lambda^2}}^{\infty} d\tau\tau^{2-\frac{d}{2}}\exp\left(-\tau\right))=$$
$$\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{2}{4-d}((m^2)^{\frac{d}{2}-2}(\int_0^{\infty}-\int_0^{\frac{m^2}{\Lambda^2}})\tau^{2-\frac{d}{2}}\exp\left(-\tau\right)d\tau-(\Lambda^2)^{\frac{d}{2}-2}\exp\left(-\frac{m^2}{\Lambda^2}\right)). \tag{17}$$

For $\epsilon=4-d<<1$ and $m^2<<\Lambda^2$ the cutoff integral becomes

$$\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{2}{4-d}((m^2)^{\frac{d}{2}-2}\int_0^{\infty}\tau^{2-\frac{d}{2}}\exp\left(-\tau\right)d\tau-(\Lambda^2)^{\frac{d}{2}-2}\exp\left(-\frac{m^2}{\Lambda^2}\right))=$$
$$\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{2}{4-d}((m^2)^{\frac{d}{2}-2}-(\Lambda^2)^{\frac{d}{2}-2}). \tag{18}$$

If at fxed $d<4$ we send the cutoff momentum Λ to ∞ we obtain the dimensionally continued expression with the pole at $d=4$. If at fixed Λ we take the limit $d\to 4$ we get a finite result with $\ln\Lambda^2$ replacing the pole in $4-d$.

Indeed, one can find the limit of $\frac{2}{4-d}((m^2)^{\frac{d}{2}-2}-(\Lambda^2)^{\frac{d}{2}-2})$ when $\epsilon=4-d\to 0$:

$$\lim_{\epsilon\to 0}\frac{2}{\epsilon}((m^2)^{\frac{-\epsilon}{2}}-(\Lambda^2)^{\frac{-\epsilon}{2}})=\lim_{\epsilon\to 0}\frac{2}{1}\frac{d}{d\epsilon}(((m^2)^{\frac{-\epsilon}{2}}-(\Lambda^2)^{\frac{-\epsilon}{2}})=$$
$$2\lim_{\epsilon\to 0}\frac{d}{d\epsilon}(\exp\left(-\frac{\ln m^2}{2}\epsilon\right)-\exp\left(-\frac{\ln\Lambda^2}{2}\epsilon\right))=$$
$$2\lim_{\epsilon\to 0}(-\frac{\ln m^2}{2}\exp\left(-\frac{\ln m^2}{2}\epsilon\right)+\frac{\ln\Lambda^2}{2}\exp\left(-\frac{\ln\Lambda^2}{2}\epsilon\right))=$$
$$\ln\left(\frac{\Lambda^2}{m^2}\right). \tag{19}$$

With this understanding one can often use dimensional continuation to do the calculations efficiently and then interpret the results in terms of cutoff regularization. Instead of giving the bare parameters m_0^2, λ_0, Z dependence on the cutoff momentum Λ within the framework of dimensional regularization **one can adjust their dependence on $\epsilon = 4 - d$, such that the correlation functions of renormalized fields ϕ have finite limit** $d \to 4$.

2. Renormalization schemes.

Counterterms must cancell divergences of diagrams **but this requirement does not fixes the counterterms completely because we can arbitrarily change the finite parts of counterterms**. For example ϕ can be additionally renormalized by a finite value

$$\phi \to Z_{fin}^{\frac{1}{2}} \phi. \tag{20}$$

One can similarly renormalize the mass m and the coupling constant λ by a finite values leading to another renormalized perturbation theory where all divergences are absorbed again. **The different renormlized perturbation theories are called renormalization schemes**.

If ϕ, m^2, λ and $\tilde{\phi}$, $\tilde{m}^2$, $\tilde{\lambda}$ are parameters in two different renormalization schemes, the corresponding proper vertices are related by

$$\Gamma^n(p_i|m^2, \lambda) = Z_{fin}(m^2, \lambda)^{-\frac{n}{2}} \Gamma^n(p_i|\tilde{m}^2(m^2, \lambda), \tilde{\lambda}(m^2, \lambda)). \tag{21}$$

These **two different renormalization schemes are two different perturbative descriptions of the same QFT. The parameters m^2, λ can be understood as a coordinates in the space of ϕ^4 QFT's**.

3. Renormalization conditions.

3.1. RENORMALIZATION IN ϕ^4 (REMINDER).

Recall our **renormalization program** for the ϕ^4 theory.

1.

Start with the action

$$A = \int d^4x (\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4) \tag{22}$$

containing the bare field, bare mass, and bare coupling constant.

2.

Introduce some cutoff with a cutoff momentum Λ (this can be done many ways).

3.

We expect that one can give parameters m_0^2, λ_0, and the field renormalization constant Z certain dependence on Λ:

$$m_0^2 = m_0^2(\Lambda)\ ,\ \lambda_0 = \lambda_0(\Lambda)\ ,\ Z = Z(\Lambda) \tag{23}$$

such that the correlation functions of the renormalized field

$$\phi = Z^{-\frac{1}{2}}(\Lambda)\phi_0 \tag{24}$$

have finite $\Lambda \to \infty$ limit.

We reformulated this program in terms of **renormalized perturbation theory**. Namely, we write the action with counterterms

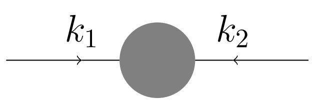
$$A = \int d^4x (\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4) \tag{25}$$

where m is an actual mass and λ is suitably defined finite coupling constant. The identity with the original action implies

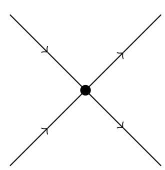
$$1 + \delta Z = Z \ , \ m^2 + \delta m^2 = Z m_0^2 \ , \ \lambda + \delta\lambda = Z^2 \lambda_0. \tag{26}$$

The renormalized perturbation theory is the expansion in renormalized coupling constant λ with the following Faynman rules:

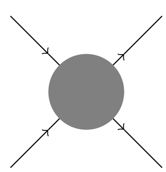
$$= \frac{1}{k^2 + m^2} \tag{27}$$



$$= (k_1^2 \delta Z - \delta m^2)(2\pi)^4 \delta(k_1 + k_2) \tag{28}$$



$$= \lambda \tag{29}$$



$$= \delta\lambda \tag{30}$$

Therefore we assume the counterterm coefficients themselves depend perturbatively (i.e. as power series) on λ:

$$\begin{aligned} \delta Z &= Z_1 \lambda + Z_2 \lambda^2 + ... \\ \delta m^2 &= b_1 \lambda + b_2 \lambda^2 ... \\ \delta\lambda &= a_1 \lambda + a_2 \lambda^2 + ... \end{aligned} \tag{31}$$

In the cutoff regularization the coefficients Z_i, b_i, a_i **depend on** Λ, **while in the dimensional regularization the coefficients** Z_i, b_i, a_i **depend on** $\epsilon = 4 - d$.

3.2. PHYSISCAL MASS AND NORMALIZATION OF FIELD.

The renormalization conditions define the relation between the parameters m^2, λ **and** ϕ **with the physical values**. It is natural for example to choose parameter m^2 as a square of physical mass. Namely, we have seen that $\Gamma^2(p^2)$ becomes zero at some point $p^2 = -m^2$ so that the physical mass is given by this value of p^2.

It does not fix the normalization of the field ϕ. It is convenient to fix the field normalization demanding

$$\Gamma^2(p^2) = p^2 + m^2 + O((p^2 + m^2)^2), \ when \ p^2 + m^2 \to 0. \tag{32}$$

In other words **we require the 2-point correlation function**

$$W^2(p^2) = \frac{1}{p^2 + m^2} + O(1) \tag{33}$$

has pole at $p^2 = -m^2$ **and the residue at this point is equal 1**.

3.3. COUPLING CONSTANT NORMALIZATION.

The coupling constant has to be normalized also. It can be done by fixing the value of vertex Γ^4. The standart way to do that is to choose

$$\Gamma^4(p, p, -p, -p)|_{p^2=-m^2} = \lambda. \tag{34}$$

The equations (32), (34) is one of the possibilities to choose the **Renormalization Scheme** (RS).

One the same footing we could choose another conditions

$$\Gamma^{(2)}(p^2) = \tilde{m}^2 + p^2 + O(p^4),$$
$$\Gamma^{(4)}(0,0,0,0) = \tilde{\lambda}, \tag{35}$$

where $\tilde{m}$ is not a physical mass ($p^2 = -m^2$ is not a solution of the eq. $\tilde{m}^2 + p^2 + O(p^4) = 0$).

4. Renormalization of ϕ^4 at 1 loop by dimensional regularization.

Now we renormalize ϕ^4 theory at one loop fixing the RS by the equations (35) and using dimensional regularization.

4.1. Γ^2 *RENORMALIZATION AT 1 LOOP.*

We start from Γ^2 renormalization.

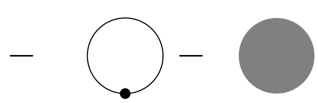
This vertex can be written as follows

$$\Gamma^2(p^2) = p^2 + \tilde{m}^2 + \Sigma(p^2) \tag{36}$$

The normalization conditions $\Gamma^2(0) = \tilde{m}^2$, $\frac{d\Gamma^2}{dp^2}|_{p^2=0} = 1$ takes the form

$$\Sigma(0) = 0, \ \frac{d\Sigma(p^2)}{dp^2}|_{p^2=0} = 0. \tag{37}$$

At one loop we have two contributions

$\Sigma(p) =$ 

(38)

where the counterterm diagram contributes to the renormalization of field and mass

$$-(Z_1 p^2 + b_1). \tag{39}$$

The first diagram does not depend on p^2, therefore

$$Z_1 = 0 \tag{40}$$

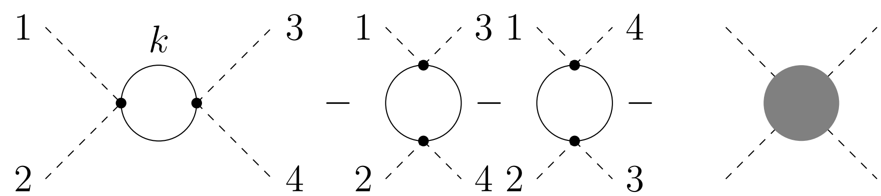
and

$$b_1 = -\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\tilde{m}^2} = -\frac{(\tilde{m}^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\Gamma(1-\frac{d}{2}). \tag{41}$$

4.2. Γ^4 *RENORMALIZATION AT 1 LOOP.*

The second condition from (35) will be fulfilled at the leading order if we set $a_1 = 0$ in (31). Thus $\delta\tilde{\lambda} = O(\tilde{\lambda}^2)$ and

$$\Gamma^4(p_1,p_2,p_3,p_4) = \tilde{\lambda} -$$



(42)

where the last, counterterm diagram is equal to

$$-\tilde{\lambda}^2 a_2 \tag{43}$$

The first second and third contributions are given by

$$\frac{\tilde{\lambda}^2}{2}(I(p_{12}^2)+I(p_{13}^2)+I(p_{14}^2)),\ \textit{where}$$
$$I(p^2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+\tilde{m}^2)((p+k)^2+\tilde{m}^2)}. \tag{44}$$

Hence, in our RS the normalization condition takes the form

$$\left(\frac{\tilde{\lambda}^2}{2}(I(p_{12}^2)+I(p_{13}^2)+I(p_{14}^2))-\tilde{\lambda}^2 a_2\right)|_{p_i=0}=0 \Leftrightarrow$$
$$a_2=\frac{3}{2}I(0). \tag{45}$$

Then

$$\Gamma^4(p_i)=\tilde{\lambda}-\frac{\tilde{\lambda}^2}{2}(I_r(p_{12}^2)+I_r(p_{13}^2)+I_r(p_{14}^2))+O(\tilde{\lambda}^3)$$
$$where$$
$$I_r(p^2)=I(p^2)-I(0). \tag{46}$$

We now show that $I_r(p^2)$ has finite limit at $d=4$. Using Feynman parametrization

$$\frac{1}{AB}=\int_0^1 du\frac{1}{(uA+(1-u)B)^2}, \tag{47}$$

one can write the integrals above as

$$I(p^2)=\int_0^1 du\int\frac{d^dk}{(2\pi)^d}\frac{1}{(\tilde{m}^2+k^2+2u(kp)+up^2)^2}=$$
$$\int_0^1 du\int\frac{d^dk}{(2\pi)^d}\frac{1}{(\tilde{m}^2+u(1-u)p^2+(k+up)^2)^2}=$$
$$\int_0^1 du\int\frac{d^dk}{(2\pi)^d}\frac{1}{(\tilde{m}^2+u(1-u)p^2+k^2)^2}=$$
$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 du(\tilde{m}^2+u(1-u)p^2)^{\frac{d}{2}-2} \tag{48}$$

(see (11)-(15)). In the limit $d\rightarrow 4$ the integral has a pole because $\Gamma(2-\frac{d}{2})=$

$\frac{2}{\epsilon} - \gamma + ...$ But the renormalized integral I_r is finite in the limit $d \to 4$:

$$I_r(p^2) = -\frac{1}{(4\pi)^2}\int_0^1 du \log\left(1 + u(1-u)\frac{p^2}{\tilde{m}^2}\right) =$$
$$-\frac{1}{16\pi^2}\left(\sqrt{\frac{p^2+4\tilde{m}^2}{p^2}} \log\left(\frac{\sqrt{p^2+4\tilde{m}^2}+\sqrt{p^2}}{\sqrt{p^2+4\tilde{m}^2}-\sqrt{p^2}}\right) - 2\right). \tag{49}$$

It gives

$$a_2 = \frac{3}{2}I(0) = \frac{3}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(\tilde{m}^2)^{\frac{d}{2}-2}. \tag{50}$$

One can show that the counterterms fixing procedure we have just carried out in the leading approximation to make the vertices $\Gamma^{(n)}$ finite in the limit $d \to 4$, can be extended to the all perturbation series (see Zinn-Zustin's book).

5. Chebishev's regularization in the RS.

Now we apply Chebishev's regularization to the same RS working at $d = 4$ with the cutoff Λ. We also assume that $\tilde{m} = 0$.

5.1. CHEBISHEV'S POLYNOMIALS.

The generation function for Chebishev's polynomials is

$$\frac{1}{1-2xz+z^2} = \sum_n P_n(x)z^n, \;\; |z| < 1. \tag{51}$$

They constitute the orthonormal basis of functions

$$\int_0^\pi P_n(\cos(\psi))P_m(\cos(\psi))\sin^2(\psi)d\psi = \frac{\pi}{2}\delta_{n,m}. \tag{52}$$

5.2. CHEBISHEV'S REGULARIZATION OF $\Gamma^{(4)}$.

We can use the properties of Chebishev's polynomials to evaluate 4-dimensional integral

$$I(p) = \int_0^{|k|=\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(p-k)^2} = \frac{1}{16\pi^2}(\log\frac{\Lambda^2}{p^2} + 1). \tag{53}$$

Indeed

$$\frac{1}{(k-p)^2} = \frac{1}{k^2 - 2|k||p|\cos(\psi) + p^2} = \\ \frac{1}{p^2}\sum_{n=0} P_n(\cos(\psi))(\frac{|k|}{|p|})^n,\ \ |k| < |p| \\ \textit{or} \\ \frac{1}{k^2}\sum_{n=0} P_n(\cos(\psi))(\frac{|p|}{|k|})^n,\ \ |k| > |p|. \tag{54}$$

Then

$$(2\pi)^4 I(p) = \int_0^{|k|=\Lambda} d|k||k|^3 \frac{\sin^2(\psi)\sin(\theta)d\psi d\theta d\phi}{k^2(k^2 - 2|k||p|\cos(\psi) + p^2)} = \\ 4\pi[\int_0^{|p|} \frac{|k|^3 d|k|}{k^2 p^2} \int d\psi(\sum_n P_n(\cos(\psi))(\frac{|k|}{|p|})^n \sin^2(\psi)) + \\ \int_{|p|}^{\Lambda} \frac{|k|^3 d|k|}{k^4} \int d\psi(\sum_n P_n(\cos(\psi))(\frac{|p|}{|k|})^n \sin^2(\psi))] = \\ 2\pi^2(\int_0^{|p|} \frac{|k|d|k|}{p^2} + \int_{|p|}^{\Lambda} \frac{d|k|}{|k|}) = 2\pi^2(\log\frac{\Lambda}{|p|} + \frac{1}{2}) = \\ \pi^2(\log\frac{\Lambda^2}{p^2} + 1). \tag{55}$$

Using this result one can write the Γ^4 in terms of bare coupling constant

$$\Gamma^{(4)}(p_i) = \lambda_0 - \frac{\lambda_0^2}{2}(I(p_{12}) + I(p_{13}) + I(p_{23})) =$$
$$\lambda_0 - \frac{\lambda_0^2}{32\pi^2}(\log(\frac{\Lambda^2}{p_{12}^2}) + \log(\frac{\Lambda^2}{p_{13}^2}) + \log(\frac{\Lambda^2}{p_{13}^2})) + ... \tag{56}$$

Now we consider more general renormalization conditions fixing arbitrary some scale μ and setting

$$\Gamma^{(4)}(p_i)|_{p_i^2=\mu^2} = \lambda(\mu). \tag{57}$$

It gives the relation

$$\lambda_0 - \frac{3\lambda_0^2}{32\pi^2}\log(\frac{\Lambda^2}{\mu^2}) = \lambda(\mu) + O(\lambda^3). \tag{58}$$

5.3. TWO POINTS OF VIEW FOR BARE COUPLING CONSTANT.

There are two points of view on the relation (58).

1. $\lambda_0(\Lambda)$ depends on the cutoff Λ in such a way that coupling constant λ does not depend on Λ. We followed just this point of view when the renormalized perturbation theory was formulated in such a way to have finite $\Lambda \to \infty$ limit for the renormalized correlation functions.

2. Coupling constant $\lambda(\mu)$ depends on the scale μ in such a way that λ_0 does not (depend on μ).

Using this point of view one can define a function

$$\mu\frac{d\lambda(\mu)}{d\mu} = \beta(\lambda, \frac{\Lambda}{\mu}) = \beta(\lambda). \tag{59}$$

In the case at hand we find at the leading order:

$$\frac{d\lambda(\mu)}{d\log(\mu)} = \frac{3}{16\pi^2}\lambda(\mu)^2. \tag{60}$$

6. Callan-Symanzik equation for massles ϕ^4.

Let us fix the normalization conditions as

$$\Gamma^{(2)}(p^2)|_{p^2=\mu^2} = 0,$$
$$\frac{d\Gamma^{(2)}}{dp^2}|_{p^2=\mu^2} = 1,$$
$$\Gamma^{(4)}(p_i)|_{p_i^2=\mu^2} = \lambda \tag{61}$$

and consider the connected N-points correlation function

$$G^{(N)} =< \phi ... \phi >_c = Z^{-\frac{N}{2}} < \phi_0 ... \phi_0 >_c . \tag{62}$$

How does this function change when we shift the scale μ?

To answer this question let us rewrite this equation in the opposit form

$$< \phi_0 ... \phi_0 >_c = Z^{\frac{N}{2}} G^{(N)}. \tag{63}$$

The correlation function of bare fields depends on the bare coupling constant λ_0 and is determined at some cutoff Λ. **It obviously does not depend on the renormalization scale μ.** But **the correlation function on the right hand side does depend on μ because we renormalized the theory by taking the limit $\Lambda \to \infty$**, replacing the bare coupling constant λ_0 with the renormalized coupling constant λ as well as replacing the bare fields ϕ_0 by the rescaled ones ϕ.

From the other hand, under the shift $\mu \to \mu + \delta\mu$ we have $\lambda \to \lambda + \delta\lambda$, $\phi \to (1 + \delta\eta)\phi$ and

$$G^{(N)} \to (1 + N\delta\eta) G^{(N)}. \tag{64}$$

Hence, one can write

$$\delta G^{(N)} = \frac{\partial G^{(N)}}{\partial \mu}\delta\mu + \frac{\partial G^{(N)}}{\partial \lambda}\delta\lambda = \delta\eta N G^{(N)}. \tag{65}$$

It is convenient to define dimensionless parameters

$$\beta = \frac{\mu}{\delta\mu}\delta\lambda,\ \gamma = -\frac{\mu}{\delta\mu}\delta\eta. \tag{66}$$

Substituting them into (65) and multiplying on $\frac{\mu}{\delta\mu}$ gives Callan-Symanzik equation

$$(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + N\gamma(\lambda))G^N(x_1, ...x_N; \lambda) = 0, \tag{67}$$

where $\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu}$, $\gamma = \mu\frac{\partial\eta}{\partial\mu}$.

The Callan-Symanzik equation (67) says that there are functions $\beta(\lambda)$ and $\gamma(\lambda)$ related to the coupling constant and field normalization shifts which compensate the scale shift in the renormalization conditions (61).