Lecture 1.

Plan.

1. Why do we need in Quantum FT?

1.1 Lagrangian, action and equations of motion in Classical mechanics and FT.

The equations of motion in Classical Mechanics follow from extremal action principle: we have an action

$$S = \int L(q(t), \dot{q}(t))dt \qquad (1)$$

where the Lagrangian $L(q(t), \dot{q}(t))$ is a function of coordinate of a particle $q(t)$ and its velocity $\dot{q}(t)$.

Notice also that we do not consider Lagrangians which include highest derivatives like $\ddot{q}(t)$ because it would contradicts to Lagrangian causality principle: **the evolution $q(t)$ is determined by the initial position of particle $q(0)$ and its initial velosity $\dot{q}(0)$.**

The equations of motion follow if we demand that trajectory is extremal

$$\delta S = 0 \tag{2}$$

In Classical FT an analog of $q(t)$ is a **field** $\phi(\vec{x}, t)$. The action is determined by the Lagrangian density

$$S = \int \Lambda(\phi(\vec{x}, t), \dot{\phi}(\vec{x}, t), \nabla\phi(\vec{x}, t)) d^3x dt \tag{3}$$

where, as in classical mechanics **we do not include the highest derivatives like $\ddot{\phi}(\vec{x}, t)$ due to the Lagrangian causality principle**.

The equations of motion follow if we demand that "'trajectory"' $\phi(\vec{x}, t)$ is extremal

$$\delta S = 0 \Leftrightarrow \partial_\mu \left(\frac{\partial \Lambda}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \Lambda}{\partial \phi} = 0 \tag{4}$$

1.2 Lorentz invariance, locality and causality.

Recall that Lorentz group is a set of linear transformations of Minkowski space-time coordinates $(\vec{x}, t)$ which leave the interval

$$ds^2 = dt^2 - (d\vec{x})^2 \tag{5}$$

unchanged.

Then the action (3) will be **Lorentz invariant** if the Lagrangian density Λ is Lorentz invariant because the mesure $d^3x dt$ is Lorentz invariant clearly.

The action must be local. It means that Lagrangian desity is local. This in turn means that all the quantities $\phi(\vec{x},t)$, $\dot{\phi}(\vec{x},t)$, $\nabla\phi(\vec{x},t)$ are taken in one point $(\vec{x},t)$. This reflects (Faraday's) principle of short-range action:

the field degree of freedom $\phi(\vec{y},t)$ does not interract immediately with $\phi(\vec{x},t)$ if $|\vec{y}-\vec{x}|$ =finite.

Allowing such interractions would lead to possible terms in Lagrangian like

$$\int F(\phi(\vec{x},t),\phi(\vec{y},t))d^3xd^3y \tag{6}$$

But Lorentz invariance of the action S would require also the terms which are nonlocal in time as well, like

$$\int G(\phi(\vec{x},t),\phi(\vec{y},t'))d^3xd^3ydtdt' \tag{7}$$

which evidently **violate causuality**. The state in the future affects the dynamics at present. **Locality and causuality are deeply connected**.

1.3 Symmetries of the action and Noether theorem.

The Lagrangian approach opens up a natural way to relate the symmetries of the action to the consrvation lows. It is given by

Noether theorem: suppose we have a continued set of transformations

$$\phi(x)\to\tilde{\phi}(x)=F_s(x,\phi(x)) \tag{8}$$

parametrized by a parameter s, such that $F_0(x,\phi(x))=\phi(x)$ and the action is invariant

$$S[\phi(x)]=S[\tilde{\phi}(x)] \tag{9}$$

Consider the infinitesimal transformation

$$\begin{gathered}\phi(x)\to\phi(x)+\epsilon E(x,\phi(x))\Rightarrow\\ \Lambda(\tilde{\phi},\partial_\mu\tilde{\phi})=\Lambda(\phi,\partial_\mu\phi)+\epsilon[\frac{\partial\Lambda}{\partial\phi}E+\frac{\partial\Lambda}{\partial(\partial_\mu\phi)}\partial_\mu E)]\end{gathered} \tag{10}$$

Because of action is invariant

$$\frac{\partial\Lambda}{\partial\phi}E + \frac{\partial\Lambda}{\partial(\partial_\mu\phi)}\partial_\mu E = \partial_\mu K^\mu \Leftrightarrow$$
$$\partial_\mu(\frac{\partial\Lambda}{\partial(\partial_\mu\phi)}E) + (\frac{\partial\Lambda}{\partial\phi} - \partial_\mu\frac{\partial\Lambda}{\partial(\partial_\mu\phi)})E = \partial_\mu K^\mu \tag{11}$$

But the second term on the r.h.s. is zero due to the equations of motion, therefore we obtain the conservation low

$$\partial_\mu(\frac{\partial\Lambda}{\partial(\partial_\mu\phi)}E) - \partial_\mu K^\mu \equiv \partial_\mu J^\mu = 0$$
$$J^\mu = \frac{\partial\Lambda}{\partial(\partial_\mu\phi)}E(x,\phi) - K^\mu(\phi,\partial\phi) \tag{12}$$

This leads to the **conserved charges** for the solutions which satisfy the equations of motion (on shell):

$$Q_{t_1} \equiv \int d^3x J^0(\vec{x},t_1) = Q_{t_2} \equiv \int d^3x J^0(\vec{x},t_2) \tag{13}$$

where it is implied that classical solution is such that $\phi(\vec{x},t) \to 0$ when $\vec{x} \to \infty$. The equation (13) is proved by Gauss theorem.

1.4 Stress-energy tensor.

The most general symmetry taking place in FT is **translational symmetry**. If the Lagrangian density

$$\Lambda(\phi(x),\partial\phi(x)) \tag{14}$$

has no explicit dependence on x_μ the action is invariant w.r.t the shifts by a constant vector a:

$$x \to \tilde{x} = x + a,$$
$$\phi(x) \to \tilde{\phi}(x) = \phi(x+a) \tag{15}$$

The infinitesimal version of the shift is

$$x \to \tilde{x} = x + \epsilon,$$
$$\phi(x) \to \tilde{\phi}(x) = \phi(x) + \epsilon^\nu \partial_\nu \phi(x), \tag{16}$$

so that $E(\phi, \partial\phi) = \partial_\nu \phi(x)$ is vector and

$$K_\mu = \partial_\nu \Lambda \tag{17}$$

Now the expression (12) gives the stress-energy tensor conservation low

$$T^\mu_\nu = \frac{\partial \Lambda}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \Lambda \delta^\mu_\nu, \ \partial_\mu T^\mu_\nu = 0 \tag{18}$$

The conservation low (18) leads to 4 conserved quantities:

$$E = \int T^{00} d^3x, \ P^i = \int T^{0i} d^3x \Leftrightarrow P^\mu = \int g^{\mu\nu} T^0_\nu d^3x,$$
$$i = 1, ..., 3, \ \nu, \mu = 0, ..., 3, \ g^{\mu\nu} = diag(1, -1, -1, -1) \tag{19}$$

E is an energy and $\vec{P}$ is a momentum.

1.5 KG theory.

It is a single component (scalar) field $\phi(\vec{x}, t)$ with the action

$$S = \int \frac{1}{2}[(\dot{\phi})^2 - (\nabla\phi)^2 - m^2\phi^2] d^3x dt =$$
$$\int \frac{1}{2}[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2] d^4x \tag{20}$$

The action is Lorentz invariant because the scalar field transforms under the Lorentz tralsformation $x^\mu \to \tilde{x}^\mu = R^\mu_\nu x^\nu$ as $\tilde{\phi}(x) = \phi(\tilde{x})$. The equation of motion:

$$\delta S = 0 \Leftrightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{21}$$

is Lorentz invariant and m^2 is a mass of the field ϕ.

The stress-energy tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\phi\partial^\lambda\phi - m^2\phi^2) \tag{22}$$

Hence, the energy and momentum densities are given by

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}(\dot\phi)^2 + (\nabla\phi)^2 + m^2\phi^2), \\ \vec{\mathcal{P}} &= \dot\phi\nabla\phi \end{aligned} \tag{23}$$

2. Hamiltonian formalism.

2.1. Canonical variables and Hamiltonian in Classical Mechanics.

To pass from Lagrangian formalism to Hamiltonian formalism in classical mechanics, it is necessary to introduce canonical momenta

$$p^i = \frac{\partial L(q,\dot q)}{\partial \dot q_i} \tag{24}$$

and then exclude $\dot q$ in favor of p: $\dot q = f(q,p)$. Then the Hamiltonian function appears as the Legandre transform

$$H(p,q) = \sum_i p^i \dot q_i - L(q,p) \tag{25}$$

Then the equations of motion take the Hamiltonian form

$$\begin{aligned} \dot q_i &= \frac{\partial H}{\partial p^i} = \{q_i, H\}, \\ \dot p^i &= -\frac{\partial H}{\partial q_i} = \{p^i, H\}, \\ \{f,g\} &\equiv \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} \end{aligned} \tag{26}$$

2.2. Canonical variables and Hamiltonian in Classical FT.

In the field theory we have $\vec x$ instead of i, hence the canonical momenta is

$$\pi(\vec x) = \frac{\delta L}{\delta \dot\phi} \tag{27}$$

where

$$L = \int \Lambda(\phi, \partial\phi) d^3x \tag{28}$$

2.3. Canonical variables and Hamiltonian in KG theory.

Applying the definition above to the case of KG Lagrangian we find

$$\pi(\vec{x}) = \dot{\phi}(\vec{x}) \tag{29}$$

Hence the Hamiltonian is

$$H = \int d^3x \pi(\vec{x})\dot{\phi}(\vec{x}) - L = \\ \int d^3x \frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2) \tag{30}$$

3. Quantization procedure in Hamiltonian formalism.

3.1. Quantization procedure in mechanics.

Classical mechanics: phase space parametrized by the canonical coordinates p^i, q_i, endowed with the canonical Poisson brackets

$$\{q_i, p^j\} = \delta_i^j \tag{31}$$

Quantum mechanics: Hilbert space of states, $\mathcal{H}$, $\hat{p}^i, \hat{q}_i$-operators with canonical brakcets

$$[q_i, p^j] = \imath\delta_i^j \tag{32}$$

Classical mechanics: Hamiltonian $H(p, q)$- function on the phase space.

Quantum mechanics: Hamiltonian $\hat{H}(\hat{p}, \hat{q})$-operator acting on the space of states.

Classical mechanics: equations of motion

$$\dot{q}_i = \{q_i, H\},\ \dot{p}^i = \{p^i, H\} \tag{33}$$

Quantum mechanics: Heisenberg's equations of motion

$$\dot{\hat{q}}_i = [\hat{q}_i, H],\ \dot{\hat{p}}^i = [\hat{p}^i, H] \tag{34}$$

or **Schr$\ddot{o}$dinger's representation**: $\mathcal{H}$ is the space of quadraticaly integrable functions $\Psi(q)$ such that

$$\hat{q}_i \Psi(q) = q_i \Psi(q),\ \hat{p}^i \Psi(q) = \imath \frac{\partial}{\partial q^i} \Psi(q)$$
$$\imath \frac{\partial}{\partial t} \Psi(q) = \hat{H} \Psi \tag{35}$$

3.2. Quantization of KG field in Schrödinger picture.

An analog of quantum mechanical wave function is a functional $\Psi[\phi(\vec{x}, t_0)]$ of the field configuration $\phi(\vec{x}, t_0)$ which we can observe at some moment of time t_0. This is an element of the Hilbert space of states $\mathcal{H}$ of KG QFT $\Psi[\phi(\vec{x})] \in \mathcal{H}$. The scalar product in $\mathcal{H}$ is determined by the functional integral

$$(\Psi_1, \Psi_2) = \int D[\phi(\vec{x})] \Psi_1[\phi(\vec{x}, t_0)] \Psi_2^*[\phi(\vec{x}, t_0)] \tag{36}$$

By definition, the functional $\Psi[\phi]$ satisfy the equations

$$\hat{\phi}(\vec{x}) \Psi[\phi(\vec{x}, t)] = \phi(\vec{x}, t) \Psi[\phi(\vec{x}, t)],$$
$$\hat{\pi}(\vec{x}) \Psi[\phi(\vec{x}, t)] = -\imath \frac{\delta}{\delta \phi(\vec{x})} \Psi[\phi(\vec{x}, t)], \tag{37}$$

where the operators $\hat{\phi}(\vec{x})$, $\hat{\pi}(\vec{x})$ does not depend on time and satisfy the canonical commutation relations

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = \imath \delta^3(\vec{x} - \vec{y}) \tag{38}$$

The evolution of $\Psi[\phi]$ in time is given by the Schrödinger equation

$$\imath\frac{\partial}{\partial t}\Psi = \hat{H}\Psi, \tag{39}$$

where the Haniltonian is

$$\hat{H} = \int d^3x\frac{1}{2}(\hat{\pi}(\vec{x})^2 + (\nabla\hat{\phi}(\vec{x}))^2 + m^2\hat{\phi}(\vec{x})^2) \tag{40}$$

3.3. KG as a set of harmonic oscillators.

The KG Hamiltonian (70) is very similar to the harmonic oscillator Hamiltonian

$$H_{osc} = \frac{1}{2}(p^2 + \omega^2 q^2). \tag{41}$$

The difference is that in case of KG field we have a continuum of harmonic oscillators parametrized by a vectors $\vec{p}$.

For the harmonic oscillator, the operators

$$a = \frac{\imath}{\sqrt{2\omega}}(p - \imath\omega\phi),\ a^\dagger = -\frac{\imath}{\sqrt{2\omega}}(p + \imath\omega\phi) \tag{42}$$

diagonalize the Hamiltonian

$$H_{osc} = \frac{\omega}{2}(a^\dagger a + aa^\dagger),\ [\hat{H}, a^\dagger] = \omega a^\dagger,\ [\hat{H}, a] = -\omega a \tag{43}$$

because

$$[a, a^\dagger] = 1 \tag{44}$$

(which in turn follows from the canonical commutator for p and q).

The same is true for KG theory:

$$[\hat{H}, a^\dagger_{\vec{p}}] = \omega_{\vec{p}} a^\dagger_{\vec{p}},\ [\hat{H}, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}} \tag{45}$$

The KG field momentum operator is given by

$$\hat{\vec{P}} = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}}{2}(a^\dagger_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} a^\dagger_{\vec{p}}) \tag{46}$$

and we have

$$[\hat{P}, a_{\vec{p}}] = -\vec{p} a_{\vec{p}}, \; [\hat{P}, a^*_{\vec{p}}] = \vec{p} a^*_{\vec{p}}. \tag{47}$$

For the harmonic oscillator the states $(a^\dagger)^n|0>$ form the basis of H_{osc}-eigenstates

$$H_{osc}(a^\dagger)^n|0>= (n+\frac{1}{2})\omega(a^\dagger)^n|0> . \tag{48}$$

and the space of states of harmonic oscillator is spanned by these eigenstates.

Now we can construct physiscally acceptable solutions of quantum KG theory. To do that one needs to find appropriate representation of the commutators from (69), (45), (47) in the Hilbert space of states $\mathcal{H}$.

We demand that **the energy spectrum is bounded from below:** $E \geq E_0$. It means that there is a state $|0>$ with minimal energy E_0 such that

$$a_{\vec{p}}|0>= 0 \; for \; all \; \vec{p} \tag{49}$$

Then the space $\mathcal{H}$ is spanned by the vectors

$$\begin{aligned} |\vec{p}_1, ..., \vec{p}_N> &= a^*_{\vec{p}_1} ... a^*_{\vec{p}_N}|0>, \\ \hat{H}|\vec{p}_1, ..., \vec{p}_N> &= (\omega_{\vec{p}_1} + ... + \omega_{\vec{p}_N} + E_0)|\vec{p}_1, ..., \vec{p}_N> \\ \hat{P}|\vec{p}_1, ..., \vec{p}_N> &= (\vec{p}_1 + ... + \vec{p}_N + \vec{P}_0)|\vec{p}_1, ..., \vec{p}_N> \end{aligned} \tag{50}$$

This space is called Fock space.

The ground state energy

$$E_0 = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\frac{1}{2}[a_{\vec{p}}a^*_{\vec{p}}] = \int d^3p \frac{\omega_{\vec{p}}}{2}\delta(0) \tag{51}$$

is divergent. But the expression for E_0 can be rewritten as follows

$$E_0 = \int \frac{d^3p'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}\int d^3x \exp\left(\imath(\vec{p}-\vec{p}')\vec{x}\right)\delta(\vec{p}-\vec{p}') = \epsilon_0 V^3, \tag{52}$$

where

$$\mathcal{E}_0 = \int d^3p \frac{\omega_{\vec{p}}}{2} =< 0|T^{00}|0 > \tag{53}$$

is the density of vacuum energy. Due to Lorentz invariance it means

$$< 0|T^{\mu\nu}|0 >= \mathcal{E}_0 g^{\mu\nu}. \tag{54}$$

Hence Lorentz invariance does not require

$$\vec{P}_0 \neq 0 \tag{55}$$

if $E_0 \neq 0$.

3.4. Vacuum energy, normal ordering and particle interpretation.

Though the **infinite vacuum energy can cause the problems, it can be ignored as long as the difference between the energy of a given state and vacuum energy matters**. Therefore it makes sense to redefine $\hat{H}$ by subtracting E_0:

$$: \hat{H} := \hat{H} - E_0 = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} a^*_{\vec{p}} a_{\vec{p}} \tag{56}$$

and do similar subtraction for the momentum operator (though the $P_0 = 0$)

$$: \hat{\vec{P}} := \int \frac{d^3p}{(2\pi)^3} \vec{p} a^*_{\vec{p}} a_{\vec{p}}. \tag{57}$$

It allows us to interprate the operator $a^*_{\vec{p}}$ as creating a particle with the energy $\omega_{\vec{p}}$ and momentum $\vec{p}$ so that the vector $|\vec{p}_1, ..., \vec{p}_N >= a^*_{\vec{p}_1} ... a^*_{\vec{p}_N}|0 >$ is an N-particles state with the momenta $\vec{p}_1, ..., \vec{p}_N$ because of the correct relation $\omega_{\vec{p}_i} = \sqrt{\vec{p}_i^2 + m^2}$ between the momentum and energy of each particle.

3.5. Vacuum energy regularization and QFT at small distances.

Though the subtraction (56), (57) is very convenient and allows to study the spectrum of excited states it does not solve the vacuum energy problem. E_0 **is invisible in infinite flat space-time but it is impodtant when we consider gravity**. That is vacuum energy contributes to the cosmological constant which as we know is astronomically small.

As one can see from (53) the divergence occurs due to **small distances (large momenta)**. **It can be assumed that the theory does not apply to very small scales and must be modifyed** so that instead of (53) we would have

$$\epsilon_0 = \int d^3p \frac{\omega_{\vec{p}}}{2} \Phi(\frac{\vec{p}^2}{\lambda^2}), \;\; \Phi(0) = 1 \tag{58}$$

where λ is some large momentum, where a new physics emerge. If we assume that $\Phi(x) \to 0$ as $x \to \infty$ sufficiently fast, the integral above would converge. But $\Phi(x)$ and λ must be determined by small scales physics. Thus, the vacuum energy divergence problem is a physical problem.

3.6. Conserved charges and KG field quantization.

Here we introduce creation-annihilation operators using the Noether theorem.

The action of KG field is invariant under some special symmetry. Indeed, doing the infinitesimal field transformation

$$\phi(x) \to \tilde{\phi}(x) = \phi(x) + f(x) \tag{59}$$

where the function $f(x)$ is an arbitrary solution of KG equation, we see that the Lagrangian density changes by

$$\Lambda(\bar{\phi}) = \Lambda(\phi) + \partial^\mu K_{\mu f}, \;\; K_{\mu f} = \phi \partial_\mu f. \tag{60}$$

Hence one can use the Noether theorem to conclude that

$$\partial_\mu J^\mu_f = 0, \;\; J^\mu_f = f \partial^\mu \phi - \phi \partial^\mu f. \tag{61}$$

So the corresponding conserved charges (integrals of motion) are given by

$$A_f = \int d^3x(\dot{\phi} f - \dot{f}\phi) = \int d^3x(\pi f - \dot{f}\phi) \tag{62}$$

The last formula allows to calculate the Poisson brackets

$$\{A_f, A_g\} = \int d^3x(\dot{f} g - f\dot{g}) \tag{63}$$

Using the plane waves basis of solutions of KG equations

$$\begin{aligned} A_{\vec{p}} \equiv A_{f_{\vec{p}}},\ \ f_{\vec{p}} = \exp\left(\imath(\omega_{\vec{p}} t - \vec{p}\vec{x})\right), \\ A^*_{\vec{p}} \equiv A_{f^*_{\vec{p}}},\ \ f^*_{\vec{p}} = \exp\left(-\imath(\omega_{\vec{p}} t - \vec{p}\vec{x})\right), \\ \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2} \end{aligned} \tag{64}$$

we find the following algebra

$$\begin{aligned} \{A_{\vec{p}}, A^*_{\vec{p}'}\} = \imath(2\pi)^3 2\omega_{\vec{p}}\delta(\vec{p} - \vec{p}'), \\ \{A_{\vec{p}}, A_{\vec{p}'}\} = \{A^*_{\vec{p}}, A^*_{\vec{p}'}\} = 0 \end{aligned} \tag{65}$$

Thus, in quantum theory we must postulate

$$\begin{aligned} A_{\vec{p}} \to \hat{A}_{\vec{p}},\ \ A^*_{\vec{p}} \to \hat{A}^\dagger_{\vec{p}} \\ [\hat{A}_{\vec{p}}, \hat{A}^\dagger_{\vec{p}'}] = (2\pi)^3 2\omega_{\vec{p}}\delta(\vec{p} - \vec{p}'), \\ [\hat{A}_{\vec{p}}, \hat{A}_{\vec{p}'}] = [\hat{A}^\dagger_{\vec{p}}, \hat{A}^\dagger_{\vec{p}'}] = 0 \end{aligned} \tag{66}$$

It is important to note that operators $\hat{A}_{\vec{p}}$, $\hat{A}^\dagger_{\vec{p}}$ **are Lorentz invariants** because one can use the covariant form of their definition:

$$A_{\vec{p}} = \int d\Sigma_\mu (f_{\vec{p}}\partial^\mu \phi - \phi\partial^\mu f_{\vec{p}}), \tag{67}$$

where Σ is any space-like 3-dim. surface. Therefore the **quantization of KG theory represented above is Lorentz covariant**.

Let us renormalize the operators above:

$$\hat{A}_{\vec{p}} = \sqrt{2\omega_{\vec{p}}}a_{\vec{p}},\ \ \hat{A}^\dagger_{\vec{p}} = \sqrt{2\omega_{\vec{p}}}a^\dagger_{\vec{p}}. \tag{68}$$

The nonzero commutators for $a_{\vec{p}}, a^{\dagger}_{\vec{p}}$ are given by

$$[a_{\vec{p}}, a^{\dagger}_{\vec{p'}}] = (2\pi)^3\delta(\vec{p} - \vec{p'}) \tag{69}$$

Then the KG Hamiltonian takes the form

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\vec{p}}}{2}(a^{\dagger}_{\vec{p}}a_{\vec{p}} + a_{\vec{p}}a^{\dagger}_{\vec{p}}). \tag{70}$$

Appendix. Noether theorem.

In this appendix we consider so called first Noether theorem.

Let us consider the most general transformation of Minkowski space-time that do not change the metric:

$$x^{\mu} \to x'^{\mu} = \Omega^{\mu}_{\nu}x^{\nu} + a^{\mu}, \tag{71}$$

where Ω^{μ}_{ν} is a Lorentz tramsformation and a^{μ} determine a shift. For the infinitesimal transformation we can write

$$x^{\mu} \to x'^{\mu} = \delta\omega^{\mu}_{\nu}x^{\nu} + \epsilon^{\mu}, \tag{72}$$

where $\delta\omega^{\nu\mu} = -\delta\omega^{\mu\nu}$ determine infinitesimal Lorentz transformation, while ϵ^{μ} determine translation, so we have infinitesimal Poincare group transformation.

How the fields transform under this transformation?

Scalar field.

By the definition of scalar field we have

$$\tilde{\phi}(x') = \phi(x). \tag{73}$$

Vector valied field.

By the definition the vector valued field $\phi^{\nu}(x)$ transforms as

$$\tilde{\phi}^{\nu}(x') = \Omega^{\nu}_{\mu}\phi^{\mu}(x'). \tag{74}$$

Covector valued field (vector poteltial).

By the definition the covector valued field $\phi_\nu(x)$ transforms as

$$\tilde{\phi}_\nu(x') = (\Omega^{-1})^\mu_\nu \phi_\mu(x'). \tag{75}$$

Tensor field.

$$\tilde{\phi}^{\nu_1 \dots \nu_p}_{\mu_1 \dots \mu_q}(x') = \Omega^{\nu_1}_{\lambda_1} \dots \Omega^{\mu_p}_{\lambda_p} (\Omega^{-1})^{\sigma_1}_{\mu_1} \dots (\Omega^{-1})^{\sigma_q}_{\mu_q} \phi^{\lambda_1 \dots \lambda_p}_{\sigma_1 \dots \sigma_q}(x'). \tag{76}$$

For the infinitesimal transformations (72) we find

$$\begin{aligned}
\delta\phi(x) = \phi(x') - \tilde{\phi}(x') = \phi(x') - \phi(x) = \\
(\phi(x) + \xi^\nu \phi_{,\nu}(x) + \dots) - \phi(x) = \xi^\nu \phi_{,\nu}(x), \\
\delta\phi^\mu(x) = \phi^\mu(x') - \tilde{\phi}^\mu(x') = \phi^\mu(x') - \Omega^\mu_\nu \phi^\nu(x) = \\
(\phi^\mu(x) + \xi^\nu \phi^\mu_{,\nu}(x) + \dots) - (\delta^\mu_\nu + \delta\omega^\mu_\nu + \dots)\phi^\nu(x) = \\
\xi^\nu \phi^\mu_{,\nu}(x) - \delta\omega^\mu_\nu \phi^\nu, \\
\delta\phi_\mu(x) = \phi_\mu(x') - \tilde{\phi}_\mu(x') = \phi_\mu(x') - (\Omega^{-1})^\nu_\mu \phi_\nu(x) = \\
(\phi_\mu(x) + \xi^\nu \phi_{\mu,\nu}(x) + \dots) - (\delta^\nu_\mu - \delta\omega^\nu_\mu + \dots)\phi_\nu(x) = \\
\xi^\nu \phi_{\mu,\nu}(x) + \delta\omega^\nu_\mu \phi_\nu(x), \\
\delta\phi^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}(x) = \xi^\sigma \phi^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q,\sigma}(x) - \delta\omega^{\mu_1}_\sigma \phi^{\sigma\mu_2 \dots \mu_p}_{\nu_1 \dots \nu_q}(x) - \dots - \delta\omega^{\mu_p}_\sigma \phi^{\mu_1 \dots \mu_{p-1}\sigma}_{\nu_1 \dots \nu_q}(x) + \\
\delta\omega^\sigma_{\nu_1} \phi^{\mu_1 \dots \mu_p}_{\sigma\nu_2 \dots \nu_q}(x) + \dots + \delta\omega^\sigma_{\nu_q} \phi^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_{q-1}\sigma}(x)
\end{aligned} \tag{77}$$

where

$$\xi^\sigma = \epsilon^\sigma + x^\lambda \delta\omega^\lambda_\sigma. \tag{78}$$

Thus, we see that the total variation of the tensor field has two contributions. The first contribution is coused by the shifts of Minkowski space-time points, and the second one comes from the variation coused by the tensor properties of the field.

Stress-energy tensor conservation.

Let us consider a variation of the scalar field action

$$S = \int d^4x \Lambda(\phi, \phi_{,\mu}) \tag{79}$$

under the infinitesimal shift transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu, \tag{80}$$

First of all notice that d^4x is invariant under the Poincare transformations.

Similar to the variation of tensor field, the variation of Lagrangian $\Lambda(x)$ is given by two contributions. The fist one is coused by the shift of point $x \to x'$. The second one is coused by the variations of fields $\phi(x)$, $\phi_{,\mu}(x)$:

$$\delta\Lambda(x) = \epsilon^\sigma \Lambda_{,\sigma}(x) - \frac{\partial\Lambda}{\partial\phi}\delta\phi(x) - \frac{\partial\Lambda}{\partial\phi_{,\mu}}\delta\phi_{,\mu}. \tag{81}$$

According to the (77) we have

$$\delta\phi(x) = \epsilon^\nu \phi_{,\nu}(x),\ \delta\phi_{,\mu}(x) = \epsilon^\nu \phi_{,\nu\mu}(x). \tag{82}$$

Hence

$$\delta S = \int d^4x (\Lambda_{,\sigma} - \frac{\partial\Lambda}{\partial\phi}\phi_{,\sigma} - \frac{\partial\Lambda}{\partial\phi_{,\mu}}\phi_{,\sigma\mu})\epsilon^\sigma. \tag{83}$$

Using the equations of motion

$$\frac{\partial\Lambda}{\partial\phi} = \partial_\mu \frac{\partial\Lambda}{\phi_{,\mu}} \tag{84}$$

we find

$$\delta S = \int d^4x \epsilon^\sigma \partial_\mu (\delta^\mu_\sigma \Lambda - \frac{\partial \Lambda}{\partial \phi_{,\mu}} \phi_{,\sigma}) \tag{85}$$

Because the action is invariant under the translations we obtain the conservation low

$$\partial_\mu T^\mu_\sigma = 0, \; T^\mu_\sigma = \frac{\partial \Lambda}{\partial \phi_{,\mu}} \phi_{,\sigma} - \delta^\mu_\sigma \Lambda \tag{86}$$

of stress-energy tensor.

Orbital momentum tensor conservation.

Now we consider Lorentz transformation:

$$\delta x^\mu = \delta \omega^{\mu\lambda} x_\lambda. \tag{87}$$

The variation of action takes the form

$$\delta S = \int d^4x (\delta \omega^{\sigma\lambda} x_\lambda \Lambda_{,\sigma} - \frac{\partial \Lambda}{\partial \phi} \delta \omega^{\sigma\lambda} x_\lambda \phi_{,\sigma} - \frac{\partial \Lambda}{\partial \phi_{,\mu}} \partial_\mu (\delta \omega^{\sigma\lambda} x_\lambda)). \tag{88}$$

Using the equations of motion we obtain

$$\delta S = \int d^4x \delta \omega^{\sigma\lambda} (x_\lambda \Lambda_{,\sigma} - \partial_\mu (\frac{\partial \Lambda}{\partial \phi_{,\mu}} \phi_{,\sigma}) x_\lambda - \frac{\partial \Lambda}{\partial \phi_{,\mu}} \phi_{,\sigma} \partial_\mu x_\lambda) = \frac{1}{2} \int d^4x \delta \omega^{\sigma\lambda} \partial_\mu (x_\lambda T^\mu_\sigma - x_\sigma T^\mu_\lambda). \tag{89}$$

Because of the action is invariant under the Lorentz transformations we obtain the conservation low

$$\partial_\mu M^\mu_{\lambda\sigma} = 0, \; M^\mu_{\lambda\sigma} = x_\lambda T^\mu_\sigma - x_\sigma T^\mu_\lambda \tag{90}$$

of orbital momentum.

Orbital momentum tensor conservation for vector potential field.

One can imagine a Lorentz invariant action for the vector potential field $\phi_\mu(x)$. It can for example be the action of EM field

$$S = \int d^4x \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, F_{\mu\nu} = \phi_{\mu,\nu} - \phi_{\nu,\mu}. \tag{91}$$

Under the infinitesimal Lorentz transformation we have

$$\begin{aligned} \delta\phi_\mu(x) = \delta^{\sigma\lambda} x_\lambda \phi_{\mu,\sigma} + \delta\omega_\mu^\sigma \phi_\sigma(x), \\ \delta\phi_{\mu,\nu}(x) = \delta^{\sigma\lambda} x_\lambda \phi_{\mu,\sigma\nu} + \delta\omega_\mu^\sigma \phi_{\sigma,\nu}(x) + \delta\omega_\nu^\sigma \phi_{\mu,\sigma}(x). \end{aligned} \tag{92}$$

The action variation is given by

$$\begin{aligned} \delta S = \int d^4x [\delta\omega^{\sigma\lambda} x_\lambda \Lambda_{,\sigma} - \frac{\partial\Lambda}{\partial\phi_\mu}\delta\phi_\mu - \frac{\partial\Lambda}{\partial\phi_{\mu,\nu}}\delta\phi_{\mu,\nu}] = \\ \int d^4x \delta\omega^{\sigma\lambda}[x_\lambda\Lambda_{,\sigma} - \frac{\partial\Lambda}{\partial\phi_\mu}(x_\lambda\phi_{\mu,\sigma} + g_{\mu\lambda}\phi_\sigma) - \frac{\partial\Lambda}{\partial\phi_{\mu,\nu}}\partial_\nu(x_\lambda\phi_{\mu,\sigma} + g_{\mu\lambda}\phi_\sigma)]. \end{aligned} \tag{93}$$

Using the equations of motion we come to

$$\delta S = \frac{1}{2}\int d^4x \delta\omega^{\sigma\lambda}\partial_\nu[x_\lambda T_\sigma^\nu - x_\sigma T_\lambda^\nu + \frac{\partial\Lambda}{\partial\phi_{\mu,\nu}}(g_{\mu\sigma}\phi_\lambda - g_{\mu\lambda}\phi_\sigma)]. \tag{94}$$

Because of the action is Lorentz invariant we obtain the conservation low

$$\begin{aligned} \partial_\nu M_{\lambda\sigma}^\nu = 0, \\ M_{\sigma\lambda}^\nu = x_\lambda T_\sigma^\nu - x_\sigma T_\lambda^\nu + \frac{\partial\Lambda}{\partial\phi_{\mu,\nu}}(g_{\mu\sigma}\phi_\lambda - g_{\mu\lambda}\phi_\sigma). \end{aligned} \tag{95}$$

of spin-orbital momentum.

Conserved currents for sotopic (unternal) symmetries.

Let us consider a Lagrangian $\Lambda(\phi_a, \partial_\mu \phi_a)$ of the scalar fields $\phi_a(x)$ which are endowed with an isotopic (internal) idex a. Suppose the corresponding action is invariant under the continues group G of internal symmetries

$$x \to x' = x, \ \phi_a(x) \to \tilde{\phi}_a(x) = R^b_a \phi_b(x), \tag{96}$$

where $R \in G$ does not depend on x.

Under the infinitesimal transformation $R \approx 1 + \delta t$ we have

$$\delta\phi_a(x) = \phi_a(x') - \tilde{\phi}_a(x') = \phi_a(x) - (\delta^b_a + \delta t^b_z)\phi_b(x) = -\delta t^b_a \phi_b(x). \tag{97}$$

Hence, the variation of the action is

$$\delta S = \int d^4x \delta\Lambda(x) = -\int d^4x (\frac{\partial \Lambda}{\partial \phi_a}\delta\phi_a(x) + \frac{\partial \Lambda}{\partial \phi_{a,\mu}}\delta\phi_{a,\mu}). \tag{98}$$

Using the equations of motion we obtain

$$\delta S = -\int d^4x \delta t^b_a \partial_\mu (\frac{\partial \Lambda}{\partial \phi_{a,\mu}}\phi_{b,\mu}). \tag{99}$$

Because the action is invariant under the transformations (97) we obtain the conservation low

$$\partial_\mu (J^\mu)^a_b(x) = 0, \ (J^\mu)^a_b(x) = -\frac{\partial \Lambda}{\partial \phi_{a,\mu}}\phi_{b,\mu}. \tag{100}$$