Lecture 11.

Plan.

1. Divergencies in ϕ^4 theory (continuation).

1.1. Divergence of Σ and mass renormalization (reminder).

1.2. Superficial degree of divergence of diagrams in ϕ^4 theory (remainder).

1.3. Divergence of Γ^4, bare coupling constant and renormalization of λ (remainder).

1.4. Field renormalization.

1.5. Renormalization program.

2. Divergences and renormalization in general field theories of ϕ.

2.1. Superficial degree of divergence of diagrams.

2.2. Analysis of divergent diagrams for different interaction terms.

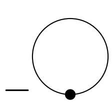
2.3. Coupling constants dimensions and renormalizability.

1. Divergencies in ϕ^4 theory (continuation).

1.1. DIVERGENCE OF Σ AND MASS RENORMALIZATION (reminder).

We saw in the last lecture that due to the diagram:

$$\tilde{\Sigma}(p) = -$$



$$= \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \tag{1}$$

the first order contribution to $\tilde{\Gamma}^2(p)$ is badly divergent at short distances $(k \to \infty)$.

To make this integral finite we introduced the cutoff Λ, by replacing

$$\frac{1}{k^2+m^2} \to \frac{1}{k^2+m^2} \Phi(\frac{k^2}{\Lambda^2}) \tag{2}$$

where the function $\Phi(\frac{k^2}{\Lambda^2})$ tends to zero fast enough as $k^2 \to \infty$ to make the integral convergent. At the same time $\Phi \approx 1$ for the values $k^2 << \Lambda^2$. Using this regularization we obtained at the first order

$$\tilde{\Gamma}^2(p) = p^2 + m^2 + \frac{\lambda m^2}{2} F(\frac{\Lambda^2}{m^2}) \tag{3}$$

and realized then that no matter what Φ and Λ were, the parameter m was not actual mass of the particles.

For this reason, we changed the notations, denoting by m_0^2 (bare mass) the coefficient in front of ϕ^2 in the action of ϕ^4 theory and **assumed then that the bare mass parameter m_0^2 must be dependent on Λ in such a way that the actual mass**

$$m^2 = m_0^2 + \frac{\lambda m_0^2}{2} F(\frac{\Lambda^2}{m_0^2}) + O(\lambda^2) \tag{4}$$

remained finite as $\Lambda \to \infty$. Then, to this order we found that

$$\tilde{\Gamma}^2(p) = p^2 + m^2 + O(\lambda^2) \tag{5}$$

has finite limit, which was independent on Φ.

1.1.2. MASS COUNTERTERM.

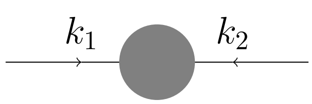
This idea was reformulated then in terms of **mass counterterm**. Namely, the initial action was rewritten as

$$A = \int d^4x(\frac{1}{2}(\partial\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4) =$$
$$\int d^4x(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4) = A_0 + A_I$$
$$A_0 = \int d^4x(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2)\ ,\ A_I = \int d^4x(\frac{\delta m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4) \tag{6}$$

where A_I was treated as a perturbation. The propagator had also been changed:

$$= \frac{1}{k^2+m^2}\ ,\ \mathit{not}\ \frac{1}{k^2+m_0^2} \tag{7}$$

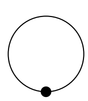
However, we got an additional vertex (mass counterterm):



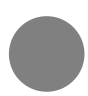
$$= -\delta m^2(2\pi)^4\delta(k_1+k_2) \tag{8}$$

so that in this modified perturbation theory we got

$$\tilde{\Sigma}(p) =$$



$$+$$



$$= \delta m^2 + \frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\Phi(\frac{k^2}{\Lambda^2}) =$$
$$\delta m^2 + \frac{\lambda m^2}{2}F(\frac{\Lambda^2}{m^2}). \tag{9}$$

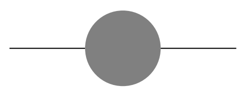
The value of the counterterm was fixed by the relation

$$\delta m^2 = -\frac{\lambda m^2}{2}F(\frac{\Lambda^2}{m^2}) + O(\lambda^2) \tag{10}$$

because of we insisted **that** m **was actual mass and it could not depends on the cutoff** Λ.

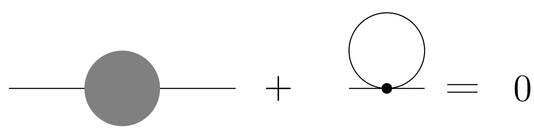
1.1.3. RENORMALIZED PERTURBATION THEORY.

Thus, we found that by the choice of δm^2 in the modified perturbation theory, the counterterm diagram



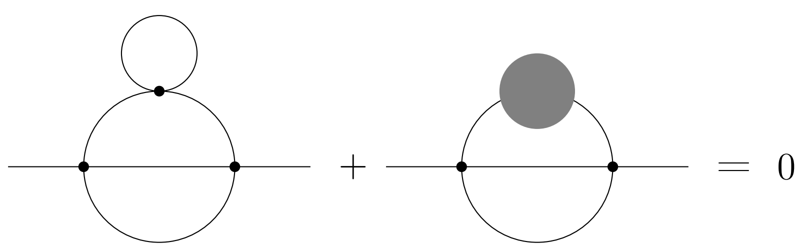
(11)

exactly cancels the cutoff dependent buble diagram



(12)

so that the dependence on Λ **and** Φ **disappeared**. We also noticed that this cancellation occurs inside the more complicated diagrams like:



(13)

1.2. SUPERFICIAL DEGREE OF DIVERGENCE OF DIAGRAMS IN ϕ^4 *MODEL.*

We introduced **superficial** degree of divergence of diagram as

$$D = 4I - 2P, \tag{14}$$

where I is a number of integrations and P is a number of internal propagators of a diagram. The integral diverges if $D \geq 0$. For ϕ^4 theory we found the relations

$$2P = 4V - n$$
$$4I = 4V - 2n + 4, \tag{15}$$

where n is a number of external legs of a diagram. Then we found

$$D = 4V - 2n + 4 - 4V + n = 4 - n. \tag{16}$$

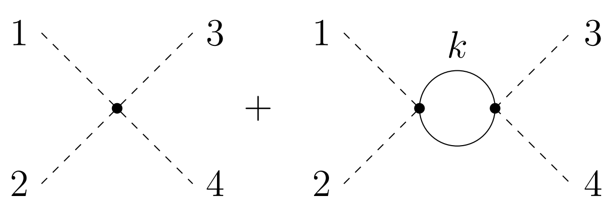
We saw that the surface degree of divergence depends only on the number of external lines n only. Thus, for $n > 4$ $(n = 6, 8, ...)$ $D < 0$ and the diagrams are superficialy convergent.

1.3. DIVERGENCE OF Γ^4, BARE COUPLING CONSTANT AND RENIRMALIZATI

According to (16) the diagrams for Γ^4 have the superficial divergence $D = 0$. It means that these diagrams are **logarithmically divergent**.

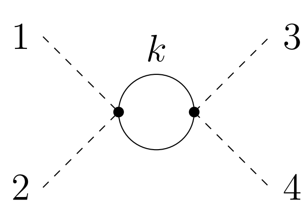
We considered the simplest diagram contributing to Γ^4

$$-\tilde{\Gamma}^4 =$$



$+\ permutations\ of\ 2, 3, 4 \quad (17)$

For the diagram



(18)

we found that corresponding integral reduced to

$$\frac{\lambda^2 2\pi^2}{2(2\pi)^4}\int\frac{d|k||k|^3}{|k|^4}=\frac{\lambda^2}{16\pi^2}\int\frac{d|k|}{|k|} \tag{19}$$

as soon as $|k| >> |p_1+p_2|$ and $|k| >> m$. Then we introduced the cutoff again

$$\frac{1}{k^2+m^2}\to\frac{1}{k^2+m^2}\Phi(\frac{k^2}{\Lambda^2}) \tag{20}$$

and found that the diagram has finite contribution

$$\frac{\lambda^2}{16\pi^2}(\ln\frac{\Lambda^2}{m^2}+f(p_1+p_2)) \tag{21}$$

(where $f(p)$ has finite limit as $\Lambda\to\infty$).

As a result we came to the finite expression for (17):

$$\tilde{\Gamma}^4=\lambda-\frac{\lambda^2}{16\pi^2}(3\ln\frac{\Lambda^2}{m^2}+f(p_1+p_2)+f(p_1+p_3)+f(p_1+p_4)). \tag{22}$$

Then we realized that in this order **the Λ-dependent part could be absorbed into some redefinition of the coupling constant**.

We made this absorbtion assuming that bare constant λ_0 was not a constant but depends on the cutof Λ in such a way that $\tilde{\Gamma}^4$ was finite when $\Lambda\to\infty$:

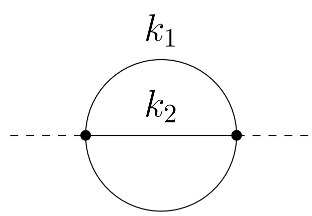
$$\lambda_0(\Lambda)=\lambda+\frac{3\lambda^2}{16\pi^2}(\ln(\frac{\Lambda^2}{m^2})+C). \tag{23}$$

The new parameter λ was called **renormalized coupling constant** and C was an arbitrary number. Then we obtained

$$\tilde{\Gamma}^4 = \lambda_0 - \frac{\lambda_0^2}{16\pi^2}(3\ln\frac{\Lambda^2}{m^2} + \tilde{f}(p_i)) + O(\lambda_0^4) =$$
$$\lambda + \frac{3\lambda^2}{16\pi^2}(\ln\frac{\Lambda^2}{m^2} + C) - \frac{\lambda^2}{16\pi^2}(3\ln\frac{\Lambda^2}{m^2} + \tilde{f}(p_i)) + O(\lambda^4). \qquad (24)$$

1.4. FIELD RENORMALIZATION.

It turns out that renormalizations of mass and coupling constant are not enough to cancell all divergences in ϕ^4 theory. Indeed, let us consider the diagram



(25)

contributing to $-\tilde{\Sigma}(p)$ at λ^2 order. This diagram has superficial degree of divergence $D = 2$, i.e. with the cutoff introduced it behaves like Λ^2 as $\Lambda \to \infty$. One can see this writing the contribution explicitly

$$\frac{(-\lambda)^2}{3!}\int \frac{d^4k_1}{(2\pi)^2}\frac{d^4k_2}{(2\pi)^2}\frac{1}{(k_1^2+m^2)(k_2^2+m^2)((p-k_1-k_2)^2+m^2)}. \qquad (26)$$

For $k_{1,2} >> p, m$ this is

$$\approx \int \frac{d^4k_1}{(2\pi)^2}\frac{d^4k_2}{(2\pi)^2}\frac{1}{k_1^2k_2^2(k_1+k_2)^2} \approx \frac{\Lambda^2}{m^2}. \qquad (27)$$

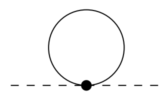
Let us denote this contribution as $\Sigma_2(p)$. One can write

$$\Sigma_2(p) = \Sigma_2(0) + (\Sigma_2(p) - \Sigma_2(0)). \qquad (28)$$

Here

$$\Sigma_2(0) = \frac{(-\lambda)^2}{3!} \int \frac{d^4k_1}{(2\pi)^2} \frac{d^4k_2}{(2\pi)^2} \frac{1}{(k_1^2 + m^2)(k_2^2 + m^2)((k_1 + k_2)^2 + m^2)}. \tag{29}$$

This Λ^2-divergent contribution does not depend on p. In this respect it is similar to the contribution of the diagram



(30)

It can be absorbed into the renormalization of the mass parameter by suitable modification of the counterterm

$$\frac{\delta m^2(\Lambda)}{2} \phi^2 \tag{31}$$

Now the difference

$$\begin{aligned} \Sigma_2(p) - \Sigma_2(0) = \\ \frac{(-\lambda)^2}{3!} \int \frac{d^4k_1}{(2\pi)^2} \frac{d^4k_2}{(2\pi)^2} \frac{1}{(k_1^2 + m^2)(k_2^2 + m^2)} \\ (\frac{1}{((p - k_1 - k_2)^2 + m^2)} - \frac{1}{((k_1 + k_2)^2 + m^2)}) \end{aligned} \tag{32}$$

with the last factor being

$$\frac{2p(k_1 + k_2) - p^2}{((p - k_1 - k_2)^2 + m^2)((k_1 + k_2)^2 + m^2)}, \tag{33}$$

diverges logarithmically, i.e. it is

$$\approx p^2 \ln \frac{\Lambda^2}{m^2} + \ finite \tag{34}$$

This can not be cancelled by a mass renormalization. At the same time the contribution to $\Gamma^2(p)$

$$\Gamma^2 = p^2 + ... + \lambda^2(ap^2 \ln \frac{\Lambda^2}{m^2} + \ finite) = \\ Z(\Lambda)p^2 + ... \tag{35}$$

can be absorbed by a **field renormalization**.

Recall that the term p^2 in Γ^2 originates from the kinetic term

$$\frac{1}{2}(\partial\phi)^2 \tag{36}$$

in the original action. **The above Λ-dependent factor appearing in Γ^2 suggests that the field ϕ entering the original action must be thought of as the bare field**, which will be denoted by ϕ_0, and it differs from the field ϕ appearing in the correlation functions by a Λ-dependent factor

$$\phi_0(\Lambda) = Z^{\frac{1}{2}}(\Lambda)\phi \tag{37}$$

and **it is the correlation functions of the renormalized field** ϕ

$$< \phi(x_1)...\phi(x_n) > \tag{38}$$

that have finite limit at $\Lambda = \infty$.

1.5. RENORMALIZATION PROGRAM.

We come out with **renormalization program**, which can be described as follows.

1.

We start with the action

$$A = \int d^4x (\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4) \tag{39}$$

containing the **bare field, bare mass, and bare coupling constant**.

2.

We introduce some cutoff with a cutoff momentum Λ (this can be done by many ways).

3.

We expect that one can give parameters m_0^2, λ_0, and the field renormalization constant Z certain dependence on Λ:

$$m_0^2 = m_0^2(\Lambda)\ ,\ \lambda_0 = \lambda_0(\Lambda)\ ,\ Z = Z(\Lambda) \tag{40}$$

such that **the correlation functions of the renormalized field**

$$\phi = Z^{\frac{-1}{2}}(\Lambda)\phi_0 \tag{41}$$

have finite $\Lambda \to \infty$ limit.

Because the only way to study QFT so far is the perturbation expansion, the above program can be further reformulated as **renormalized perturbation theory**.

Namely, we write the action with counterterms

$$A = \int d^4x (\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4), \tag{42}$$

where m **is an actual mass and λ is suitably defined finite coupling constant**. The identity with the original action implies

$$1 + \delta Z = Z \ , \ m^2 + \delta m^2 = Z m_0^2 \ , \ \lambda + \delta\lambda = Z^2 \lambda_0. \tag{43}$$

The renormalized perturbation theory is the expansion in renormalized coupling constant λ. Therefore we assume the counterterm coefficients themselves depend perturbatively (i.e. as power series) on λ.

The project is to give these counterterms certain dependence on Λ such that the renormalized correlation functions are Λ-independent order-by-order in λ.

2. Divergences and renormalization in general field theories of ϕ.

Let us consider the divergences and renormalizations in a scalar field theory whose interaction term is more general polynomial in ϕ. It is also helpful to study such theory in the space of d dimensions. The action is

$$A = \int d^d x \left(\frac{1}{2}(\partial\phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \sum_{n=3}^{N} \frac{\lambda_{0,n}}{n!}\phi^n\right). \tag{44}$$

The Feynman rules remain the same as in the case $d = 4$, except for the momentum integration:

$$\frac{d^4 k}{(2\pi)^4} \to \frac{d^d k}{(2\pi)^d} \tag{45}$$

and the diagrams contain n-leg vertices associated with the couplings $\lambda_{0,n}$.

2.1. SUPERFICIAL DEGREE OF DIVERGENCE OF DIAGRAMS.

Generic giagram contributing to $\tilde{\Gamma}^n$, contains P-propagators and I d-momentum integrations. The superficial degree of divergence for such a diagram is given by

$$D = dI - 2P \tag{46}$$

because now the momenta are d-dimensional. Assume that the diagram contains V_m m-legs vertices. Analysis similar to that we have made for ϕ^4 theory reveals two identities

$$\begin{aligned} 2P + n &= \sum_m mV_m, \\ I &= P - \sum_m V_m + 1 \end{aligned} \tag{47}$$

These equations give the following expression

$$D = \sum_m (\frac{d-2}{2}m - d)V_m - \frac{d-2}{2}n + d. \tag{48}$$

2.2. ANALYSIS OF DIVERGENT DIAGRAMS FOR DIFFERENT INTERACTION T

2.2.1. Suppose $N = 3$ so that we have ϕ^3 theory. Then

$$D = \frac{d-6}{2}V_3 + n + d(1 - \frac{n}{2}). \tag{49}$$

If $d > 6$ then for any Γ^n there are diagrams with big enough V_3 which diverge. It means that ϕ^3 theory is **not renormalizable for** $d > 6$.

If $d = 6$ then

$$D = 6 - 2n. \tag{50}$$

There is a divergence only for Γ^2, Γ^3. The theory is **renormalizable**.

If $2 < d < 6$

$$D = \frac{d-6}{2}V_3 + n + d(1-\frac{n}{2}) \leq \frac{d-6}{2} + n + d(1-\frac{n}{2}) = \frac{(d-2)(3-n)}{2} \Leftrightarrow$$
$$D \leq \frac{(d-2)(3-n)}{2}. \tag{51}$$

The divergence appears only for Γ^2, Γ^3 (?) and hence the theory is **renormalizable**.

2.2.2 Consider now ϕ^4 theory in different dimensions.

$$D = (d-4)V_4 + d - \frac{d-2}{2}n. \tag{52}$$

If $d > 4$ for all $\tilde{\Gamma^n}$ there are divergences (because for given n one can find diagram with sufficiently large V_4) and hence the theory is **not renormalizable**.

If $d = 4$

$$D = 4 - n \tag{53}$$

and the divergences appear only in $\tilde{\Gamma}^2$ and $\tilde{\Gamma}^4$ so the theory is **renormalizable**.

If $d = 3$ then

$$D = (1 - V_4) + 2 - \frac{n}{2} \leq 2 - \frac{n}{2} \tag{54}$$

thus, only $\tilde{\Gamma}^2$ is divergent and the theory is **renormalizable**.

Summarizing we conclude

$d > 6$: all theories are **nonrenormalizable**.

$d = 6$ ϕ^3 is **renormalizable**.

$d = 4$ ϕ^3 and ϕ^4 are **renormalizable**.

$d = 3$ ϕ^3, ϕ^4, ϕ^5, ϕ^6 are **renormalizable**.

$d = 2$ the theory is renormalizable for any polynomial of variable ϕ, **superrenormalizable** theory.

2.3. COUPLING CONSTANTS DIMENSIONS AND RENORMALIZABILITY.

The formula (48) admits very simple interpretation in terms of dimensional counting. Note that in our units $c = \hbar = 1$ so that there is only one independent unit, which we take to be mass unit. Let us denote by $[X]$ the mass dimension of a quantity X, for example

$$[mass] = 1 \ , \ [lenght] = -1 \tag{55}$$

The action is dimensionless and therefore it follows from (44)

$$[\phi_0] = \frac{d-2}{2} \ , \ [m_0^2] = 2 \ , \ [\lambda_{0,n}] = d - \frac{d-2}{2}n. \tag{56}$$

Notice that this simple dimensional analysis is applied to the **bare quantities**. We will see later that due to renormalization constant $Z(\Lambda)$ the renormalized field ϕ can have different dimension. By this reason the dimensions in (56) are called **canonical** (or engeneering) dimensions. It is easy to check that

$$[\tilde{\Gamma}^n] = d - \frac{d-2}{2}n \tag{57}$$

(as the coefficient standing in front ϕ^n.) Therefore (48) can be rewritten as

$$D = [\tilde{\Gamma}^n] - \sum_m [\lambda_{0,m}] V_m. \tag{58}$$

At $\Lambda >> |p_i|$ dominating contribution of the diagram with V_m vertices λ_m is

$$\tilde{\Gamma}^n \approx (\prod_m (\lambda_{0,m})^{V_m})\Lambda^D \tag{59}$$

and (58) simply describes the ballance of dimensions.

As it follows from (58) the **mass dimensions of the coupling constants $\lambda_{0,m}$ play the key role in the analysis of the perturbative divergences**.

Suppose some coupling constant $\lambda_{0,m}$ have **strictly negative** mass dimension. Then there are divergent contributions to $\tilde{\Gamma}^n$ with any n from the diagrams with sufficiently large V_m. In other words such theory has infinitely many primitive divergences which can not be obsorbed by any finite number of counterterms. QFT of this type are called (perturbatively) **nonrenormalizable**. Overall consistency of nonrenormalizable theories is very questionable. From purely pragmatic point of view, **the necessity to introduce infinitely many counterterms brings in also infinitely many free parameters, and predictive power of such theories is limited**.

If the mass dimensions of all coupling constants in (44) are **non-negative**, the equation (58) shows that there is only finite number of primitively divergent proper vertices if $d > 2$ ($d = 2$ case is special and must be analysed separetely). In tis case the divergences can be absorbed by finitely many counterterms. Such theories are called **renormalizable**.

If all $\lambda_{0,m}$ have **strictly positive** mass dimensions there is only finite number of divergent diagrams. The theories of this kind are refered to as **super-renormalizable**. Thus, **the renormalizable field theories contain infinite number of divergent diagrams but finite number of**

primitive divergences. This happens when at least one of the coupling constants is dimensinless (see (58)). Overall consistency of a renormalizable theories require more subtle analyses, but at least **they make sense perturbatively**.

It is important to note also that nonexistence of perturbatively renormalizable field theories in high dimensionalities does not imply that consistent QFT are limited to low space-time dimensions, **there may exist perfectly consistent QFT which are just too far from free field theory to admit meaningfull perturbative interpretation**.