Lecture notes on classical field theory (September 23, 2026)

For comments: litvinov@itp.ac.ru

Contents

Lecture 1: Representations of $SO(N)$, $\mathfrak{so}(N)$, and $\mathrm{Spin}(N)$ **2**

Lecture 1: Representations of $SO(N)$, $\mathfrak{so}(N)$, and $\mathrm{Spin}(N)$

Before introducing classical field theory, let us recall some basic facts about the continuous symmetry group $SO(N)$ and its representations. We shall proceed from concrete to abstract. We first construct tensor representations directly from vectors. We then introduce commuting rotations and weights, use the adjoint representation to discover roots and Dynkin diagrams, and finally introduce spinor representations and explain why they require the double cover $\mathrm{Spin}(N)$.

The rotation group $SO(N)$.

Euclidean space $\mathbb{R}^N$ carries the scalar product

$$(x, y) = x^T y = \delta_{ij} x_i y_j.$$

The orthogonal group is the set of real matrices preserving this scalar product,

$$O(N) = \{R \in M_N(\mathbb{R}) \mid R^T R = \mathbf{1}\}.$$

Taking determinants gives $(\det R)^2 = 1$, hence $\det R = \pm 1$. The subgroup with determinant $+1$ is

$$\mathrm{SO}(N) = \{R \in O(N) \mid \det R = 1\}.$$

It is the group of orientation-preserving rotations of $\mathbb{R}^N$.

The most elementary representation of $SO(N)$ is its action on vectors,

$$x_i \longmapsto x_i' = R_i{}^j x_j.$$

We call this the *vector* or *defining* representation and denote it by V,

$$V \simeq \mathbb{R}^N, \qquad \dim V = N.$$

For example, for $N = 2$, every rotation is

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

If we combine the two real coordinates into

$$z = x_1 + i x_2,$$

then

$$z \longmapsto e^{i\theta} z.$$

The Lie algebra $\mathfrak{so}(N)$.

So far we have described rotations as finite transformations. For representation theory, however, it is often more useful to study infinitesimal rotations. They form the Lie algebra $\mathfrak{so}(N)$, and much of the structure of the group is already visible at this infinitesimal level.

To study transformations close to the identity write

$$R = \mathbf{1} + \varepsilon X + O(\varepsilon^2).$$

The condition $R^T R = \mathbf{1}$ gives, to first order,

$$X^T + X = 0.$$

Therefore

$$\mathfrak{so}(N) = \{X \in M_N(\mathbb{R}) \mid X^T = -X\}.$$

An antisymmetric matrix has one independent entry for every pair $i < j$, so

$$\dim \mathfrak{so}(N) = \frac{N(N-1)}{2}.$$

A convenient basis is given by matrices $M_{ij} = -M_{ji}$ with

$$(M_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

The generator M_{ij} produces a rotation in the ij-plane. Their commutation relations are

$$[M_{ij}, M_{kl}] = \delta_{jk}M_{il} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \delta_{il}M_{jk}. \tag{1}$$

What is a representation?

A representation of a group G on a vector space W is a rule

$$R \longmapsto D(R) \in \mathrm{GL}(W)$$

such that

$$D(R_1 R_2) = D(R_1)D(R_2).$$

Infinitesimally, the generators M_{ij} are represented by matrices t_{ij} satisfying the same commutation relations (1).

A subspace $U \subset W$ is *invariant* if

$$D(R)U \subset U$$

for every $R \in SO(N)$. A representation is *irreducible* if its only invariant subspaces are 0 and W.

If a representation is reducible, it contains a proper invariant subspace. For a general group, an invariant complement need not to exist. However, for compact groups such as $SO(N)$, every finite-dimensional representation is completely reducible: one can always choose an invariant complement, and therefore every representation decomposes into a direct sum of irreducible representations.

Tensor representations of $SO(N)$.

The abstract definition tells us what a representation is, but we now want concrete examples. Starting from the defining vector representation V, the simplest way to build new ones is to let $SO(N)$ act simultaneously on several vector indices. This leads naturally to tensor products.

For example, a rank-two tensor transforms as

$$T_{ij} \longmapsto T'_{ij} = R_i{}^k R_j{}^l T_{kl}, \quad T \to T' = RTR^T.$$

This is the representation $V \otimes V$. Every rank-two tensor decomposes uniquely into symmetric and antisymmetric parts,

$$T_{ij} = T_{(ij)} + T_{[ij]},$$

where

$$T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji}), \qquad T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}).$$

Both subspaces are preserved by rotations, so

$$V \otimes V = S^2 V \oplus \Lambda^2 V.$$

Their dimensions are

$$\dim S^2 V = \frac{N(N+1)}{2}, \qquad \dim \Lambda^2 V = \frac{N(N-1)}{2}.$$

The trace of a symmetric tensor,

$$S^i{}_i = \delta^{ij} S_{ij},$$

is invariant under $SO(N)$. Therefore the symmetric representation is itself reducible:

$$S_{ij} = \left(S_{ij} - \frac{1}{N} \delta_{ij} S^k{}_k \right) + \frac{1}{N} \delta_{ij} S^k{}_k.$$

The first term is symmetric and traceless, while the second is proportional to the invariant tensor δ_{ij}. Hence

$$S^2 V = S_0^2 V \oplus \mathbf{1}.$$

Combining the two decompositions, one has

$$V \otimes V = S_0^2 V \oplus \Lambda^2 V \oplus \mathbf{1}.$$

The dimensions are

$$\dim S_0^2 V = \frac{N(N+1)}{2} - 1, \qquad \dim \Lambda^2 V = \frac{N(N-1)}{2}.$$

Antisymmetric tensors and the adjoint representation.

The decomposition of $V \otimes V$ has produced three invariant pieces. The antisymmetric piece is particularly important: it is not merely another tensor representation, but is naturally identified with the Lie algebra itself. This brings us to the adjoint representation. Indeed, an antisymmetric tensor $A_{ij} = -A_{ji}$ has exactly the same index structure as an infinitesimal rotation. In fact, we have a stronger statement

$$\Lambda^2 V \simeq \mathfrak{so}(N).$$

This identification has a direct algebraic meaning. A Lie group acts on its own Lie algebra by conjugation,

$$\mathrm{Ad}_R(X) = RXR^{-1}, \qquad X \in \mathfrak{so}(N). \tag{2}$$

For $R \in SO(N)$, the matrix RXR^{-1} is again antisymmetric, so (2) really defines a representation on the vector space $\mathfrak{so}(N)$. This is the *adjoint representation.*

To obtain its infinitesimal form, put $R = e^{tA}$. Then

$$\mathrm{Ad}_{e^{tA}}(X) = e^{tA}Xe^{-tA} = X + t[A, X] + O(t^2).$$

Hence the Lie algebra acts on itself by

$$\mathrm{ad}_A(X) = [A, X].$$

This is a representation because the Jacobi identity gives

$$[\mathrm{ad}_A, \mathrm{ad}_B]X = [A, [B, X]] - [B, [A, X]] = [[A, B], X] = \mathrm{ad}_{[A,B]}\, X.$$

Thus[1]

$$[\mathrm{ad}_A, \mathrm{ad}_B] = \mathrm{ad}_{[A,B]}\,.$$

For $SO(N)$, conjugation acts on an antisymmetric tensor as

$$A_{ij} \longmapsto R_i{}^k R_j{}^l A_{kl}, \quad A \longmapsto RAR^{-1} = RAR^T.$$

This is precisely the transformation law of a two-form (antisymmetric rank two tensor). Therefore

$$\mathrm{Adj} \simeq \mathfrak{so}(N) \simeq \Lambda^2 V.$$

For a simple Lie algebra the adjoint representation is irreducible. The familiar exception $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is not simple; correspondingly,

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$$

splits into self-dual and anti-self-dual parts.

Higher rank tensors

One may similarly consider $\Lambda^p V$, i.e. the space of completely antisymmetric p-tensors. Its dimension is

$$\dim \Lambda^p V = \binom{N}{p}.$$

[1] If T_a is a basis of a Lie algebra and

$$[T_a, T_b] = f_{ab}{}^c T_c,$$

then the matrices of the adjoint representation are simply the structure constants,

$$(\mathrm{ad}_{T_a})_b{}^c = f_{ab}{}^c.$$

As p increases, the representations $\Lambda^p V$ may look unrelated. For $SO(N)$, however, there is an additional invariant tensor—the Levi-Civita tensor—which produces a remarkable relation between p-forms and $(N-p)$-forms. This is known as Hodge duality

$$\Lambda^p V \simeq \Lambda^{N-p} V.$$

In order to understand the Hodge duality, we note the following. Besides the metric δ_{ij}, the group $SO(N)$ preserves the completely antisymmetric Levi–Civita tensor

$$\epsilon_{i_1 \dots i_N}, \qquad \epsilon_{12\dots N} = 1. \tag{3}$$

Under a general orthogonal transformation

$$R_{i_1}{}^{j_1} \cdots R_{i_N}{}^{j_N} \epsilon_{j_1 \dots j_N} = (\det R) \epsilon_{i_1 \dots i_N} \overset{R \in SO(N)}{=} \epsilon_{i_1 \dots i_N}. \tag{4}$$

Note that if we use in (4) orientation-reversing element of $O(N)$ with $\det R = -1$, then ϵ would change sign. Hence $\epsilon_{i_1 \dots i_N}$ is invariant under $SO(N)$, but it is a pseudotensor under $O(N)$.

A completely antisymmetric rank-p tensor

$$A_{i_1 \dots i_p} = A_{[i_1 \dots i_p]}$$

belongs to the representation $\Lambda^p V$. The epsilon tensor (3) converts a p-index antisymmetric tensor into an $(N-p)$-index antisymmetric tensor by the *Hodge star*

$$(*A)_{i_{p+1} \dots i_N} \overset{\text{def}}{=} \frac{1}{p!} \epsilon_{i_{p+1} \dots i_N j_1 \dots j_p} A_{j_1 \dots j_p}.$$

Geometrically, if A describes an oriented p-dimensional plane, $*A$ describes its oriented orthogonal $(N-p)$-dimensional plane.

Because both δ_{ij} and $\epsilon_{i_1 \dots i_N}$ are $SO(N)$-invariant, the Hodge star commutes with rotations:

$$*(R \cdot A) = R \cdot (*A).$$

Therefore it is an intertwining operator between two $SO(N)$ representations,

$$\Lambda^p V \simeq \Lambda^{N-p} V \quad \text{as} \quad SO(N) \quad \text{representations.}$$

One can apply Hodge star twice. One immediately finds

$$*^2 = (-1)^{p(N-p)} \quad \text{on } \Lambda^p V. \tag{5}$$

Several familiar low-dimensional coincidences follow immediately. For $N = 3$,

$$\Lambda^2 V \simeq \Lambda^1 V = V.$$

Explicitly,

$$a_k = \frac{1}{2} \epsilon_{kij} A_{ij}, \qquad A_{ij} = \epsilon_{ijk} a_k.$$

This is the representation-theoretic origin of the cross product and of the special isomorphism $\mathfrak{so}(3) \simeq V$.

For $N = 4$, the situation is more subtle. The Hodge star maps two-forms to two-forms and satisfies $*^2 = 1$. Hence $\Lambda^2 V$ is reducible and splits into two

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V, \qquad *A_\pm = \pm A_\pm,$$

with

$$\dim \Lambda^2_+ V = \dim \Lambda^2_- V = 3.$$

At the level of algebras, this is equivalent to the exceptional isomorphism

$$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

From the example above, we see that the most important special case of Hodge duality occurs in even dimension. Let

$$V = \mathbb{R}^{2n}.$$

Then the middle exterior power is $\Lambda^n V$. Because $2n - n = n$, the Hodge star is now an endomorphism of the same representation,

$$* : \Lambda^n V \longrightarrow \Lambda^n V.$$

It commutes with the action of $SO(2n)$, so its eigenspaces are automatically invariant representations, exactly as it was for $SO(4)$. From (5), one has

$$*^2 = (-1)^{n^2} = (-1)^n \qquad \text{on } \Lambda^n V.$$

There are therefore two different cases.

Even n. If n is even, $*^2 = 1$. Hence

$$\Lambda^n \mathbb{R}^{2n} = \Lambda^n_+ \oplus \Lambda^n_-, \qquad *A_\pm = \pm A_\pm.$$

These are called the *self-dual* and *anti-self-dual* forms. They have equal dimension,

$$\dim \Lambda^n_+ = \dim \Lambda^n_- = \frac{1}{2}\binom{2n}{n}.$$

Odd n. If n is odd, then $*^2 = -1$. Thus the real vector space $\Lambda^n \mathbb{R}^{2n}$ does not have real ± 1 eigenspaces. It can be decomposed only after complexification,

$$\Lambda^n \mathbb{C}^{2n} = \Lambda^n_+ \oplus \Lambda^n_-, \qquad *A_\pm = \pm i A_\pm.$$

Cartan subalgebra and the rank.

Up to this point we have constructed representations explicitly using tensors. This method is concrete, but it does not yet give us a systematic way to classify representations. To do that, we now change viewpoint: instead of studying all rotations at once, we first diagonalize a maximal set of mutually commuting rotations. Their simultaneous eigenvalues are the weights of the representation.

A convenient way to study representations of Lie algebras is to find the so called *Cartan subalgebra*, i.e. the maximal set of commuting generators. Rotations in disjoint two-planes commute. For example,

$$[M_{12}, M_{34}] = 0.$$

For $N = 2r$ or $N = 2r + 1$ the real antisymmetric matrices

$$M_{2k-1,2k}, \qquad k = 1, \ldots, r,$$

span a maximal commuting subalgebra. It is the Cartan subalgebra. The number of independent commuting rotations is the rank:

$$r = \operatorname{rank} SO(N) = \left\lfloor \frac{N}{2} \right\rfloor .$$

We note that the matrices M_{ij} are real antisymmetric and therefore anti-Hermitian. For discussing weights it is convenient to use the Hermitian generators

$$H_k = iM_{2k-1,2k}, \qquad H_k^\dagger = H_k. \tag{6}$$

Because the Hermitian Cartan generators H_k commute, over $\mathbb{C}$ they can be diagonalized simultaneously. If

$$H_k v = \mu_k v,$$

then

$$\mu = (\mu_1, \ldots, \mu_r)$$

is called the *weight* of v. With the convention (6), the components μ_k are real.

There is a very concrete way to find these weights in the vector representation. Introduce

$$z_k = x_{2k-1} + ix_{2k}, \qquad \bar{z}_k = x_{2k-1} - ix_{2k}.$$

Under a rotation by an angle θ_k in the k-th plane,

$$z_k \longmapsto e^{i\theta_k} z_k, \qquad \bar{z}_k \longmapsto e^{-i\theta_k} \bar{z}_k.$$

Let e_k denote the unit vector in the k-th weight direction. Then z_k has weight $+e_k$ and $\bar{z}_k$ has weight $-e_k$.

Hence for $SO(2r)$,

$$\mathrm{Wt}(V) = \{\pm e_1, \ldots, \pm e_r\}. \tag{7}$$

For $SO(2r+1)$ there is one extra coordinate x_{2r+1} unchanged by all Cartan rotations, and therefore one extra zero weight:

$$\mathrm{Wt}(V) = \{0, \pm e_1, \ldots, \pm e_r\}.$$

Suppose v has weight μ and w has weight ν. Since the Cartan generators act on a tensor product as

$$H_k(v \otimes w) = (H_k v) \otimes w + v \otimes (H_k w),$$

we obtain

$$H_k(v \otimes w) = (\mu_k + \nu_k)(v \otimes w).$$

Therefore weights add in tensor products.

We have now learned how to assign weights to vectors in any representation. Let us apply the same idea to the most important representation introduced earlier, the adjoint representation

$$\mathrm{Adj} \simeq \Lambda^2 V.$$

Its nonzero weights are special: they are called the roots of the Lie algebra. Thus roots are nothing more than the nonzero weights of the adjoint representation. From the algebraic point of view, after complexifying the Lie algebra we have the decomposition

$$\mathfrak{so}(N)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}_{\mathbb{C}}$ is the Cartan subalgebra spanned by H_k and

$$\mathfrak{g}_\alpha = \{X : [H_k, X] = \alpha_k X \text{ for all } k\}.$$

An element $E_\alpha \in \mathfrak{g}_\alpha$ is called a *root vector*. Thus

$$[H_k, E_\alpha] = \alpha_k E_\alpha.$$

The Cartan generators themselves have weight zero because $[H_k, H_l] = 0$. Hence the zero-weight subspace of the adjoint contains the Cartan subalgebra, while the nonzero weights are the roots.

$SO(2r)$**.** Using (7), antisymmetric products give nonzero weights

$$\pm e_i \pm e_j, \qquad i \neq j.$$

Therefore

$$\Delta(D_r) = \{\pm e_i \pm e_j \mid i \neq j\}, \qquad D_r \equiv \mathfrak{so}(2r).$$

All roots have the same length.

$SO(2r+1)$

The zero weight in the vector representation allows antisymmetric products of 0 with $\pm e_i$. Thus in addition to $\pm e_i \pm e_j$ we get $\pm e_i$:

$$\Delta(B_r) = \{\pm e_i \pm e_j,\ \pm e_i\}, \qquad B_r \equiv \mathfrak{so}(2r+1).$$

The roots $\pm e_i \pm e_j$ are long and the roots $\pm e_i$ are short.

In all the examples above, whenever α occurs, so does $-\alpha$. This is not an accident. The intrinsic statement is

$$\alpha \in \Delta \quad \Longrightarrow \quad -\alpha \in \Delta.$$

For compact groups such as $SO(N)$ there is a very elementary explanation of this fact using Hermitian conjugation. Take the Hermitian Cartan generators H_k introduced in (6), and suppose that E_α is a root vector,

$$[H_k, E_\alpha] = \alpha_k E_\alpha.$$

Because $H_k^\dagger = H_k$ and the eigenvalues α_k are real, taking the Hermitian conjugate gives

$$([H_k, E_\alpha])^\dagger = \alpha_k E_\alpha^\dagger.$$

But

$$([H_k, E_\alpha])^\dagger = (H_k E_\alpha - E_\alpha H_k)^\dagger = E_\alpha^\dagger H_k - H_k E_\alpha^\dagger = -[H_k, E_\alpha^\dagger].$$

Therefore

$$[H_k, E_\alpha^\dagger] = -\alpha_k E_\alpha^\dagger.$$

So $E_\alpha^\dagger$ is a root vector of weight $-\alpha$:

$$E_\alpha^\dagger \propto E_{-\alpha}.$$

Hermiticity is a very convenient way to *see* the pairing for compact groups, but it is not the fundamental reason roots occur in opposite pairs. For every complex semisimple Lie algebra the root system is invariant under $\alpha \mapsto -\alpha$, even when no particular Hermitian realization has been chosen. One algebraic proof uses the nondegeneracy and invariance of the Killing form: a root space $\mathfrak{g}_\alpha$ can pair nontrivially only with $\mathfrak{g}_{-\alpha}$, so nondegeneracy forces $\mathfrak{g}_{-\alpha}$ to exist whenever $\mathfrak{g}_\alpha$ does.

For $SO(N)$ one can also see the pairing directly from tensors. For $SO(2r)$ the complex vector basis contains z_i of weight e_i and $\bar{z}_i$ of weight $-e_i$. Therefore the adjoint $\Lambda^2 V$ contains, for example,

$$z_i \wedge z_j \quad \text{of weight } e_i + e_j,$$

and its complex conjugate

$$\bar{z}_i \wedge \bar{z}_j \quad \text{of weight } -(e_i + e_j).$$

Likewise,

$$z_i \wedge \bar{z}_j \quad \text{has weight } e_i - e_j, \quad \bar{z}_i \wedge z_j \quad \text{has weight } -(e_i - e_j).$$

For $SO(2r+1)$, if u denotes the zero-weight vector, then

$$z_i \wedge u \quad \text{and} \quad \bar{z}_i \wedge u$$

have weights e_i and $-e_i$.

Since the roots always come in pairs, it worth to introduce the notion of positive and negative roots. The adjectives "positive" and "negative" are not intrinsic properties of a root. We choose one root from each pair $\{\alpha, -\alpha\}$ to be positive and one to be negative. A convenient choice for $SO(N)$ is based on the ordering

$$e_1 > e_2 > \cdots > e_r > 0.$$

The opposite roots are then called negative.

A small set of positive roots, called the *simple roots*, generates all positive roots by nonnegative integer combinations. For $B_r = \mathfrak{so}(2r+1)$ one may choose

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \ldots, \quad \alpha_{r-1} = e_{r-1} - e_r, \quad \alpha_r = e_r.$$

For $D_r = \mathfrak{so}(2r)$ one may choose

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{r-1} = e_{r-1} - e_r, \quad \alpha_r = e_{r-1} + e_r.$$

Once the simple roots have been chosen, it is useful to summarize their geometry by a small graph called the *Dynkin diagram*. The construction is very concrete:

- each simple root α_i is represented by one node;
- two nodes are joined when the corresponding simple roots are not orthogonal;
- the number of bonds records the angle and the relative lengths of the two roots;

- if the two roots have unequal length, an arrow is drawn toward the *shorter* root.

More algebraically, introduce the Cartan matrix

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

For the root systems that we need here, the possibilities are especially simple. If

$$A_{ij} = A_{ji} = 0,$$

the nodes are not connected. If

$$A_{ij} = A_{ji} = -1,$$

they are connected by a single bond. If the two entries are -1 and -2, the nodes are connected by a double bond, with the arrow pointing toward the shorter root.

The diagram of $B_r = \mathfrak{so}(2r+1)$

For

$$\alpha_i = e_i - e_{i+1} \quad (i = 1, \ldots, r-1), \qquad \alpha_r = e_r,$$

we have

$$(\alpha_i, \alpha_i) = 2 \quad (i < r), \qquad (\alpha_r, \alpha_r) = 1.$$

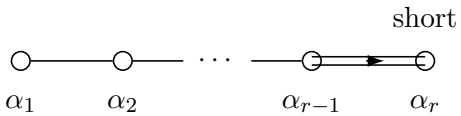
Thus $\alpha_1, \ldots, \alpha_{r-1}$ are long roots and α_r is short. Neighboring roots satisfy

$$(\alpha_i, \alpha_{i+1}) = -1,$$

while non-neighboring roots are orthogonal. In particular,

$$A_{r-1,r} = -1, \qquad A_{r,r-1} = -2,$$

so the last two nodes are joined by a double bond and the arrow points toward the short root α_r.



B_r Dynkin diagram: the arrow points toward the short simple root α_r.

The diagram of $D_r = \mathfrak{so}(2r)$

For D_r all simple roots have the same length,

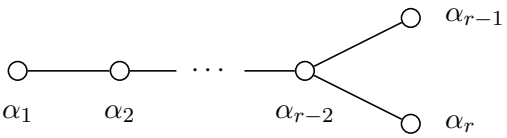
$$(\alpha_i, \alpha_i) = 2.$$

Therefore every bond is a single bond: D_r is a *simply-laced* root system. The only feature that is not a straight chain occurs at the end. With our choice of simple roots,

$$\alpha_{r-2} = e_{r-2} - e_{r-1}, \qquad \alpha_{r-1} = e_{r-1} - e_r, \qquad \alpha_r = e_{r-1} + e_r,$$

and hence

$$(\alpha_{r-2}, \alpha_{r-1}) = -1, \qquad (\alpha_{r-2}, \alpha_r) = -1, \qquad (\alpha_{r-1}, \alpha_r) = 0.$$



D_r Dynkin diagram: all roots have equal length and the last two nodes form the two branches of the fork.

Fundamental weights.

Let us now define the fundamental weights in a way that works for an arbitrary semisimple Lie algebra. Choose simple roots

$$\alpha_1, \ldots, \alpha_r.$$

The simple roots form a basis of the real vector space spanned by all roots. For each root we introduce the corresponding *coroot*

$$\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}.$$

The fundamental weights $\omega_1, \ldots, \omega_r$ are defined to be the basis dual to the simple coroots:

$$\langle \omega_i, \alpha_j^\vee \rangle = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}. \tag{8}$$

This is the general definition. It is useful because it does not assume that all roots have the same length. In a simply-laced algebra, such as D_r, all roots have length squared 2, so roots and coroots coincide. In B_r the short and long roots have different lengths, and the distinction between roots and coroots is essential.

An irreducible finite-dimensional representation contains a weight λ that cannot be raised further by positive-root operators. It is called the *highest weight*. Starting from a highest-weight vector and applying lowering operators generates the whole irreducible representation. Thus, in practice, an irreducible finite-dimensional representation is specified by its highest weight.

Finite-dimensional irreducible representations of the Lie algebra $\mathfrak{so}(N)$, or equivalently of the simply connected group $\mathrm{Spin}(N)$, are classified by dominant integral highest weights

$$\lambda = a_1\omega_1 + \cdots + a_r\omega_r, \qquad a_i \in \mathbb{Z}_{\geq 0}. \tag{9}$$

Let us explain both parts of this statement in more detail.

The fundamental weights form a basis of the real vector space spanned by the weight lattice, dual to the simple coroots. Therefore any weight can be expanded uniquely as in (9). Pairing this equation with the simple coroot $\alpha_i^\vee$ and using

$$\langle \omega_j, \alpha_i^\vee \rangle = \delta_{ji}$$

immediately gives

$$a_i = \langle \lambda, \alpha_i^\vee \rangle.$$

Why must these numbers be nonnegative integers for a finite-dimensional highest-weight representation? For each simple root α_i, the three generators

$$E_i = E_{\alpha_i}, \qquad F_i = E_{-\alpha_i}, \qquad H_i = \langle H, \alpha_i^\vee \rangle$$

form an $\mathfrak{sl}(2)$ subalgebra. With the usual normalization,

$$[H_i, E_i] = 2E_i, \qquad [H_i, F_i] = -2F_i, \qquad [E_i, F_i] = H_i.$$

If v_λ is the highest-weight vector, then

$$E_i v_\lambda = 0, \qquad H_i v_\lambda = a_i v_\lambda, \qquad a_i = \langle \lambda, \alpha_i^\vee \rangle.$$

Repeated application of F_i moves downward along the α_i-string of weights,

$$\lambda, \quad \lambda - \alpha_i, \quad \lambda - 2\alpha_i, \quad \ldots.$$

The elementary $\mathfrak{sl}(2)$ identity

$$E_i F_i^m v_\lambda = m(a_i - m + 1) F_i^{m-1} v_\lambda$$

shows why integrality appears. Since the representation is finite-dimensional, there is a last nonzero vector in this string. Suppose

$$F_i^q v_\lambda \neq 0, \qquad F_i^{q+1} v_\lambda = 0.$$

Applying E_i to the second equation gives

$$(q+1)(a_i - q) F_i^q v_\lambda = 0,$$

and hence

$$a_i = q \in \mathbb{Z}_{\geq 0}.$$

Thus the nonnegative-integrality condition in (9) is nothing but the familiar quantization of the highest spin in each embedded $\mathfrak{sl}(2)$.

Fundamental weights of B_r and D_r

For $B_r = \mathfrak{so}(2r+1)$ the first $r-1$ roots are long, while α_r is short. With $(e_i, e_j) = \delta_{ij}$,

$$\alpha_i^\vee = \alpha_i \quad (i < r), \qquad \alpha_r^\vee = 2e_r.$$

Solving (8) gives

$$\omega_k = e_1 + \cdots + e_k \quad (k = 1, \ldots, r-1), \qquad \omega_r = \frac{1}{2}(e_1 + \cdots + e_r).$$

For $D_r = \mathfrak{so}(2r)$ all roots have the same length. With the simple roots chosen above one finds

$$\omega_k = e_1 + \cdots + e_k \quad (k = 1, \ldots, r-2),$$

$$\omega_{r-1} = \frac{1}{2}(e_1 + \cdots + e_{r-1} - e_r), \qquad \omega_r = \frac{1}{2}(e_1 + \cdots + e_{r-1} + e_r).$$

The vector representation has highest weight

$$\lambda_V = e_1 = \omega_1.$$

The symmetric traceless rank-two tensor has highest weight

$$\lambda_{S_0^2 V} = 2e_1 = 2\omega_1.$$

For generic rank, the two-form/adjoint representation has highest weight

$$\lambda_{\mathrm{Adj}} = e_1 + e_2 = \omega_2.$$

For the higher exterior powers, one finds generically

$$\Lambda^p V \quad \leftrightarrow \quad \omega_p$$

for the range of p before one reaches the "end" of the Dynkin diagram. Specifically, for $B_r = \mathfrak{so}(2r+1)$

$$\Lambda^p V : \omega_p, \quad 1 \le p \le r-1, \quad \Lambda^r V : 2\omega_r,$$

and for $D_r = \mathfrak{so}(2r)$

$$\Lambda^p V : \omega_p, \quad 1 \le p \le r-2,$$

and the middle/end exterior powers require separate discussion. The weight ω_r for B_r and the weights ω_{r-1} and ω_r for D_r correspond to spinor representations.

Clifford algebra and spinor representations

Many representations are obtained by taking tensor products of the vector representation. However, they do not exhaust all the representations of $\mathfrak{so}(N)$. To construct the missing spinor representations we introduce the Clifford algebra. The complex Euclidean Clifford algebra is generated by elements γ_i, $i = 1, \ldots, N$, satisfying

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbf{1}. \tag{10}$$

Thus

$$\gamma_i^2 = 1, \qquad \gamma_i\gamma_j = -\gamma_j\gamma_i \quad (i \ne j).$$

We may choose the γ_i Hermitian. The operators

$$\Sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j] = \frac{1}{2}\gamma_i\gamma_j \qquad (i \ne j) \tag{11}$$

are anti-Hermitian and satisfy the $\mathfrak{so}(N)$ commutation relations. Hence any module of the Clifford algebra is automatically a representation space for $\mathfrak{so}(N)$.

There is a standard matrix realization of the Clifford algebra. Let $N = 2r$. Take the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They known to satisfy the relations

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_k. \tag{12}$$

The spinor space is

$$S = (\mathbb{C}^2)^{\otimes r}, \qquad \dim S = 2^r.$$

A standard choice of gamma matrices is

$$\gamma_{2a-1} = \sigma_3^{\otimes(a-1)} \otimes \sigma_1 \otimes \mathbf{1}^{\otimes(r-a)}, \quad \gamma_{2a} = \sigma_3^{\otimes(a-1)} \otimes \sigma_2 \otimes \mathbf{1}^{\otimes(r-a)}, \qquad a = 1, \ldots, r. \tag{13}$$

Because σ_1, σ_2 anticommute with σ_3, matrices belonging to different pairs anticommute, while each squares to one. Thus (13) satisfies (10).

For odd dimension $N = 2r+1$ we use the same 2^r-dimensional space and add

$$\gamma_{2r+1} \stackrel{\text{def}}{=} i^r\gamma_1 \ldots \gamma_{2r} = \sigma_3^{\otimes r}. \tag{14}$$

It anticommutes with every $\gamma_1, \dots, \gamma_{2r}$ and squares to one.

We can now derive weights of the spinor representation. Recall that for the vector representation we chose Hermitian Cartan generators

$$H_k = iM_{2k-1,2k}.$$

In the spinor representation, using (11), they are represented by

$$H_k^{(S)} = i\Sigma_{2k-1,2k} = \frac{i}{2}\gamma_{2k-1}\gamma_{2k}.$$

Substituting (13), all factors $\sigma_3^{\otimes(k-1)}$ cancel and

$$H_k^{(S)} = -\frac{1}{2}\,\mathbf{1} \otimes \cdots \otimes \sigma_3 \otimes \cdots \otimes \mathbf{1},$$

where σ_3 acts in the a-th factor.

Each $H_k^{(S)}$ has eigenvalues $\pm\frac{1}{2}$. Choose in every copy of $\mathbb{C}^2$ eigenvectors $|+\rangle_k, |-\rangle_k$ of $H_k^{(S)}$ with eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$

$$|+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The tensor-product vectors

$$|s_1, \dots, s_r\rangle = |s_1\rangle_1 \otimes \cdots \otimes |s_r\rangle_r, \qquad s_k = \pm 1,$$

are simultaneous eigenvectors:

$$H_k^{(S)}|s_1, \dots, s_r\rangle = \frac{s_k}{2}|s_1, \dots, s_r\rangle.$$

Hence the weight of this vector is

$$\mu = \frac{1}{2}(s_1 e_1 + \cdots + s_r e_r).$$

We have therefore derived

$$\mathrm{Wt}(S) = \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm \cdots \pm e_r) \right\}.$$

There are 2^r independent sign choices, exactly matching $\dim S = 2^r$.

For $SO(2r+1)$ this 2^r-dimensional spinor representation is irreducible and its highest weight is

$$\omega_r = \frac{1}{2}(e_1 + \cdots + e_r).$$

For $SO(2r)$ there is a chirality operator

$$\Gamma_* = i^r \gamma_1 \gamma_2 \cdots \gamma_{2r}. \qquad \Gamma_*^2 = 1.$$

It commutes with all Σ_{ij}, so the spinor space splits into two invariant eigenspaces,

$$S = S_+ \oplus S_-, \qquad \Gamma_*|_{S_\pm} = \pm 1, \quad S_\pm = \frac{1 \pm \Gamma_*}{2} S.$$

Each has dimension 2^{r-1}. In the basis above, chirality is determined by the parity of the number of minus signs in the weight. Thus one chiral representation contains

$$\frac{1}{2}(\pm e_1 \pm \cdots \pm e_r)$$

with an even number of minus signs and the other with an odd number. Their highest weights are ω_r and ω_{r-1} respectively.

Spin(N) **vs** $SO(N)$

The group $SO(N)$ describes ordinary rotations of vectors, but the Lie algebra alone does not remember the global topology of the group. The spin group Spin(N) is another Lie group with the same Lie algebra

$$\mathfrak{spin}(N) \simeq \mathfrak{so}(N),$$

but with different global topology. For $N \geq 3$, Spin(N) is the simply connected double cover of $SO(N)$. Thus every rotation $R \in SO(N)$ has two lifts to Spin(N), say g and $-g$, and both act in exactly the same way on ordinary vectors.

There is a very concrete construction of Spin(N) using the Euclidean Clifford algebra (10). Define a unit vector $n = n^i \gamma_i$ which satisfies $n^2 = 1$. Let another vector x be decomposed into its components parallel and perpendicular to n,

$$x = x_\parallel + x_\perp.$$

The Clifford relations (10) imply

$$n x_\parallel = x_\parallel n, \qquad n x_\perp = -x_\perp n.$$

Therefore $nxn = x_\parallel - x_\perp$, and thus $x \mapsto nxn$ is the reflection in the line spanned by n, while

$$x \longmapsto -nxn = x_\perp - x_\parallel$$

is the usual reflection in the hyperplane orthogonal to n.

Now compose two such hyperplane reflections, with unit normals n_1 and n_2:

$$x \longmapsto -n_1 x n_1 \longmapsto n_2 n_1 \, x \, n_1 n_2.$$

If

$$g = n_2 n_1,$$

then $g^{-1} = n_1 n_2$, and the composition becomes

$$x \longmapsto g x g^{-1}.$$

A product of an even number of reflections is an orientation-preserving orthogonal transformation, hence a rotation. This motivates the Clifford-algebra realization of Spin(N) as the group generated by even products of unit vectors.

It is also useful to see why conjugation by such an element sends vectors to vectors. Infinitesimally the generators of $\mathfrak{so}(N)$ are

$$\Sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j] = \frac{1}{2}\gamma_i \gamma_j \qquad (i \neq j),$$

and

$$[\gamma_i \gamma_j, \gamma_k] = 2\big(\delta_{jk}\gamma_i - \delta_{ik}\gamma_j\big),$$

which is again a linear combination of the γ's. Hence the vector subspace

$$V = \mathrm{span}\{\gamma_1, \ldots, \gamma_N\}$$

is preserved by conjugation by the elements even in γ_i. Since the Clifford product encodes the Euclidean scalar product through

$$xy + yx = 2(x \cdot y)\, \mathbf{1},$$

this conjugation preserves the scalar product and therefore defines an element of $SO(N)$.

If $g \in \mathrm{Spin}(N)$ and a vector $x = x^i\gamma_i$ is embedded into the Clifford algebra, the corresponding rotation is therefore

$$x \longmapsto gxg^{-1}.$$

Both g and $-g$ give the same transformation because

$$(-g)x(-g)^{-1} = gxg^{-1}.$$

This is the elementary origin of the two-to-one map $\mathrm{Spin}(N) \mapsto SO(N)$.

One more intuitive argument is as follows. Consider a rotation in the $(1,2)$ plane. Since

$$(\gamma_1\gamma_2)^2 = -1,$$

the element

$$S(\theta) = \exp\left(\frac{\theta}{2}\gamma_1\gamma_2\right) = \cos\frac{\theta}{2} + \gamma_1\gamma_2\sin\frac{\theta}{2} \tag{15}$$

belongs to $\mathrm{Spin}(N)$. Acting by conjugation on vectors gives an ordinary rotation through angle θ:

$$S(\theta)\gamma_1 S(\theta)^{-1} = \gamma_1\cos\theta - \gamma_2\sin\theta,$$
$$S(\theta)\gamma_2 S(\theta)^{-1} = \gamma_1\sin\theta + \gamma_2\cos\theta.$$

The important point is the factor $\theta/2$ in (15). A rotation by angle θ in vector space is represented by a motion through only half that angle in the spin group.

Now set $\theta = 2\pi$. From (15),

$$S(2\pi) = \cos\pi + \gamma_1\gamma_2\sin\pi = -1.$$

Nevertheless, on a vector the transformation is trivial $(-1)x(-1)^{-1} = x$. So the vector representation cannot distinguish $+1$ from -1 in $\mathrm{Spin}(N)$.

A spinor, by contrast, is a vector in a module on which Clifford-algebra elements act directly. Its transformation law is

$$\psi \longmapsto S(\theta)\psi.$$

Therefore a full 2π rotation gives

$$\psi \longmapsto -\psi.$$

After a second full turn,

$$S(4\pi) = +1,$$

so a spinor returns to itself after a 4π rotation.

It worth to understand the difference between $\mathrm{Spin}(N)$ and $SO(N)$ topologically. For $N \geq 3$ the group $SO(N)$ is not simply connected:

$$\pi_1(SO(N)) = \mathbb{Z}_2.$$

Let us first understand this statement geometrically for $SO(3)$, without using spinors.

Every rotation in three dimensions can be described by an axis $\mathbf{n}$ and an angle

$$0 \leq \alpha \leq \pi.$$

Represent this rotation by the point

$$\mathbf{r} = \alpha\,\mathbf{n}$$

in the closed three-dimensional ball of radius π. Points in the interior correspond uniquely to rotations. On the boundary, however,

$$\pi\,\mathbf{n} \sim -\pi\,\mathbf{n},$$

because a rotation by π around $\mathbf{n}$ is the same as a rotation by π around $-\mathbf{n}$. Thus one may picture

$$SO(3) \simeq B^3/\{\mathbf{r} \sim -\mathbf{r} \text{ on } \partial B^3\} = \mathbb{R}P^3.$$

The identity rotation is the centre of the ball.

Now consider a rotation about a fixed axis, say the third axis, with angle θ increasing from 0 to 2π. For

$$0 \le \theta \le \pi$$

the corresponding point moves from the centre to the boundary:

$$\mathbf{r}(\theta) = \theta\,\mathbf{e}_3.$$

For

$$\pi \le \theta \le 2\pi$$

we describe the same rotation by the angle $2\pi - \theta \le \pi$ around the opposite axis, so

$$\mathbf{r}(\theta) = -(2\pi - \theta)\,\mathbf{e}_3.$$

Hence the 2π rotation is represented by the path

$$0 \longrightarrow \pi\mathbf{e}_3 \sim -\pi\mathbf{e}_3 \longrightarrow 0.$$

In the quotient space this is a closed loop. In the ball picture it passes through the identified boundary exactly once.

This gives an elementary way to see why the loop cannot be shrunk to the constant loop. During a continuous deformation of a generic loop, intersections with the boundary may move, and a pair of intersections may be created or annihilated, but a single intersection cannot disappear by itself. Therefore the number of boundary crossings modulo 2 is unchanged by a continuous deformation:

$$\nu = \#(\text{boundary crossings}) \pmod 2.$$

For the constant loop $\nu = 0$, whereas for the loop corresponding to a 2π rotation, $\nu = 1$. They therefore cannot be continuously deformed into one another. A 4π rotation crosses the boundary twice, so its parity is zero; indeed that doubled loop can be contracted.

This picture is the geometric origin of

$$\pi_1(SO(3)) = \mathbb{Z}_2.$$

The two homotopy classes are simply "even" and "odd" according to this crossing parity.

The universal cover of $SO(3)$ is Spin(3). There is an exceptional isomorphism $\text{Spin}(3) \simeq SU(2)$. It can be seen explicitly in terms of treee dimensional gamma matrices.

From (13) and (14), we have

$$\gamma_i = \sigma_i, \quad i = 1, 2, 3.$$

The even Clifford algebra is spanned by

$$1, \qquad \gamma_2\gamma_3, \qquad \gamma_3\gamma_1, \qquad \gamma_1\gamma_2,$$

which can be rewritten using (12)

$$\gamma_2\gamma_3 = i\sigma_1, \qquad \gamma_3\gamma_1 = i\sigma_2, \qquad \gamma_1\gamma_2 = i\sigma_3.$$

Therefore a general even element has the form

$$g = x_0\mathbf{1} + i\, x_k\sigma_k.$$

The unit-norm condition appropriate to the spin group is

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1.$$

But every matrix of the form

$$g = x_0\mathbf{1} + i\, \mathbf{x}\cdot\boldsymbol{\sigma}, \qquad x_0^2 + \mathbf{x}^2 = 1,$$

is an element of $SU(2)$, and conversely every element of $SU(2)$ can be written in this form.Hence

$$\mathrm{Spin}(3) = SU(2) \simeq S^3,$$

since the four real numbers (x_0, x_1, x_2, x_3) lie on the unit three-sphere.

Note that for $N = 2$ the topology is slightly different: $SO(2) \simeq U(1)$ already has $\pi_1(SO(2)) = \mathbb{Z}$. Nevertheless there is still a double covering $\mathrm{Spin}(2) \simeq U(1) \to SO(2) \simeq U(1)$, given in angle variables by $\varphi \mapsto 2\varphi$.

Casimir operators

The distinction between $SO(N)$ and $\mathrm{Spin}(N)$ is a global question: it tells us which Lie-algebra representations exponentiate to which group. We now return to a purely representation-theoretic question. Once an irreducible representation has been specified by its highest weight, how can we attach useful numerical invariants to it? The simplest such invariant is the quadratic Casimir.

A Casimir operator is an operator built from the generators that commutes with every generator. Such operators give useful numerical labels for irreducible representations.

For $SO(N)$ a convenient quadratic Casimir is

$$C_2 = -\frac{1}{2}\sum_{i,j} T_{ij}T_{ij},$$

where T_{ij} are the generators in the representation under consideration. With anti-Hermitian generators this convention gives positive eigenvalues on the usual finite-dimensional representations.

By construction, by Schur's lemma,

$$[C_2, T_{ij}] = 0.$$

Therefore on an irreducible complex representation R,

$$C_2 = C_2(R)\,\mathbf{1}.$$

Choose the standard normalization in which long roots have length squared 2. If λ is the highest weight, then

$$C_2(\lambda) = (\lambda, \lambda + 2\rho), \tag{16}$$

where

$$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$$

is the Weyl vector.

For the vector representation $\lambda = e_1$, formula (16) gives

$$C_2(V) = N - 1.$$

For the adjoint representation one obtains

$$C_2(\mathrm{Adj}) = 2(N-2)$$

in the same normalization.

Exersises

1.