Lecture 10.

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1. Wick's theorem for fermions and Yukawa model.

1.1. Normal ordering and Wick's theorem for fermions.

We start with the calculation of $< 0|T(\psi(x)\bar{\psi}(y))|0 >$ for the Dirac field. In the Hesenberg picture the field operators are

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s (a^s_{\vec{p}}u^s(\vec{p})\exp(-\imath px) + b^{s\dagger}_{\vec{p}}v^s(\vec{p})\exp(\imath px))$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s ((a^s_{\vec{p}})^\dagger\bar{u}^s(\vec{p})\exp(\imath px) + (b^s_{\vec{p}})v^s(\vec{p})\exp(-\imath px)) \quad (1)$$

where the creation-annihilation operators satisfy the following anti-commutators relations

$$[a^s_{\vec{p}}, a^{r\dagger}_{\vec{q}}]_+ = [b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}]_+ = (2\pi)^3\delta^3(\vec{p} - \vec{q})\delta^{s,r} \tag{2}$$

Let us introduce the decompositions

$$\begin{aligned}
\psi^+(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s a^s_{\vec{p}} u^s(\vec{p}) \exp{(-\imath px)}, \\
\bar{\psi}^+(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s b^s_{\vec{p}} \bar{v}^s(\vec{p}) \exp{(-\imath px)}, \\
\psi^-(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s b^{s\dagger}_{\vec{p}} v^s(\vec{p}) \exp{(\imath px)}, \\
\bar{\psi}^-(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}}}\sum_s a^{s\dagger}_{\vec{p}} \bar{u}^s(\vec{p}) \exp{(\imath px)}
\end{aligned} \tag{3}$$

Suppose that $x^0 > y^0$ **in** $T(\psi(x)\bar{\psi}(y))$. Then

$$\begin{aligned}
T(\psi(x)\bar{\psi}(y)) = (\psi^+(x) + \psi^-(x))(\bar{\psi}^+(y) + \bar{\psi}^-(y)) = \\
\psi^+(x)\bar{\psi}^+(y) - \bar{\psi}^-(y)\psi^+(x) + \psi^-(x)\bar{\psi}^+(y) + \psi^-(x)\bar{\psi}^-(y) + [\psi^+(x), \bar{\psi}^-(y)]_+
\end{aligned} \tag{4}$$

where we have taken into account Fermi statistics of the Dirac field.

In each term of this expression, excluding the anti-commutator, the annihilation operators $a^s_{\vec{p}}$, $b^s_{\vec{p}}$ are to the right of the creation operators $a^{s\dagger}_{\vec{p}}$, $b^{s\dagger}_{\vec{p}}$. This way to order the fields in $T(\psi(x)\bar{\psi}(y))$ is convenient because by the vacuum definition

$$\psi^+(x)|0> = 0 = \bar{\psi}^+(x)|0>, \; <0|\psi^-(x) = 0 = <0|\bar{\psi}^-(x) \tag{5}$$

so that the all terms in $< 0|T(\psi(x)\bar{\psi}(y))|0 >$ except the anti-commutator vanish. The way to arrange the creation-annihilation operators in the operator product when the annihilation operators stand to the right of the creation operators is called **the normal ordering of operators**. The normal ordering of operators $a^s_{\vec{q}}...b^r_{\vec{p}}...a^{t\dagger}_{\vec{k}}...b^{u\dagger}_{\vec{l}}$ is denoted usualy as

$$: a^s_{\vec{q}}...b^r_{\vec{p}}...a^{t\dagger}_{\vec{k}}...b^{u\dagger}_{\vec{l}} : \tag{6}$$

The only difference from the bosonic case in the definition of normal ordering of fermions is the sign factor (due to Fermi statistics). For example

$$: a^s_{\vec{q}} a^{r\dagger}_{\vec{p}} b^t_{\vec{k}} b^{u\dagger}_{\vec{l}} := -a^{r\dagger}_{\vec{p}} b^{u\dagger}_{\vec{l}} a^s_{\vec{q}} b^t_{\vec{k}} \tag{7}$$

Suppose now that $y^0 > x^0$ **in** $T(\psi(x)\bar{\psi}(y))$. Then

$$T(\psi(x)\bar{\psi}(y)) = -\bar{\psi}(y)\psi(x) =$$
$$-\bar{\psi}^+(y)\psi^+(x) + \psi^-(x)\bar{\psi}^+(y) - \bar{\psi}^-(y)\psi^+(x) - \bar{\psi}^-(y)\psi^-(x) - [\bar{\psi}^+(y), \psi^-(y)]_+ \tag{8}$$

where the Fermi statistics has been taken into account again.

The all terms in $< 0|T(\psi(x)\bar{\psi}(y))|0 >$ are vanishing except the last anti-commutator. It makes sense therefore to define the operation which is called **contraction** for fermions:

$$\overbrace{\psi(x)\bar{\psi}}(y) = [\psi^+(x), \bar{\psi}^-(y)]_+ \; if \; x^0 > y^0$$
$$and$$
$$\overbrace{\psi(x)\bar{\psi}}(y) = -[\bar{\psi}^+(y), \psi^-(x)]_+ \; if \; y^0 > x^0$$
$$\overbrace{\psi(x)\psi}(y) = \overbrace{\bar{\psi}(x)\bar{\psi}}(y) = 0. \tag{9}$$

But the contraction so defined coincides with the definition of Feynman's propagator for Dirac fermions

$$
\begin{aligned}
\overline{\psi(x)\bar{\psi}}(y) &= S_F(x-y),\\
S_F(x-y) &=\\
\Theta(x^0-y^0) < 0|\psi_a(x)\bar{\psi}_b(y)|0> -\Theta(y^0-x^0) < 0|\bar{\psi}_b(y)\psi_a(x)|0> &=\\
\int \frac{d^4p}{(2\pi)^4}\frac{\imath(p_\mu\gamma^\mu+m)}{p^2-m^2+\imath\epsilon}\exp\left(-\imath p(x-y)\right) &
\end{aligned}
\tag{10}
$$

Hence we can write

$$
T(\psi(x)\bar{\psi}(y)) =: \psi(x)\bar{\psi}(y) : +\overline{\psi(x)\bar{\psi}}(y) \tag{11}
$$

It allows to prove by induction **Wick's Theorem:**

$$
\begin{aligned}
T(\psi(x_1)\bar{\psi}(x_2)...\psi(x_N)) =: \psi(x_1)\bar{\psi}(x_2)...\psi(x_N) : +\\
\textit{sum of } \ :\psi(x_1)\bar{\psi}(x_2)...\psi(x_N): \ \textit{with all possible contractions inside.}
\end{aligned}
\tag{12}
$$

1.2. Interaction picture for Yukawa model.

The Yukawa model can be considered as a simplifying version of QED where the foton is changed by a scalar field. The Lagrangian is the sum

$$
\begin{aligned}
\mathcal{L}_Y &= \mathcal{L}_{KG} + \mathcal{L}_{Dir} + \mathcal{L}_{int},\\
\mathcal{L}_{KG} &= \frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2],\\
\mathcal{L}_{Dir} &= \bar{\psi}(\imath\gamma^\mu\partial_\mu - m)\psi,\\
\mathcal{L}_{int} &= -g\phi\bar{\psi}\psi.
\end{aligned}
\tag{13}
$$

Similarly to the ϕ^4 model one can develop interaction picture for Yukawa model assuming that couplin constant g is small and introducing Heisenberg fields operators $\psi_I(x)$, $\phi_I(x)$ in representation picture. By the definition, these fields obey the free field theory equations of motion:

$$
\begin{aligned}
\imath\frac{\partial}{\partial t}\phi_I(\vec{x},t) &= [H_{KG},\phi_I(\vec{x},t)] \\
\imath\frac{\partial}{\partial t}\psi_I(\vec{x},t) &= [H_{Dir},\psi_I(\vec{x},t)]
\end{aligned}
\tag{14}
$$

Then we can express the Heisenberg fields $\psi(x)$, $\phi(x)$ and the vacuum state $|\Omega>$ of Yukawa theory in terms of the Heisenberg fields $\psi_I(x)$, $\phi_I(x)$ and vacuum $|0>$ of free theory similarly to the case of ϕ^4 theory. It allows to obtain the Green's function formula like this

$$
\begin{aligned}
&< \Omega|T(\psi(x_1)...\psi(x_n)\bar{\psi}(y_1)...\bar{\psi}(y_n)\phi(z_1)...\phi(z_m))|\Omega >= \\
&\lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\psi_I(x_1)...\psi_I(x_n)\bar{\psi}_I(y_1)...\bar{\psi}_I(y_n)\phi_I(x_1)...\phi_I(x_n)\exp\left(-\imath\int_{-T}^{T}dtH_I(t)\right))|0 >}{< 0|T(\exp\left(-\imath\int_{-T}^{T}dtH_I(t)\right))|0 >}
\end{aligned}
\tag{15}
$$

where

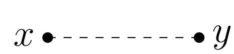
$$
H_I(t) = g\int d^3x\bar{\psi}_I(\vec{x},t)\psi_I(\vec{x},t)\phi_I(\vec{x},t)
\tag{16}
$$

1.3. Green's functions and Feynmam diagrams.

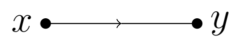
Consider first the 2-points Green's functions

$$
\begin{aligned}
< \Omega|T(\phi(x)\phi(y))|\Omega >&= \lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\phi_I(x)\phi_I(y)\exp\left(-\imath\int_{-T}^{T}dtH_I(t)\right))|0 >}{< 0|T(\exp\left(-\imath\int_{-T}^{T}dtH_I(t)\right))|0 >}, \\
< \Omega|T(\psi(x)\bar{\psi}(y))|\Omega >&= \lim_{T\to\infty(1-\imath\epsilon)} \frac{< 0|T(\psi_I(x)\bar{\psi}(y)\exp\left(-\imath\int_{-T}^{T}dtH_I(t)\right))|0 >}{< 0|T(\exp\left(-\imath\int_{-T}^{T}dtH_I(t)\right))|0 >}
\end{aligned}
\tag{17}
$$

One can use perturbation expansion in order to calculate these functions. At zero perturbation order the numerators give the free propagators

$D_F(x-y) =$ 

(18)

$$S_F(x-y) =$$ 

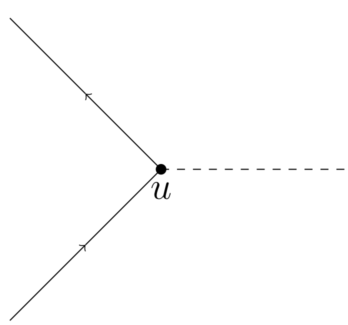
(19)

As usual, there are certainly highest order corrections which contain connected diagrams as well as unconnected fragments of vacuum diagrams, but these fragments are cancelled by the vacuum diagrams from denominator.

The interaction vertex appears when we consider 3-point Green's function $< \Omega|T(\psi(x)\bar{\psi}(y)\phi(z))|\Omega >$ at first order. In this case Wick's theorem applied to the numerator gives

$$\begin{aligned}
-\imath g \int d^4u < 0|T(\psi(x)\bar{\psi}(y)\phi(z)\bar{\psi}(u)\psi(u)\phi(u))|0 >= \\
-\imath g \int d^4u (< 0| : \psi(x)\bar{\psi}(y)\phi(z)\bar{\psi}(u)\psi(u)\phi(u) : |0 > +... \\
+ < 0| : \psi(x)\bar{\psi}(y)\phi(z)\bar{\psi}(u)\psi(u)\phi(u) : |0 >) = \\
-\imath g \int d^4u < 0|(-1)\psi(x)\bar{\psi}(u)(-1)\psi(u)\bar{\psi}(y)\phi(z)\phi(u)|0 >= \\
-\imath g \int d^4u S_F(x-u) S_F(u-y) D_F(z-u)
\end{aligned} \tag{20}$$

Thus we find the vertex diagram



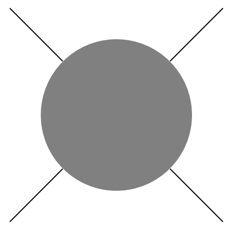
$$= -\imath g \int d^4u \tag{21}$$

The diagrams of propagators (18), (19) and the vertex diagram (21) generate all the diagrams in Yukawa theory. Therefore, one can calculate any Green's function of the theory using these Feynman rules.

2. Classification of diagrams (reminder).

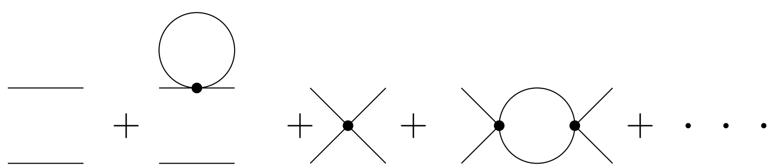
2.1. Unconnected and connected diagrams.

A general 4-point correlation function, for example, can be represented by the diagrams:



$$=$$

(22)



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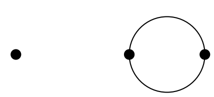
The first pair of diagrams is an example of **disconnected diagrams**, while the second pair is an example of **connected diagrams**.

2.2. Amputated diagrams.

These are the connected diagrams with amputated external lines:



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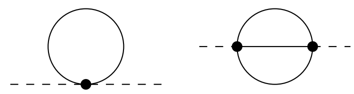
2.3. One-particle irreducible diagrams and proper vetices.

The **one-particle irreducible diagrams** are the connected diagrams which cannot be made disconnected by cutting just one line.



(25)

For $n > 2$ the n-point proper vertex $-\Gamma^{(n)}(y_1, ..., y_n)$ is the sum of all one-particle irreducible diagrams. In particular, in ϕ^4 theory $\Gamma^{(4)} = -W^{(4)}_{amp}$.

3. Momentum representation.

To actually evaluate a diagram it is often more convenient to represent the diagram in the **momentum space**.

3.1. Fourier transformed correlation functions.

$$\int d^4x_1 ... d^4x_n W^n(x_1, ..., x_n) \exp(-\imath p_1 x_1) ... \exp(-\imath p_n x_n) = (2\pi)^4 \delta(p_1 + ... + p_n) \tilde{W}^n(p_1, ... p_n) \tag{26}$$

where $W^n(x_1, ..., x_n)$ means connected n-point correlation function. δ-function appears due to the translation invariance of the correlation function.

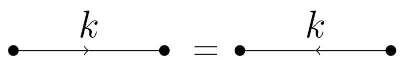
3.2. Feynman rules in the momentum space.

In lecture 7 the Feynman rules for the ϕ^4 theory in momentum representation were obtained:

1.

Internal propagator carry momenta to be integrated over

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} =$$

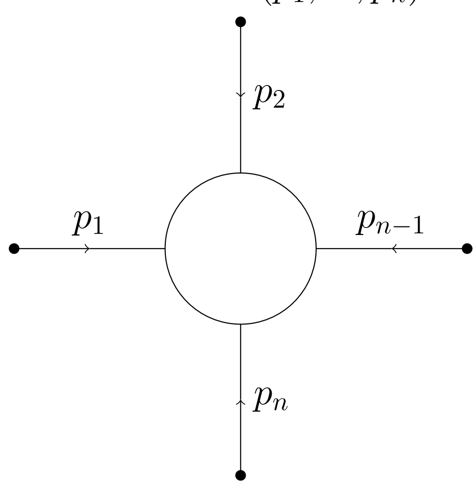


(27)

2.

External propagator carry fixed momenta

$$\tilde{W}^n(p_1, ..., p_n) =$$



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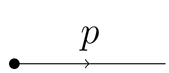
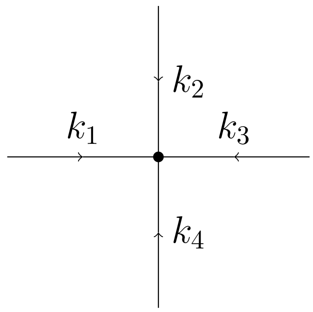


$$= \frac{1}{p^2+m^2}$$

(29)

3.

The vertices conserve the momenta



$$= -\lambda(2\pi)^4\delta(k_1 + ... + k_4) \tag{30}$$

In the momentum representation many operations simplify. For example the amputation takes very simple form

$$\tilde{W}^n(p_1, ..., p_n) = \prod_{i=1}^{n} \tilde{W}(p_i)\tilde{W}^n_{amp}(p_1, ..., p_n) \tag{31}$$

where $\tilde{W}(p_i)$ is Fourier transform of $W(x - x')$.

In the momentum space $\tilde{\Gamma}^2(p)$ is just an inverse of $\tilde{W}(p)$

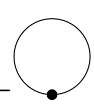
$$\tilde{\Gamma}^2(p) = \frac{1}{\tilde{W}(p)} = p^2 + m^2 + \tilde{\Sigma}(p) \tag{32}$$

4. Divergences and Λ-regularization.

4.1. Divergence of Σ, bare mass and idea of mass renormalization.

Let us calculate the leading order to $\tilde{\Sigma}(p)$ (euclidean space formulation of QFT will be implied in what follows)

$$\tilde{\Sigma}(p) = -$$



$$= \frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 + m^2} \tag{33}$$

The integral is badly divergent as $k \to \infty$. What should we do about it?

Just as we did in the Casimir effect let us introduce a cutoff, replacing

$$\frac{1}{k^2+m^2} \to \frac{1}{k^2+m^2}\Phi(\frac{k^2}{\Lambda^2}) \tag{34}$$

where $\Phi(\frac{k^2}{\Lambda^2}) \to 0$ as $k^2 \to \infty$ fast enough to make the integral convergent, while $\Phi \approx 1$ at $k^2 << \Lambda^2$. Then

$$\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\Phi(\frac{k^2}{\Lambda^2}) = \frac{\lambda m^2}{2}F(\frac{\Lambda^2}{m^2}) \tag{35}$$

As just in the case of the KG vacuum energy, the cutoff by itself brings no physical insight as we know nothing about both Φ and Λ, and can not fix F.

Let us note however, that in this case the **contribution has no momentum dependence, and enters the quantity $\tilde{\Gamma}^2(p)$ in the combination** $m^2 + \frac{\lambda m^2}{2}F(\frac{\Lambda^2}{m^2})$

$$\tilde{\Gamma}^2(p) = p^2 + m^2 + \frac{\lambda m^2}{2}F(\frac{\Lambda^2}{m^2}) \tag{36}$$

We see that no matter what Φ and Λ is, the parameter m is not actual mass of the particles. The interraction leads to a shift that happens to depend on the cutoff. We will see later that actual mass of the particle in interacting theory is determined by the equation

$$\tilde{\Gamma}^2(p)|_{p^2=-m^2} = 0 \tag{37}$$

It is reasonable therefore to change the notations, denoting by m_0^2 the coefficient in front of ϕ^2 in A. Then

$$\tilde{\Gamma}^2(p) = p^2 + m_0^2 + \frac{\lambda m_0^2}{2}F(\frac{\Lambda^2}{m_0^2}) + ... = p^2 + m^2 \tag{38}$$

Note that we have already observed similar phenomenon in our path integral representation of the free propagator D. We started with a discret approximation of the path characterized by the step Δ (cutoff) and have taken some parameter m_0 instead of m in the action

$$A = m_0 \cdot Lenght \tag{39}$$

We found that actual mass m was different from m_0 by some terms which depend on $\Delta \approx \Lambda^{-1}$. We then had to give some dependence of m_0 on Δ

$$m_0 = m_0(\Delta) \tag{40}$$

so that in the continous limit $\Delta \to 0$ the actual mass m had finite value.

Similar procedure applies here. **Assume that parameter m_0^2 in the action (usualy called bare mass parameter) depends on Λ in such a way that the actual mass**

$$m^2 = m_0^2 + \frac{\lambda m_0^2}{2} F(\frac{\Lambda^2}{m_0^2}) + o(\lambda^2) \tag{41}$$

remains finite as $\Lambda \to \infty$. Then to this order

$$\tilde{\Gamma}^2(p) = p^2 + m^2 + O(\lambda^2) \tag{42}$$

has finite limit, independent on Φ.

4.2. Counterterms.

One can improve this idea. Our (euclidean) action is

$$A = \int d^4x (\frac{1}{2}(\partial\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4) \tag{43}$$

Let us now write

$$m_0^2 = m^2 + \delta m^2 \tag{44}$$

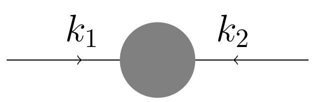
where m^2 is an actual mass. Then one can write the initial action as

$$A = \int d^4x (\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4) = A_0 + A_I$$
$$A_0 = \int d^4x (\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2) \ , \ A_I = \int d^4x (\frac{\delta m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4) \tag{45}$$

and we can treat the A_I as a perturbation. In this case the propagator is

$$= \frac{1}{k^2+m^2} \ , \ not \ \frac{1}{k^2+m_0^2} \tag{46}$$

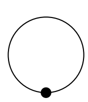
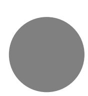
However we have an aditional vertex



$$= -\delta m^2 (2\pi)^4 \delta(k_1+k_2) \tag{47}$$

The term $\frac{\delta m^2}{2}\phi^2$ in the action is the simplest appearence of so called **counterterms** and the above extra vertex is called **counterterm vertex**.

In this modified perturbation theory we have

$$\tilde{\Sigma}(p) =$$  $$+$$ 

$$= \delta m^2 + \frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\Phi(\frac{k^2}{\Lambda^2}) =$$
$$\delta m^2 + \frac{\lambda m^2}{2}F(\frac{\Lambda^2}{m^2}) \tag{48}$$

If we insist that m is actual mass, we have to set

$$\delta m^2 = -\frac{\lambda m^2}{2}F(\frac{\Lambda^2}{m^2}) + O(\lambda^2) \tag{49}$$

That is the bare mass as a function of cuttof Λ is given by

$$m_0^2 = m^2 - \frac{\lambda m^2}{2} F(\frac{\Lambda^2}{m^2}) + O(\lambda^2) \tag{50}$$

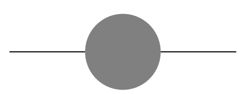
Notice that this seems differ from the relation between m_0^2 and m^2 we obtained within the original perturbation theory

$$m_0^2 = m^2 - \frac{\lambda}{2} m_0^2 F(\frac{\Lambda^2}{m_0^2}) + O(\lambda^2) \tag{51}$$

There is no contrudiction here because the difference between these two expressions is $\approx \lambda^2$.

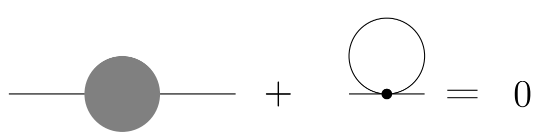
4.3. Renormalized perturbation theory.

By the choice of δm^2 in the modified perturbation theory, the counterterm diagram



(52)

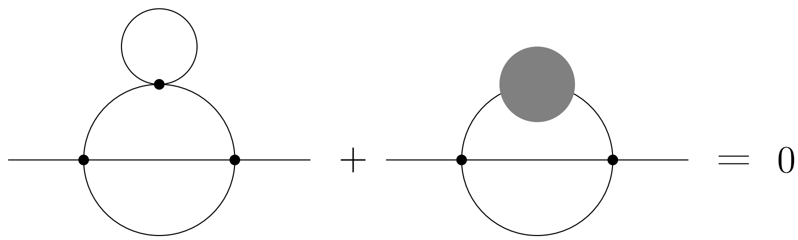
exactly cancels the cutoff dependent buble diagram



(53)

so the dependence on Λ and Φ disappears.

Note that this cancellation occures also inside more complicated diagrams:



(54)

The idea behind this modified perturbation theory can be extended to obtain so called renormalized perturbation theory in which all divergences of original perturbation theory are exterminated. This new perturbation theory with counterterms is called **renormalized perturbation theory**.