Lecture notes on CFT (March 3, 2024)
For comments: litvinov@itp.ac.ru

Lecture 1: Classical field theory, Noether theorem, $T_{\mu\nu}$, scaling and conformal invariances

Consider classical field theory in $\mathbb{R}^D$ given by the action

$$S = \int_{\mathbb{R}^D} \mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x})) d^D \boldsymbol{x}. \tag{1}$$

For simplicity we consider here the case of one bosonic field $\Phi(\boldsymbol{x})$, but it can be a collection of fields, can carry representation indices etc. The function $\mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}))$ is called the Lagrangian density. There are two conditions, which we require. First, is the locality: $\mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}))$ should contain interactions only in the same point, that is we forbid terms in the Lagrangian of the form $\Phi(\boldsymbol{x})\Phi(\boldsymbol{x}+\boldsymbol{a})$ etc. Second, we assume that the Langrangian does not include higher derivatives. This is what in principle can be violated, but we will not do it.

Having defined an action, one obtains equations of motion from the least action principle $\delta S = 0$. They are (for simplicity Φ here is just one scalar field)

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \Phi} \delta \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu \delta \Phi \right) d^D \boldsymbol{x} \implies \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) = 0. \tag{2}$$

The main task of the classical field theory is to solve equations of motion subject to certain boundary conditions, initial data etc.

Special role is played by the Integrals of Motion and conservation laws. Their existence is related to the Noether theorem, the relation between symmetries and conservation laws, which we briefly review now. Suppose our theory has a family of continuous transformations

$$\Phi(\boldsymbol{x}) \to \tilde{\Phi}(\boldsymbol{x}) = \mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})),$$

such that the action does not change $S[\Phi(\boldsymbol{x})] = S[\tilde{\Phi}(\boldsymbol{x})]$. For example, consider $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\boldsymbol{x}+\boldsymbol{a})$. This is a continuous transformation corresponding to the translation $\boldsymbol{x} \to \boldsymbol{x}+\boldsymbol{a}$. It is continuous because the vector $\boldsymbol{a}$ varies. It can be taken arbitrary small, in this case the transformation $\mathcal{F}$ will be in the vicinity of the identity transformation. Since our action is an integral over entire the space $\mathbb{R}^D$ and the Lagrangian $\mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}))$ does not depend explicitly on coordinates, which we assume, the action is invariant.

Another example: $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\Lambda \boldsymbol{x})$, where $\Lambda \in SO(D)$, corresponds to rotations (Lorentz transformation). The action is invariant if the derivatives enter the action in $SO(D)$ invariant way. The most general invariant Lagrangian of one bosonic field with at most two derivatives is

$$\mathcal{L} = (\partial_\mu \Phi)^2 + V(\Phi), \tag{3}$$

where V is an arbitrary functions.

Another important example of the symmetry involves only transformation of the fields. For example, consider complex scalar field

$$X = \Phi_1 + i\Phi_2,$$

with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu X \partial_\mu X^* + V(|X|^2) = \frac{1}{2}\left(\left(\partial_\mu \Phi_1\right)^2 + \left(\partial_\mu \Phi_2\right)^2\right) + V\left(\Phi_1^2 + \Phi_2^2\right).$$

This action is manifestly invariant under arbitrary $U(1)$ rotation

$$X \to e^{i\alpha} X, \quad X^* \to e^{-i\alpha} X^*,$$

again a continuous symmetry.

Given a symmetry, one derives Noether current. Suppose that our theory admits a family of transformations indexed by a continuous parameter ϵ

$$\Phi(\boldsymbol{x}) \to \tilde{\Phi}(\boldsymbol{x}) = \mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\boldsymbol{x}) + \epsilon f\big(\boldsymbol{x}, \Phi(\boldsymbol{x})\big) + O(\epsilon^2),$$

such that the action does not change $S[\Phi(\boldsymbol{x})] = S[\tilde{\Phi}(\boldsymbol{x})]$. Consider the following infinitesimal transformation

$$\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x}) f\big(\boldsymbol{x}, \Phi(\boldsymbol{x})\big) + O(\epsilon^2)$$

where, what is important, $\epsilon(\boldsymbol{x})$ depends on a position $\boldsymbol{x}$. Then the corresponding variation of the action S takes the form (here we assume that the action contains at most first derivatives)

$$\delta S = \int \Big(\epsilon(\boldsymbol{x}) J(\boldsymbol{x}) + \partial_\mu \epsilon(\boldsymbol{x})\, K_\mu(\boldsymbol{x})\Big) d^D\boldsymbol{x},$$

with some $J(\boldsymbol{x})$ and $K_\mu(\boldsymbol{x})$. By assumption, $\delta S = 0$ for constant ϵ. This is possible if (we assume no boundary issues here)

$$J(\boldsymbol{x}) = \partial_\mu J_\mu(\boldsymbol{x}),$$

and hence

$$\delta S = \int \Big(K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})\Big) \partial_\mu \epsilon(\boldsymbol{x}) d^D\boldsymbol{x} = -\int \partial_\mu \Big(K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})\Big) \epsilon(\boldsymbol{x}) d^D\boldsymbol{x}.$$

What we just made is an arbitrary variation of the action. It should vanish "on-shell", i.e. provided that classical equations of motion (2) are satisfied. But the function $\epsilon(\boldsymbol{x})$ is arbitrary. This implies that $j_\mu(\boldsymbol{x}) = K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})$ satisfies the continuity equation

$$\partial_\mu j_\mu(\boldsymbol{x}) = \partial_\mu \left(K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})\right) \overset{\text{on-shell}}{=} 0. \tag{4}$$

The current j_μ is usually referred as Noether current.

The continuity equation (4) then implies the conservation law. Namely, by Stokes theorem we have

$$\oint_{\partial\mathcal{M}} j_\mu(\boldsymbol{x}) d\sigma_\mu = \int_{\mathcal{M}} \partial_\mu j_\mu(\boldsymbol{x}) d^D\boldsymbol{x} = 0,$$

for any closed "surface" $\partial\mathcal{M}$. In particular, if we take $\mathcal{M}$ to be very large cylinder between the two time slices $x_1 = t_1$ and $x_1 = t_2$, we get[1]

$$Q_{t_1} = Q_{t_2}, \quad \text{where} \quad Q_t = \int j_1(\boldsymbol{x}) d^{D-1}\boldsymbol{x}\Big|_t.$$

[1]We note that the choice of the time slice is not canonically defined.

Among other Noether currents, the one, called the stress-energy tensor, will be primarily important for us. It is conserved due to invariance of the action under translations. Consider variation of the action (1) under arbitrary coordinate transformations $\boldsymbol{x} \to \boldsymbol{x} + \epsilon(\boldsymbol{x})$. In other words, we compute the response of the action (1) to the substitution $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x} + \boldsymbol{\epsilon}(\boldsymbol{x})) = \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x}) + \dots$

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} \left[\epsilon_\nu \Big(\frac{\partial\mathcal{L}}{\partial\Phi}\partial_\nu\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\mu\partial_\nu\Phi\Big) + \partial_\mu\epsilon_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi\right] d^D\boldsymbol{x}. \tag{5}$$

First term in (5) equals to $\epsilon_\nu\partial_\mu(\delta_{\mu\nu}\mathcal{L})$ as a reflection of the fact that the action does not depend on $\boldsymbol{x}$ explicitly and hence for constant ϵ_μ the variation should vanish. After integrating by parts we get

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} \partial_\mu\epsilon_\nu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta_{\mu\nu}\mathcal{L}\right) d^D\boldsymbol{x} \overset{\text{def}}{=} \int_{\mathbb{R}^D} \partial_\mu\epsilon_\nu T_{\mu\nu} d^D\boldsymbol{x} = -\int_{\mathbb{R}^D} \epsilon_\nu\partial_\mu T_{\mu\nu} d^D\boldsymbol{x}. \tag{6}$$

On shell the variation (6) should vanish. Since the function $\epsilon_\mu(\boldsymbol{x})$ is arbitrary it implies the continuity condition for $T_{\mu\nu}$

$$\partial_\mu T_{\mu\nu} = 0,$$

and hence conservation of the energy and momentum

$$E \overset{\text{def}}{=} \int T_{11} d^{D-1}\boldsymbol{x}, \qquad P_i \overset{\text{def}}{=} \int T_{1i} d^{D-1}\boldsymbol{x}, \quad i \neq 1.$$

Derivation given above leads to the definition of the stress-energy tensor as a response to the infinitesimal coordinate change

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} \partial_\mu\epsilon_\nu T_{\mu\nu} d^D\boldsymbol{x} \tag{7}$$

However, in some cases this derivation requires extra care. In fact $T_{\mu\nu}$ is not canonically defined by (7) as this definition contains intrinsic ambiguity. Indeed, in our example with one bosonic field one can change the transformation rules $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x}) + \dots$ to the more general ones

$$\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x}) + \partial_\mu\epsilon_\nu(\boldsymbol{x})\Sigma_{\mu\nu}[\Phi(\boldsymbol{x})] + \dots$$

where $\Sigma_{\mu\nu}[\Phi(\boldsymbol{x})]$ are some functions of $\Phi(\boldsymbol{x})$ and $\dots$ may contain higher derivatives of $\boldsymbol{\epsilon}(\boldsymbol{x})$. The stress-energy tensor changes as

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta_{\mu\nu}\mathcal{L} + \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\Phi)}\right)\right)\Sigma_{\mu\nu} + \dots \tag{8}$$

We note that the additional term in (8) is proportional to equations of motion and hence both tensors coincide on-shell. The stress-energy tensor without additional terms, i.e. given by

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta_{\mu\nu}\mathcal{L}$$

is known as *canonical* stress-energy tensor.

In $SO(D)$ invariant FT involving only scalar fields the canonical stress-energy tensor always comes out to be symmetric $T_{\mu\nu} = T_{\nu\mu}$. For example for the theory with multicomponent bosonic field $\boldsymbol{\Phi} = (\Phi_1, \dots, \Phi_n)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\boldsymbol{\Phi}\cdot\partial_\mu\boldsymbol{\Phi}) + V(\boldsymbol{\Phi}),$$

the stress-energy tensor has manifestly symmetric form

$$T_{\mu\nu} = (\partial_\mu \boldsymbol{\Phi} \cdot \partial_\nu \boldsymbol{\Phi}) - \delta_{\mu\nu}\mathcal{L}.$$

In general this is not the case, but in $SO(D)$ invariant theories $T_{\mu\nu}$ can always be made symmetric. It can be seen as follows. Consider $\epsilon_\nu = \omega_{\nu\lambda}x_\lambda$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, then from (7) we have

$$\delta_{\boldsymbol{\epsilon}} S = \frac{1}{2}\int_{\mathbb{R}^D} \left[\partial_\mu \omega_{\nu\lambda}\left(x_\lambda T_{\mu\nu} - x_\nu T_{\mu\lambda}\right) - \omega_{\mu\nu}\big(T_{\mu\nu} - T_{\nu\mu}\big)\right] d^D\boldsymbol{x}. \tag{9}$$

For constant $\omega_{\mu\nu}$ this variation should vanish for rotationally invariant theories, which implies

$$T_{\mu\nu} - T_{\nu\mu} = \partial_\lambda f_{\lambda\mu\nu}, \qquad f_{\lambda\mu\nu} = -f_{\lambda\nu\mu}.$$

Now we define the modified tensor

$$\tilde{T}_{\mu\nu} \overset{\text{def}}{=} T_{\mu\nu} - \partial_\lambda B_{\lambda\mu\nu} \quad \text{where} \quad B_{\lambda\mu\nu} = \frac{1}{2}\big(f_{\lambda\mu\nu} - f_{\mu\lambda\nu} - f_{\nu\lambda\mu}\big). \tag{10}$$

We note that the tensor $B_{\lambda\mu\nu}$ is antisymmetric in first two indexes $B_{\lambda\mu\nu} = -B_{\mu\lambda\nu}$, which implies

$$\partial_\mu\partial_\lambda B_{\lambda\mu\nu} \equiv 0.$$

At the same time, we have

$$B_{\lambda\mu\nu} - B_{\lambda\nu\mu} = f_{\lambda\mu\nu},$$

and hence the modified stress-energy tensor (10) is symmetric

$$\tilde{T}_{\mu\nu} = \tilde{T}_{\nu\mu}.$$

This tensor is known as Belinfante tensor. Integrating by parts (9) one finds conservation of the angular momentum current

$$\partial_\mu\big(x_\nu T_{\mu\lambda} - x_\lambda T_{\mu\nu} + f_{\lambda\mu\nu}\big) = \partial_\mu\big(x_\nu \tilde{T}_{\mu\lambda} - x_\lambda \tilde{T}_{\mu\nu}\big) \overset{\text{def}}{=} 0.$$

These difficulties can be overcome by adopting an alternative point of view and notice that the change of fields $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x} + \boldsymbol{\epsilon}(\boldsymbol{x}))$ can be supplemented by the change of coordinates $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\epsilon}(\boldsymbol{x})$ or $\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{\epsilon}(\boldsymbol{y}) + \ldots$, such that the fields do not change, but we have to replace (infinitesimally)

$$\frac{\partial}{\partial x^\mu} = (\delta_{\mu\nu} + \partial_\mu \epsilon_\nu)\frac{\partial}{\partial y^\mu}, \qquad d^D\boldsymbol{x} = (1 - \partial_\nu\epsilon_\nu)d^D\boldsymbol{y}.$$

Of course this variation leads to the same conclusion (6) with points redefinition $\boldsymbol{x} \to \boldsymbol{y}$. We note that the transformation $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{\epsilon}$ induces the following variation of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} = -(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu).$$

So, we come to an idea to define the stress-energy tensor as a response to the infinitesimal variation of the background metric. Namely, we assume that the action (1) admits a covariant extension $S[\Phi] \to S[\Phi, g]$. Then we define the stress-energy tensor as the flat space limit of gravitational stress-energy tensor

$$T_{\mu\nu} = T^g_{\mu\nu}\Big|_{g_{\mu\nu}\to\delta_{\mu\nu}}, \tag{11}$$

defined by

$$\delta S = \frac{1}{2}\int \sqrt{g}\, T^g_{\mu\nu}\delta g^{\mu\nu} d^D\boldsymbol{x}.$$

Equivalently the definition (11) can be written as

$$T_{\mu\nu} = \left(\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\right)\Bigg|_{g_{\mu\nu}\to\delta_{\mu\nu}}$$

From this definition it is clear that $T_{\mu\nu} = T_{\nu\mu}$.

Note, that in the flat space the definition (11) is still ambiguous, as one can add terms to the action, which vanish at $g_{\mu\nu} \to \delta_{\mu\nu}$. For example, one can add the so called *dilaton* term

$$\int W(\Phi) R\sqrt{g}\, d^D\boldsymbol{x}, \tag{12}$$

where R is the scalar curvature and $W(\Phi)$ is arbitrary dilaton potential. This term disappears in the flat space, however it affects the form of $T_{\mu\nu}$. The variation of the dilaton term has the form

$$\delta\int W[\Phi]R\sqrt{g}\, d^D\boldsymbol{x} = \delta\int W[\Phi]R_{\mu\nu}g^{\mu\nu}\sqrt{g}\, d^D\boldsymbol{x} = \int W[\Phi]\delta R_{\mu\nu}g^{\mu\nu}\sqrt{g}\, d^D\boldsymbol{x} + \dots \tag{13}$$

where terms shown by $\dots$ do not contribute to $T_{\mu\nu}$ at $g_{\mu\nu} \to \delta_{\mu\nu}$. Using

$$\delta R_{\mu\nu}g^{\mu\nu} = \nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \nabla^2\big(g_{\mu\nu}\delta g^{\mu\nu}\big)$$

and integrating by parts in (13), we find that dilaton term modifies $T_{\mu\nu}$ as follows

$$T_{\mu\nu} \to T_{\mu\nu} + 2\left(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)W[\Phi]$$

The most general term, which can be added to the action, and which gives a non-vanishing contribution to $T_{\mu\nu}$ in the limit $g_{\mu\nu} \to \delta_{\mu\nu}$, but vanishes in this limit, is

$$\int R^{\mu\sigma\nu\rho}Y_{\mu\sigma\nu\rho}(\Phi)\sqrt{g}\, d^D\boldsymbol{x}, \tag{14}$$

where $R^{\mu\sigma\nu\rho}$ is the Riemann tensor for the background metric and $Y_{\mu\sigma\nu\rho}(\Phi)$ is some local tensor field which is antisymmetric in $(\mu\sigma)$ and in $(\nu\rho)$, but symmetric with respect to exchange of these pairs. This term gives the following contribution to the stress-energy tensor

$$T_{\mu\nu} \to T_{\mu\nu} + 2\partial_\sigma\partial_\rho Y_{\mu\sigma\nu\rho}.$$

The theory in which the terms like (14) and similar are absent called minimal covariant extension.

From now we assume that the theory has a symmetric stress-energy tensor defined by (7). Important class of theories obey the property of scale invariance. Let us probe if the Poincaré invariant action (3) is scale invariant as well. Namely, let $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \lambda^\Delta\Phi(\lambda\cdot\boldsymbol{x})$, where Δ is the so called scaling dimension. Then we immediately see that the first term in the action (3) is invariant if

$$\Delta = \frac{D-2}{2}.$$

Then it is clear that $V(\Phi)$ has to be a powerlike Φ^n, where

$$n = \frac{D}{\Delta} = \frac{2D}{D-2}.$$

We see, that n is rarely an integer. The only exceptions are: $n = 6$ for $D = 3$, $n = 4$ for $D = 4$ and $n = 3$ for $D = 6$. The case $D = 2$ is exceptional because scalar field is dimensionless in this case and we can not built scale invariant theory with power like potential. Instead we have Liouville theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 + e^{\Phi}, \tag{15}$$

which is scale invariant with $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\lambda \cdot \boldsymbol{x}) + 2 \log \lambda$.

Another scale invariant theory in two dimensions is known as non-linear sigma model

$$\mathcal{L} = G_{ab}(\boldsymbol{\Phi}) \partial_\mu \Phi^a \partial_\mu \Phi^b.$$

Here $\boldsymbol{\Phi} = (\Phi^1, \dots, \Phi^N)$ is the $N-$component bosonic field and $G_{ab}(\boldsymbol{\Phi})$ is some matrix function (since Φ is dimensionless this function is dimensionless). In string theory one interprets $\boldsymbol{\Phi}$ as coordinates on some "target" Riemanian manifold $\mathcal{M}$ and $G_{ab}(\boldsymbol{\Phi})$ as a metric on it.

The last, but not the least example is the Yang-Mills theory (here fundamental fields A_μ take values in some simple Lie algebra, say $\mathfrak{su}(N)$)

$$S = \int \mathrm{Tr}(F_{\mu\nu}^2)\, d^D x \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

To insure scale invariance both terms in $F_{\mu\nu}$ should have the same scale dimension. From this condition we find $\Delta = 1$. Then the action transforms

$$S \to \lambda^{4-D} S,$$

and hence this theory is scale invariant only in 4 dimensions.

Going back to the definition (7) we see that if the theory is scale invariant then we should have

$$\int \Theta \, d^D \boldsymbol{x} = 0 \quad \text{where} \quad \Theta \stackrel{\text{def}}{=} T_{\mu\mu},$$

which requires $\Theta = \partial_\mu \theta_\mu$, so that

$$D_\mu = \theta_\mu - x_\nu T_{\mu\nu}$$

is a conserved current called the scale current.

The scale invariance might imply the extended symmetry known as the *conformal* symmetry. For example, one can notice that if θ_μ in turn is a gradient $\theta_\mu = \partial_\mu L$ then one can redefine $T_{\mu\nu}$

$$T_{\mu\nu} \to \tilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{D-1} \left(\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2 \right) L \tag{16}$$

to make it traceless[2]. This redefinition of $T_{\mu\nu}$ corresponds to the dilaton term (12) in the curved space

$$W = \frac{1}{2(1-D)} L.$$

[2]For $D > 2$ it is enough to have $\Theta = \partial_\mu \partial_\nu L_{\mu\nu}$ then the improved tensor

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{D-2} \left(\partial_\mu \partial_\lambda L_{\lambda\nu} + \partial_\nu \partial_\lambda L_{\lambda\mu} - \partial^2 L_{\mu\nu} - \delta_{\mu\nu} \partial_\lambda \partial_\rho L_{\lambda\rho} \right) + \frac{1}{(D-2)(D-1)} \left(\delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right) L_{\lambda\lambda}$$

is traceless (see [1] for more details)

Let us consider $\lambda\Phi^4$ theory in four dimensions

$$S = \int \left(\frac{1}{2}(\partial_\mu \Phi)^2 + \lambda \Phi^4 \right) d^4\boldsymbol{x}.$$

While computing the stress-energy tensor, we are free to choose the transformation rule for the field Φ. The only condition is that it is reduced to $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}+\boldsymbol{\epsilon})$ for constant $\boldsymbol{\epsilon}$. As we saw above Φ has dimension 1 in $D=4$. It suggests to consider the replacement

$$\Phi(\boldsymbol{x}) \to \left(1 + \frac{\partial_\mu \epsilon_\mu}{4}\right) \Phi(\boldsymbol{x}+\boldsymbol{\epsilon}).$$

Then formula (8) implies that

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \delta_{\mu\nu} \left(\frac{1}{2}(\partial\Phi)^2 + \lambda\Phi^4 \right) + \frac{1}{4}(4\lambda\Phi^3 - \partial^2\Phi)\Phi\delta_{\mu\nu},$$

where the last term corresponds to $\Sigma_{\mu\nu}$ in (8). We see that this stress-energy tensor satisfies conditions specified above, that is

$$\Theta = -\frac{1}{2}\partial^2\Phi^2,$$

and hence the improved stress-energy tensor (16) is traceless. The corresponding dilaton field has the form $W = \frac{\Phi^2}{6}$.

The vanishing of Θ implies the invariance under conformal transformations, whose infinitesimal form is

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(\boldsymbol{x})\delta_{\mu\nu}. \tag{17}$$

It will be studied in the next lecture.

Problems:

1. Consider electrodynamics

$$S = \frac{1}{4}\int F_{\mu\nu}^2 d^D\boldsymbol{x}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

   - Find canonical stress-energy tensor (i.e. pretending that A_μ's are scalars).
   - Find modified stress-energy tensor (i.e. treat $\boldsymbol{A}$ as a vector)
   - Find gravitational stress-energy tensor

   discuss the results.

2. Consider Liouville theory (15)

   - Compute stress-energy tensor and show that it can be made traceless
   - Consider embedding of (15) into background metric. Adjust dilaton term (12) in such a way that $\Theta = 0$.

Lecture 2: The conformal group

The transformation (17) is the infinitesimal form of the conformal transformation, that is an invertible map $\boldsymbol{x} \to \boldsymbol{x}'$ which leaves the metric $\delta_{\mu\nu}$ invariant up to a scale

$$\delta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(\boldsymbol{x}) \delta_{\mu\nu}. \tag{18}$$

We note that $\Lambda(x) = 1$ corresponds to the Poincaré group consisting of rotations and translations. These transformations preserve the distances, while the general conformal transformations only preserve the angles.

In infinitesimal form we have the condition

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = f(\boldsymbol{x})\delta_{\mu\nu}.$$

Contracting this equation with $\delta^{\mu\nu}$ we find that $f(\boldsymbol{x}) = \frac{2}{D}(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon})$ and hence

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{D}(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon})\delta_{\mu\nu}. \tag{19}$$

Applying ∂^{ν} we get

$$\left(1 - \frac{2}{D}\right) \partial_{\mu}(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) + \partial^2 \epsilon_{\mu} = 0.$$

Furthermore we take ∂_{ν} and symmetrize $\mu \leftrightarrow \nu$ to find

$$\left(1 - \frac{2}{D}\right) \partial_{\mu}\partial_{\nu}(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) + \frac{1}{2}\partial^2 \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right) = 0.$$

Finally, using (19), we find[3]

$$\big((D-2)\partial_{\mu}\partial_{\nu} + \delta_{\mu\nu}\partial^2\big)(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) = 0 \Longrightarrow (D-1)\partial^2(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) = 0 \overset{D>2}{\Longrightarrow} \partial_{\mu}\partial_{\nu}(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) = 0. \tag{20}$$

Another useful identity is obtained from (19) by taking $\partial_{\mu}(19)_{\nu\rho} + \partial_{\nu}(19)_{\mu\rho} - \partial_{\rho}(19)_{\mu\nu}$

$$2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \frac{2}{D}\left(\delta_{\rho\mu}\partial_{\nu} + \delta_{\rho\nu}\partial_{\mu} - \delta_{\mu\nu}\partial_{\rho}\right)(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}). \tag{21}$$

We see from (20) that the cases $D = 2$ and $D > 2$ are different. Let us consider the case $D > 2$ first. Equation (20) implies that the function $(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon})$ is linear[4] and then equation (21) implies that $\partial_{\mu}\partial_{\nu}\epsilon_{p}$ is a constant. Henceforth $\epsilon_{\mu}(\boldsymbol{x})$ are quadratic functions

$$\epsilon_{\mu}(\boldsymbol{x}) = a_{\mu} + b_{\mu\nu}x^{\nu} + c_{\mu\nu\lambda}x^{\nu}x^{\lambda}, \tag{22}$$

---

[3] From (20) we see that for conformal invariance it is enough to have $\Theta = \partial_{\mu}\partial_{\nu}L_{\mu\nu}$ for $D > 2$ and $\Theta = \partial^2 L$ for $D = 2$. It follows from the fact that for conformal transformations the variation (7) takes the form

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} (\boldsymbol{\partial} \cdot \boldsymbol{\epsilon})\Theta \, d^D\boldsymbol{x},$$

and hence in the virtue of (20) it can be integrated to zero if either $\Theta = \partial_{\mu}\partial_{\nu}L_{\mu\nu}$ or $\Theta = \partial^2 L$ holds.

[4] Indeed $\partial_{\mu}\partial_{\nu}f(\boldsymbol{x}) = 0$ for $\mu \neq \nu$ implies that $f(\boldsymbol{x}) = \sum_{\mu} f_{\mu}(x_{\mu})$. And then $\partial_{\mu}\partial_{\mu}f(\boldsymbol{x})$(no summation in μ) implies that $f_{\mu}(x_{\mu})$ is a linear function.

subject to the condition (19). The constant term a_μ is not constrained at all. It represents the infinitesimal translations. The linear term $b_{\mu\nu}$ obeys

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D}\delta_{\mu\nu} b_{\lambda\lambda}. \tag{23}$$

General solution to (23) is

$$b_{\mu\nu} = \alpha\delta_{\mu\nu} + \omega_{\mu\nu} \quad \text{where} \quad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

The antisymmetric part represents infinitesimal rotations, while the pure trace part corresponds to the scale transformation $x'^\mu = (1+\alpha)x^\mu$.

So, we are left with the quadratic term $c_{\mu\nu\lambda}$. Inserting (22) into (21), one finds

$$c_{\mu\nu\rho} = \delta_{\nu\rho}\zeta_\mu - \delta_{\mu\rho}\zeta_\nu - \delta_{\mu\nu}\zeta_\rho \quad \text{where} \quad \zeta_\mu = -\frac{1}{D}c^{\rho}_{\rho\mu},$$

which corresponds to the infinitesimal transformation

$$x'^\mu = x^\mu - 2(\boldsymbol{x}\cdot\boldsymbol{\zeta})x^\mu + \zeta^\mu \boldsymbol{x}^2 + \ldots,$$

called the *Special Conformal Transformation.*

Finite conformal transformation can be obtained by exponentiation. They and the corresponding generators acting on functions are summarized in the following table[5]

	Transformation	Generators
Translation	$x'^\mu = x^\mu + a^\mu$	$\mathcal{P}_\mu = -i\partial_\mu,$
Rotation	$x'^\mu = \Omega^\mu_\nu x^\nu$	$\mathcal{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu),$
Dilatation	$x'^\mu = \lambda x^\mu$	$\mathcal{D} = -i(\boldsymbol{x}\cdot\boldsymbol{\partial}),$
SCT	$x'^\mu = \dfrac{x^\mu + \boldsymbol{x}^2\zeta^\mu}{1+2(\boldsymbol{\zeta}\cdot\boldsymbol{x}) + \boldsymbol{\zeta}^2\boldsymbol{x}^2}$	$\mathcal{K}_\mu = -i(\boldsymbol{x}^2\partial_\mu - 2x_\mu(\boldsymbol{x}\cdot\boldsymbol{\partial}))$

One can check that SCT is indeed a conformal transformation with the scale factor $\Lambda = (1 + 2(\boldsymbol{\zeta}\cdot\boldsymbol{x}) + \boldsymbol{\zeta}^2\boldsymbol{x}^2)^2$. More intuitive way of thinking about the SCT comes from the formula

$$\frac{x'^\mu}{\boldsymbol{x}'^2} = \frac{x^\mu}{\boldsymbol{x}^2} + \zeta^\mu.$$

It means that we can define the special conformal transformations by combining an inversion with a translation and then another inversion

$$x^\mu \longrightarrow \frac{x^\mu}{\boldsymbol{x}^2} \longrightarrow \frac{x^\mu}{\boldsymbol{x}^2} + \zeta^\mu \longrightarrow \frac{\frac{x^\mu}{\boldsymbol{x}^2} + \zeta^\mu}{(\frac{\boldsymbol{x}}{\boldsymbol{x}^2} + \boldsymbol{\zeta})^2} = \frac{x^\mu + \boldsymbol{x}^2\zeta^\mu}{1 + 2(\boldsymbol{\zeta}\cdot\boldsymbol{x}) + \boldsymbol{\zeta}^2\boldsymbol{x}^2}. \tag{24}$$

[5] We note that

$$\mathcal{D} = -i(\boldsymbol{x}\cdot\boldsymbol{\partial}) = -i\sum_{\mu=1}^{D}\frac{\partial}{\partial \log x_\mu} \implies e^{i\epsilon\mathcal{D}}f(\boldsymbol{x}) = f\left(e^\epsilon \boldsymbol{x}\right)$$

and

$$\mathcal{K}_\mu = -i(\boldsymbol{x}^2\partial_\mu - 2x_\mu(\boldsymbol{x}\cdot\boldsymbol{\partial})) \overset{\boldsymbol{y}=\frac{\boldsymbol{x}}{\boldsymbol{x}^2}}{=} -i\frac{\partial}{\partial y^\mu} \implies e^{i\xi^\mu\mathcal{K}_\mu}f(\boldsymbol{x}) = f\left(\frac{\frac{x^\mu}{\boldsymbol{x}^2} + \zeta^\mu}{(\frac{\boldsymbol{x}}{\boldsymbol{x}^2} + \boldsymbol{\zeta})^2}\right)$$

One can easily check that the inversion obeys (18), so does the SCT.

We note that the SCT is not globally well defined in $\mathbb{R}^D$. In particular the point

$$y^\mu = -\frac{\zeta^\mu}{\boldsymbol{\zeta}^2}$$

is mapped to infinity. Therefore in order to define SCT globally, one usually considers conformal compactification of $\mathbb{R}^D$ which is S^D, or the D-sphere.

It is interesting to identify the conformal algebra for $D > 2$, that is the Lie algebra of $(\mathcal{P}_\mu, \mathcal{L}_{\mu\nu}, \mathcal{D}, \mathcal{K}_\mu)$. First, we compute the number of generators. Keeping in mind that $\mathcal{L}_{\mu\nu}$ is antisymmetric, we have

$$D + \frac{D(D-1)}{2} + 1 + D = \frac{(D+2)(D+1)}{2}.$$

We note that, it is equal to the size of $SO(D+2)$. The generators $\mathcal{D}$, $\mathcal{K}_\mu$ and $\mathcal{P}_\mu$, $\mathcal{L}_{\mu\nu}$ admit the following commutation relations

$$\begin{aligned}
&[\mathcal{D}, \mathcal{P}_\mu] = i\mathcal{P}_\mu, \qquad &&[\mathcal{D}, \mathcal{L}_{\mu\nu}] = 0, \qquad &&[\mathcal{D}, \mathcal{K}_\mu] = -i\mathcal{K}_\mu, \\
&[\mathcal{K}_\mu, \mathcal{K}_\nu] = 0, \qquad &&[\mathcal{K}_\mu, \mathcal{P}_\nu] = -2i\left(\delta_{\mu\nu}\mathcal{D} + \mathcal{L}_{\mu\nu}\right), \qquad &&[\mathcal{K}_\lambda, \mathcal{L}_{\mu\nu}] = i(\delta_{\lambda\mu}\mathcal{K}_\nu - \delta_{\lambda\nu}\mathcal{K}_\mu),
\end{aligned} \tag{25}$$

plus those of Poincaré algebra

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0, \quad [\mathcal{P}_\lambda, \mathcal{L}_{\mu\nu}] = i(\delta_{\lambda\mu}\mathcal{P}_\nu - \delta_{\lambda\nu}\mathcal{P}_\mu), \quad [\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] = i(\delta_{\nu\rho}\mathcal{L}_{\mu\sigma} + \delta_{\mu\sigma}\mathcal{L}_{\nu\rho} - \delta_{\mu\rho}\mathcal{L}_{\nu\sigma} - \delta_{\nu\sigma}\mathcal{L}_{\mu\rho}). \tag{26}$$

These commutation relations can be brought to the convenient form by defining

$$\mathcal{J}_{\mu\nu} = \mathcal{L}_{\mu\nu}, \quad \mathcal{J}_{D+2\,D+1} = \mathcal{D}, \quad \mathcal{J}_{D+1\,\mu} = \frac{1}{2}(\mathcal{P}_\mu + \mathcal{K}_\mu), \quad \mathcal{J}_{D+2\,\mu} = \frac{1}{2}(\mathcal{P}_\mu - \mathcal{K}_\mu),$$

where $\mathcal{J}_{MN} = -\mathcal{J}_{NM}$. Then the new generators satisfy the relations of $SO(D+1,1)$ Lie algebra

$$[\mathcal{J}_{MN}, \mathcal{J}_{RS}] = i\left(\eta_{NR}\mathcal{J}_{MS} + \eta_{MS}\mathcal{J}_{NR} - \eta_{MR}\mathcal{J}_{NS} - \eta_{NS}\mathcal{J}_{MR}\right),$$

where η_{MN} is the diagonal matrix with Minkowski signature $(1, 1, \ldots, 1, -1)$.

It can be seen by explicit calculations and we leave it as an exercise, but it is better to derive it from the following arguments. Consider the vector

$$\boldsymbol{X} = (x^1, \ldots, x^D, \frac{1-\boldsymbol{x}^2}{2}, \frac{1+\boldsymbol{x}^2}{2}) \in \mathbb{R}^{D+1,1}$$

It is not arbitrary, but subject to two additional constraints. First it is a "light-like" vector $\boldsymbol{X}^2 = \boldsymbol{x}^2 + (\frac{1-\boldsymbol{x}^2}{2})^2 - (\frac{1+\boldsymbol{x}^2}{2})^2 = 0$. Second constraint is the condition $X^{D+1} + X^{D+2} = 1$, which defines the section of the light-cone. This section is parameterized by $\boldsymbol{x}$, which are our original coordinates in $\mathbb{R}^D$. One can easily check that the induced metric on $\mathbb{R}^D$ coincides with the flat metric. The group $SO(D+1,1)$ acts on $\mathbb{R}^{D+1,1}$ as

$$X^M \to \Lambda^M_N X^N. \tag{27}$$

We want to project this action to our section: $\boldsymbol{X}^2 = 0$, $X^{D+1} + X^{D+2} = 1$. The first constraint is preserved by this action, while the second transforms

$$1 = X^{D+1} + X^{D+2} \to \lambda(\boldsymbol{X}),$$

where $\lambda(\boldsymbol{X})$ is some linear function. So, we just replace (27) by the transformation

$$X^M \to \lambda^{-1}(\boldsymbol{X})\Lambda^M_N X^N, \tag{28}$$

which certainly preserves both constraints. It remains to show that the transformation $X^M \to \lambda(\boldsymbol{X})X^M$ is a conformal transformation of the light-cone (and hence of the section as well). Indeed

$$\Big(d(\lambda\boldsymbol{X})\cdot d(\lambda\boldsymbol{X})\Big) = \Big(\big(\lambda d\boldsymbol{X} + (\boldsymbol{\nabla}\lambda\cdot d\boldsymbol{X})\boldsymbol{X}\big)\cdot\big(\lambda d\boldsymbol{X} + (\boldsymbol{\nabla}\lambda\cdot d\boldsymbol{X})\boldsymbol{X}\big)\Big) = \lambda^2\Big(d\boldsymbol{X}\cdot d\boldsymbol{X}\Big),$$

where we have used $\boldsymbol{X}^2 = 0$, $(\boldsymbol{X}\cdot d\boldsymbol{X}) = 0$. That is, that linear transformations $\Lambda \in SO(D+1,1)$ corresponds via (28) to the conformal transformations of $\mathbb{R}^D$, which we summarized in (25)-(26). For example

$$\begin{pmatrix}
1 & 0 & 0 & \ldots\ldots & 0 & 0 \\
0 & 1 & 0 & \ldots\ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & 0 & 0 & 0 \\
\ldots & 0 & 1 & 0 & 0 & 0 \\
\ldots & \ldots & 0 & 1 & 0 & 0 \\
\ldots & \ldots & 0 & 0 & \frac{\lambda+\lambda^{-1}}{2} & \frac{\lambda-\lambda^{-1}}{2} \\
\ldots & \ldots & 0 & 0 & \frac{\lambda-\lambda^{-1}}{2} & \frac{\lambda+\lambda^{-1}}{2}
\end{pmatrix} \tag{29}$$

corresponds to dilations $\boldsymbol{x} \to \lambda^{-1}\boldsymbol{x}$ etc.

It is interesting to construct conformal invariants, that is functions $F(x_1,\ldots,x_N)$ which are invariant with respect to all conformal transformations. Poincare invariance implies that $F(x_1,\ldots,x_N)$ may depend only on relative distances $|\boldsymbol{x}_i - \boldsymbol{x}_j|$, then the scale invariance implies that it can only depend on the ratios

$$\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{|\boldsymbol{x}_k - \boldsymbol{x}_l|}.$$

Finally, to insure the invariance under the SCT it is enough due to (24) to ensure the invariance with respect to inversions. Using

$$(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \xrightarrow{x^\mu \to \frac{x^\mu}{\boldsymbol{x}^2}} \frac{(\boldsymbol{x}_i - \boldsymbol{x}_j)^2}{\boldsymbol{x}_i^2\boldsymbol{x}_j^2},$$

we see that we can built an invariant only through 4 points

$$u = \frac{|\boldsymbol{x}_1 - \boldsymbol{x}_2||\boldsymbol{x}_3 - \boldsymbol{x}_4|}{|\boldsymbol{x}_1 - \boldsymbol{x}_3||\boldsymbol{x}_2 - \boldsymbol{x}_4|}, \quad v = \frac{|\boldsymbol{x}_1 - \boldsymbol{x}_2||\boldsymbol{x}_3 - \boldsymbol{x}_4|}{|\boldsymbol{x}_2 - \boldsymbol{x}_3||\boldsymbol{x}_1 - \boldsymbol{x}_4|}.$$

It means that the conformaly invariant function of four points is actually a function of two invariants $F(u,v)$. In general, there are $N(N-3)/2$ invariants for N points. Indeed, the number of $|\boldsymbol{x}_i - \boldsymbol{x}_j|$'s is $N(N-1)/2$. Then write a monomial

$$\prod_{i<j} |\boldsymbol{x}_i - \boldsymbol{x}_j|^{m_{ij}}.$$

Conformal invariance demands that each individual degree in $\boldsymbol{x}_k$ is 0. That is

$$\sum_{j=1}^{k-1} m_{jk} + \sum_{j=k+1}^{N} m_{kj} = 0.$$

So we have N equations for $N(N-1)/2$ unknowns: $N(N-1)/2 - N = N(N-3)/2$.

We note that these monomials are not algebraically independent for $N \geq D+2$. Indeed, after all we have N points in D-dimensional space constrained by the conformal group. Hence the number of algebraically independent cross-rations has to be

$$ND - \frac{(D+2)(D+1)}{2}.$$

In particular, one has only $2(N-3)$ independent cross-ratios for $D=2$ and $N-3$ for $D=1$. For example for $D=1$ and for $x_1 > x_2 > x_3 > x_4$

$$u = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_2-x_4)}, \quad v = \frac{(x_1-x_2)(x_3-x_4)}{(x_2-x_3)(x_1-x_4)} \implies v = \frac{u}{1-u}.$$

The counting above does not work for $N < D+2$, since there is a residual subgroup of the conformal group that leaves N points invariant. So that we have, $N(N-3)/2$ invariants for $N < D+2$ and $ND - \frac{(D+2)(D+1)}{2}$ for $N \geq D+2$.

Now, we consider the conformal group in two dimensions. We already saw that $D=2$ is special (see eqs (19), (20), (21)). Namely, the condition (19) reads

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1, \tag{30}$$

which are nothing else as the Cauchy-Riemann equations in complex analysis: the complex function whose real and imaginary parts satisfy (30) is holomorphic. Namely, we introduce the notations

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \quad \partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2), \quad \epsilon = \epsilon_1 + i\epsilon_2, \quad \epsilon = \epsilon_1 - i\epsilon_2.$$

Then (30) is equivalent to the statement

$$\partial\bar{\epsilon} = \bar{\partial}\epsilon = 0. \tag{31}$$

What we just obtained is merely the simple fact that in two dimensions any holomorphic function $f(z)$ give rise to the conformal transformation

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dzd\bar{z} \xrightarrow{z=f(w),\, \bar{z}=\bar{f}(\bar{w})} \left|\frac{df}{dw}\right|^2 dwd\bar{w},$$

or in infinitesimal form $f(z) = z + \epsilon(z)$. We note that doing holomorphic maps we regard the variables z and $\bar{z}$ as complex conjugated, so that the metric remains real.

We see that in $D=2$ the conformal group is infinite dimensional and consists of all holomorphic maps with the group multiplication being the composition of maps. This is to be compared to the finite-dimensional conformal group in $D > 2$ dimensions, which is isomorphic to $SO(D+1,1)$. It is precisely this infiniteness, which makes the $D=2$ case so special. Imposing this infinite symmetry, we got infinitely many constraints on correlation functions and in some cases can compute them exactly. There is, however, some subtlety which is related to the global definition of holomorphic maps. In fact, the Cauchy-Riemann conditions (30)-(31) are defined only locally. They just kinematically guarantee that the function $\epsilon(z)$ depends only on one variable z, but do not demand the corresponding map to be defined everywhere and be invertible. By definition, the conformal group consists of all invertible and globally defined maps (keeping in mind that SCT requires to add "infinity" point to the manifold). We will therefore distinguish between global and local conformal transformations in two dimensions. Let us construct holomorphic invertible globally defined mappings $f(z)$. Clearly, $f(z)$ could not have any essential singularities of branch points. Hence the only admissible singularities are the poles and then $f(z)$ is a rational function

$$f(z) = \frac{P(z)}{Q(z)}.$$

The polynomial $P(z)$ could not have distinct zeroes because in this case the inverse image of 0 is not well defined. The multiple zeroes are also not allowed, because the inverse function will be multiple valued. So, the only possibility is the linear functions. The same arguments apply to the denominator of $f(z)$ when looking at the behavior near ∞. We conclude that

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ad-bc=1.$$

The last condition has been applied in order to fix the freedom $(a,b,c,d) \to (\lambda a, \lambda b, \lambda c, \lambda d)$ which does not change $f(z)$. For each global map $f(z)$ one associates a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}),$$

where $\mathrm{SL}(2,\mathbb{C})$ is the group of complex 2×2 matrices with unit determinant. One can easily verify, that the composition of maps $f_2(f_1(z))$ corresponds to the matrix multiplication M_2M_1

$$f_2(f_1(z)) = \frac{a_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right)+b_2}{c_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right)+d_2} = \frac{(a_1a_2+b_2c_1)z+(a_2b_1+b_2d_1)}{(a_1c_2+c_1d_2)z+(c_2b_1+d_2d_1)}.$$

Furthermore, we note that even after imposing the condition $ad-bc=1$ there is still a redundant symmetry $(a,b,c,d) \to (-a,-b,-c,-d)$ that does not change $f(z$ and we have to eliminate it. We conclude that the group of global conformal transformations in two dimensions coincides with the Möbius group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

The Möbius group is a continuous 6-parametric group, which is known to coincide with the Lorentz group in four dimensions group $SO(3,1)$ (more precisely its identity component) according to the spinor map. Namely, we note that there is a natural map from $\mathbb{R}^{1,3}$ to the space of 2×2 Hermitian matrices

$$(x_0,x_1,x_2,x_3) \to H = \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} \tag{32}$$

such that the quadratic form becomes the determinant $x_0^2-x_1^2-x_2^2-x_3^2 = \det H$. Any matrix $M \in \mathrm{SL}(2,\mathbb{C})$ acts on Hermitian matrices by conjugation (the spin homomorphism)

$$H \to M^+HM.$$

The kernel of the spin homomorphism consists of two matrices $M = \pm\mathbb{I}$ and hence we come to the conclusion that the Möbius group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ is isomorphic to the identity component of $SO(1,3)$.

We note that $SL(2,\mathbb{C})$ contains compact subgroup $SU(2) \in SL(2,\mathbb{C})$

$$z \to \frac{az-b^*}{bz+a^*}, \quad |a|^2+|b|^2=1, \tag{33}$$

and hence the Möbius group contains the compact subgroup $SU(2)/\mathbb{Z}_2 \sim SO(3)$. This group acts transitively on a two-sphere. It is convenient to realize it as a surface in 3D space parameterized by spherical angles $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\varphi \in [0, 2\pi]$

$$x_1 = \cos\theta\cos\varphi, \quad x_2 = \cos\theta\sin\varphi, \quad x_3 = \sin\theta \implies x_1^2 + x_2^2 + x_3^2 = 1$$

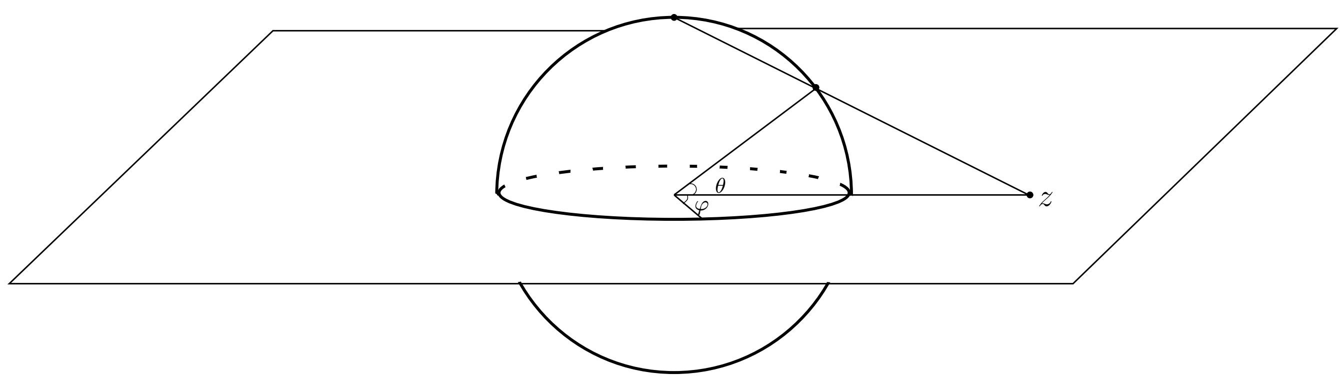
The action of $SO(3)$ is inherited from (32)

$$\begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \to \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \tag{34}$$

To see it geometrically, we switch to stereographic coordinates $(z, \bar{z})$

$$z = \frac{\cos\theta}{1 - \sin\theta} e^{i\varphi} \qquad \bar{z} = \frac{\cos\theta}{1 - \sin\theta} e^{-i\varphi}.$$

explained by the picture



The stereographic projection provides the map from S^2 to $\mathbb{C} + \{\infty\}$ such that the north pole of S^2 is mapped to ∞.

The sphere S^2 is homogeneous space for $SO(3)$. Consider the element of $SO(3)$: the rotation on angle α around the axis which pierce the sphere at the points (θ, φ) and $(-\theta, \pi + \varphi)$ (antipodal point). This element of $SO(3)$ should correspond to some a and b in (34). Demanding that the points

$$z = z(\theta, \varphi) \quad \text{and} \quad z = z(-\theta, \varphi + \pi),$$

are the fixed points of the map (33), we find

$$a^* = a - 2b\tan\theta e^{i\varphi}, \quad b^* = -be^{2i\varphi},$$

which can be solved by (remember the condition $|a|^2 + |b|^2 = 1$)

$$a = \cos\beta + i\sin\theta\sin\beta, \quad b = i\cos\theta\sin\beta e^{-i\varphi},$$

for some β. Simple analysis shows that $\beta = \alpha/2$.

Problems:

1. Find Noether current corresponding to the special conformal transformation (24).
2. Show, that (29) corresponds to dilations. Identify other elements of the conformal group as elements of $SO(D+1, 1)$.

Lecture 3: Stress-energy tensor in QFT, conformal Ward identities

We consider Euclidean, $SO(D)$ invariant field theory with *symmetric* stress-energy tensor $T_{\mu\nu} = T_{\nu\mu}$ defined by (7). Quantization of the theory amounts to consider functional integrals of the form

$$\langle X \rangle \overset{\text{def}}{=} \frac{1}{Z} \int X e^{-S[\Phi]} [\mathcal{D}\Phi], \tag{35}$$

where X is a composite field. Usually, we take it in the form

$$X = \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_N(\boldsymbol{x}_N),$$

where $\mathcal{O}_r(\boldsymbol{x}_r)$'s are some local fields

$$\mathcal{O}(\boldsymbol{x}) = \mathcal{F}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}), \dots)$$

In principle, function $\mathcal{F}(\dots)$ can be arbitrary. The only property which we require, is that the fields finitely separated in the space are not allowed. The collection of all local fields is usually thought of as a vector space. One can imagine it as

$$\mathcal{A} = \text{span}\{\Phi^N(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu \Phi(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu \partial_\nu \Phi(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu \Phi(\boldsymbol{x})\partial_\nu \Phi(\boldsymbol{x}), \dots\} \tag{36}$$

As we learn in Quantum Field Theory course the composite fields like $\Phi^N(\boldsymbol{x})$ require renormalization. So, the fields in (36) can be regarded as symbols for the true quantum fields. We will usually denote them as $\mathcal{O}_j(\boldsymbol{x})$, $j = 1, \dots, \infty$, meaning that they form a basis in infinitedimensional vector space $\mathcal{A}$.

At the moment we do not specify $\mathcal{O}_j(\boldsymbol{x})$'s in (35) and try to work in general. The symbolic integration in the right hand side in (35) is known to lack mathematically rigorous definition. Nevertheless, we assume that the functional integral (35) exists and shares some properties of ordinary integral. In particular, since in (35) we integrate over all functions $\Phi(\boldsymbol{x})$ we assume that the measure of integration is invariant with respect to translations

$$\mathcal{D}(\Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x})) = \mathcal{D}(\Phi(\boldsymbol{x})),$$

where $\epsilon(\boldsymbol{x})$ is an arbitrary function. The value of the functional integral (35) should not change. It leads to the following identity

$$\sum_{k=1}^{N} \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \delta_\epsilon \mathcal{O}_k(\boldsymbol{x}_k) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle = \int_{\mathbb{R}^D} \epsilon(\boldsymbol{x}) \langle \text{EOM}(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle \, d^D\boldsymbol{x}, \tag{37}$$

where

$$\text{EOM}(\boldsymbol{x}) = \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right)$$

is the composite field which vanishes on-shell in classical field theory. Now, we note that the function $\epsilon(\boldsymbol{x})$ is arbitrary. In particular, it can be taken to have no support at the point $\boldsymbol{x}_k$. Then the left hand side of (37) should vanish by assumption of locality. Thus we have

$$\langle \text{EOM}(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle = 0 \quad \text{for} \quad \boldsymbol{x} \neq \boldsymbol{x}_k. \tag{38}$$

A field with this property, that is any correlation function involving this field vanishes unless its position $\boldsymbol{x}$ coincides with one of the other insertion points, is called the redundant field. Equation of the form (38) is usually referred as vanishing of correlation function up to contact terms.

There are, in principle, infinitely many redundant fields in QFT. Formally, their existence is related to the more general transformations of integration variable in (35)

$$\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x}) F[\Phi(\boldsymbol{x})]. \tag{39}$$

Generally, we do not known if the measure transforms covariantly under this change. If we would known a Jacobian of this transformation we would find a new redundant field similar to $\text{EOM}(\boldsymbol{x})$. An important class of transformations (39) comes from the symmetries of the theory. Natural assumption would be that if the action has some symmetry, then the measure should share the same symmetry as well. For example, we expect that

$$\mathcal{D}(\Phi(\boldsymbol{x}+\boldsymbol{\epsilon})) = \mathcal{D}(\Phi(\boldsymbol{x})), \tag{40}$$

as a manifestation of the fact that the change $\boldsymbol{x} \to \boldsymbol{x}+\boldsymbol{\epsilon}$ just relabels the coordinates in the functional integral. It is easy to justify the invariance (40) for a constant $\boldsymbol{\epsilon}$. But what if $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{x})$ is a function, as in $2D$ CFT? In general, this is the source of anomaly. We will discuss it later in our course.

Exactly, for the transformation $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}+\boldsymbol{\epsilon}(\boldsymbol{x})) = \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x}) + \dots$ we do not expect measure issues. Therefore we have an identity

$$\sum_{k=1}^{N} \langle \mathcal{O}_1(\boldsymbol{x}_1)\dots\delta_{\boldsymbol{\epsilon}}\mathcal{O}_k(\boldsymbol{x}_k)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle = \int_{\mathbb{R}^D} \partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x}. \tag{41}$$

In quantum field theory we take (41) as a definition of the stress-energy tensor.

From very general grounds one can assume that variation of local field can depend on $\boldsymbol{\epsilon}(\boldsymbol{x})$ and *finitely* many its derivatives

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(\boldsymbol{x}) = \epsilon_\mu(\boldsymbol{x})\partial_\mu\mathcal{O}(\boldsymbol{x}) + \partial_\mu\epsilon_\nu\mathcal{O}^{\mu\nu}(\boldsymbol{x}) + \dots, \tag{42}$$

where $\mathcal{O}^{\mu\nu}(\boldsymbol{x})$ etc are some local fields. The fact that there are only derivatives of $\boldsymbol{\epsilon}(\boldsymbol{x})$ in (42) reflects general assumption of locality. The fact that there are finitely many terms in (42) is an assumption that the spectra of dimensions of local fields is bounded from below.

Now, let $\mathbb{B}_k$ be the small ball surrounding the point $\boldsymbol{x}_k$, such that $\mathbb{B}_i \cap \mathbb{B}_j = \varnothing$. Then we split the integral in the r.h.s. in (41) as

$$\int_{\mathbb{R}^D} = \sum_{k=1}^{N}\int_{\mathbb{B}_k} + \int_{\bar{\mathbb{R}}^D},$$

where $\bar{\mathbb{R}}^D \cup \mathbb{B}_1 \cup \dots \cup \mathbb{B}_N = \mathbb{R}^D$. The last integral can be transformed by parts

$$\int_{\bar{\mathbb{R}}^D} \partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x} = -\int_{\bar{\mathbb{R}}^D} \epsilon_\nu(\boldsymbol{x})\langle \partial_\mu T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x} + \text{b.t.} \tag{43}$$

Where by b.t. we denoted the boundary terms. They are the sum of integrals over the boundaries of all balls $\mathbb{B}_k$. Now, let us take $\boldsymbol{\epsilon}(\boldsymbol{x})$ of very special form (with no support at $\boldsymbol{x} = \boldsymbol{x}_k$)

$$\boldsymbol{\epsilon}(\boldsymbol{x})\Big|_{\mathbb{B}_k} = 0 \quad \text{for all} \quad k = 1,\dots,N.$$

Then the first term in the right hand side of (43) is the only one who contributes and hence we have

$$\langle \partial_\mu T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle = 0 \quad \text{if} \quad \boldsymbol{x}\in\bar{\mathbb{R}}^D. \tag{44}$$

We note that we can take the balls $\mathbb{B}_k$ arbitrary small and hence (44) is valid for all $\boldsymbol{x}\neq\boldsymbol{x}_k$, i.e. the correlation function (44) vanishes everywhere except for some delta functions supported at the insertion points $\boldsymbol{x}_1\dots\boldsymbol{x}_N$. That is $\partial_\mu T_{\mu\nu}$ is a redundant field.

Having in mind (44), we conclude that

$$\sum_{k=1}^N \langle \mathcal{O}_1(\boldsymbol{x}_1)\dots\delta_{\boldsymbol{\epsilon}}\mathcal{O}_k(\boldsymbol{x}_k)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle = \sum_{k=1}^N \int_{\mathbb{B}_k}\partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x} + \text{b.t.}$$

Now, we specify everything to the case of $D=2$ and scale $T_{\mu\nu}\to\frac{1}{2\pi}T_{\mu\nu}$ for future convenience. Using the Green theorem

$$\int_{\mathcal{D}}\partial_\mu A_\mu d^2\boldsymbol{x} = \oint_{\partial\mathcal{D}}\varepsilon_{\mu\nu}A_\mu dx^\nu,$$

we find

$$\begin{aligned}\sum_{k=1}^N \langle \mathcal{O}_1(\boldsymbol{x}_1)\dots\delta_{\boldsymbol{\epsilon}}\mathcal{O}_k(\boldsymbol{x}_k)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle &= \frac{1}{2\pi}\sum_{k=1}^N \int_{\mathbb{B}_k}\partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^2\boldsymbol{x} - \\ &- \frac{1}{2\pi}\sum_{k=1}^N \oint_{\partial\mathbb{B}_k}\epsilon_\nu(\boldsymbol{x})\varepsilon_{\mu\lambda}\langle T_{\lambda\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)dx^\mu, \end{aligned} \tag{45}$$

where the contour integral goes in the counterclockwise direction. Since, $\boldsymbol{\epsilon}$ is arbitrary we can take it non-zero only in the vicinity of the point $\boldsymbol{x}_k$. In this case only one term of the sum contributes in (45). We can rewrite (45), formally erasing an average sign, as

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathcal{D}_{\boldsymbol{x}}}\partial_\mu\epsilon_\nu(\boldsymbol{y})T_{\mu\nu}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})d^2\boldsymbol{y} - \frac{1}{2\pi}\oint_{\mathcal{C}_{\boldsymbol{x}}}\epsilon_\nu(\boldsymbol{y})\varepsilon_{\mu\lambda}T_{\lambda\nu}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})dy^\mu, \tag{46}$$

where $\mathcal{D}_{\boldsymbol{x}}$ is a small disk surrounding the point $\boldsymbol{x}$ and $\mathcal{C}_{\boldsymbol{x}}$ is its boundary.

Now, suppose that our theory is conformally invariant, that is $T_{\mu\nu}$ is traceless. In this case the first term in (46) does not contribute for conformal transformations $\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu\sim\delta_{\mu\nu}$. Moreover, taking into account peculiarities of $2D$ geometry

$$\begin{aligned} \epsilon_1(\boldsymbol{x})+i\epsilon_2(\boldsymbol{x}) &= \epsilon(z), \qquad & \frac{1}{4}\left(T_{11}(\boldsymbol{x})-T_{22}(\boldsymbol{x})-2iT_{12}(\boldsymbol{x})\right) &\overset{\text{def}}{=} T(z), \\ \epsilon_1(\boldsymbol{x})-i\epsilon_2(\boldsymbol{x}) &= \bar{\epsilon}(\bar{z}), \qquad & \frac{1}{4}\left(T_{11}(\boldsymbol{x})-T_{22}(\boldsymbol{x})+2iT_{12}(\boldsymbol{x})\right) &\overset{\text{def}}{=} \bar{T}(\bar{z}), \end{aligned}$$

we find that

$$\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_N((z_1,\bar{z}_1))\rangle \quad \text{and} \quad \langle \bar{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_N((z_1,\bar{z}_1))\rangle \tag{47}$$

are holomorphic and antiholomorphic functions respectively. Moreover variation of the field $\mathcal{O}(z,\bar{z})$ under the conformal change of coordinates $\boldsymbol{\epsilon}=(\epsilon,\bar{\epsilon})$: $z\to z+\epsilon(z)$, $\bar{z}\to\bar{z}+\bar{\epsilon}(\bar{z})$ is

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z}\epsilon(\zeta)\,T(\zeta)\mathcal{O}(z,\bar{z})d\zeta + \frac{1}{2\pi i}\oint_{\mathcal{C}_{\bar{z}}}\bar{\epsilon}(\bar{\zeta})\,\bar{T}(\bar{\zeta})\mathcal{O}(z,\bar{z})d\bar{\zeta},$$

where both contours $\mathcal{C}_z$ and $\mathcal{C}_{\bar{z}}$ go in the counterclockwise direction. It is important, that correlation functions (47) not only holomorphic (antiholomorphic), but also single valued. It allows us to define the holomorphic variation of local fields (assuming that $\epsilon(\zeta)$ is single-valued as well)

$$\delta_\epsilon \mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} \epsilon(\zeta)\, T(\zeta)\mathcal{O}(z,\bar{z})d\zeta.$$

Consider infinitesimal transformation of the very special form (note that here z is exactly the insertion point $\mathcal{O}(z,\bar{z})$)

$$\epsilon_n(\zeta) = \alpha(\zeta - z)^{n+1} \qquad \alpha \ll 1, \quad n \geq -1.$$

Variation of $\mathcal{O}(z,\bar{z})$ under this special conformal transformation we denote by $L_n\mathcal{O}(z,\bar{z})$: $\delta\mathcal{O} = \alpha L_n \mathcal{O}(z,\bar{z})$. For generic ϵ the variation $\delta_\epsilon\mathcal{O}$ can be expressed in terms of $(L_k\mathcal{O})$ as

$$\delta_\epsilon \mathcal{O} = \epsilon(L_{-1}\mathcal{O}) + \epsilon'(L_0\mathcal{O}) + \frac{\epsilon''}{2!}(L_1\mathcal{O}) + \frac{\epsilon'''}{3!}(L_2\mathcal{O}) + \dots \tag{48}$$

At least two of these new fields $L_n\mathcal{O}(z,\bar{z})$ we can identify

$$L_{-1}\mathcal{O}(z,\bar{z}) = \partial\mathcal{O}(z,\bar{z}), \qquad L_0\mathcal{O}(z,\bar{z}) = \Delta_{\mathcal{O}}\mathcal{O}(z,\bar{z}),$$

where $\Delta_{\mathcal{O}}$ is called the conformal dimension of the field $\mathcal{O}$[6]. Other fields $L_n\mathcal{O}(z,\bar{z})$ are some new fields which a priori are unrelated to the original one $\mathcal{O}(z,\bar{z})$. In general, we expect that (48) contains only finitely many derivative terms, that is there should exists such $N > 0$, that

$$L_n\mathcal{O}(z,\bar{z}) = 0 \quad \text{for} \quad n > N.$$

It is clear that conformal dimensions of the fields $L_k\mathcal{O}$ are given by

$$\Delta_{L_k O} = \Delta_{\mathcal{O}} - k.$$

We assume that the spectra of conformal dimensions $\{\Delta_j\}$ is bounded from below. Actually, we might require even more and forbid negative conformal dimensions at all. It guaranties for example that the two-point functions

$$\langle \mathcal{O}(z,\bar{z})\mathcal{O}(z',\bar{z}')\rangle \sim \frac{1}{|z-z'|^{4\Delta_{\mathcal{O}}}},$$

will fall at infinity. In any case, this restriction implies existence of *primary* fields, which we denote as Φ, having the most simple variation (48)

$$\delta_\epsilon\Phi(z) = \epsilon(z)\partial\Phi(z) + \Delta\epsilon'(z)\Phi(z),$$

or $L_n\Phi = 0$ for all $n > 0$. Under generic, not infinitesimal, holomorphic transformation primary fields behave as generalized tensor fields

$$\Phi(z) \to \left(\frac{dw}{dz}\right)^{\Delta} \Phi(w).$$

[6] Similarly, one can define antiholomorphic conformal dimension $\bar{\Delta}$ as $\bar{L}_0\mathcal{O} = \bar{\Delta}\mathcal{O}$. Altogether it corresponds to the following transformation rules

$$\mathcal{O}(z,\bar{z}) \to \lambda^{\Delta}\bar{\lambda}^{\bar{\Delta}}\mathcal{O}(\lambda z, \bar{\lambda}\bar{z}).$$

So that $\Delta + \bar{\Delta}$ can be identified with the scaling dimension and $\Delta - \bar{\Delta}$ with the spin.

From now on the notation $\Phi(z)$ will stick for primary field.

Consider the Ward identity

$$\sum_{j=1}^{N}\langle\mathcal{O}_1(z_1)\dots\delta_\epsilon\mathcal{O}_j(z_j)\dots\mathcal{O}_N(z_N)\rangle=\frac{1}{2\pi i}\sum_{j=1}^{N}\oint_{\mathcal{C}_{z_j}}\epsilon(\zeta)\langle T(\zeta)\mathcal{O}_1(z_1)\dots\mathcal{O}_N(z_N)\rangle d\boldsymbol{\zeta}, \tag{49}$$

We assume that the correlation function $\langle T(\zeta)\mathcal{O}_1(z_1)\dots\mathcal{O}_N(z_N)\rangle$ is a single-valued function of ζ falling at infinity ($T(\zeta)\underset{\zeta\to\infty}{\to}0$) with only possible singularities, the poles at the insertion point z_j. Then, taking

$$\epsilon(\zeta)=\frac{\alpha}{z-\zeta},\quad \alpha\ll 1 \tag{50}$$

where $z\neq z_j$ and using (49) and (48), we find

$$\langle T(z)\mathcal{O}_1(z_1)\dots\mathcal{O}_N(z_N)\rangle=\sum_{j=1}^{N}\sum_{k=0}^{\nu_j}\frac{1}{(z-z_j)^{k+1}}\langle\mathcal{O}_1(z_1)\dots\mathcal{O}_{j-1}(z_{j-1})L_{k-1}\mathcal{O}_j(z_j)\mathcal{O}_{j+1}(z_{j+1})\mathcal{O}_N(z_N)\rangle. \tag{51}$$

It is important to mention that in (49) the contours $\mathcal{C}_{z_j}$ are very small circles, so that z in (50) lies outside of all $\mathcal{C}_{z_j}$'s. The formula (51) is known under the name of conformal Ward identity. It has a particularly neat form for primary fields

$$\langle T(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle=\sum_{k=1}^{N}\left(\frac{\Delta_k}{(z-z_k)^2}+\frac{\partial_k}{z-z_k}\right)\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle. \tag{52}$$

One can rewrite (52) in the form of operator product expansion (OPE)

$$T(\zeta)\Phi(z)=\frac{\Delta\Phi(z)}{(\zeta-z)^2}+\frac{\partial\Phi(z)}{\zeta-z}+\dots \tag{53}$$

where by $\dots$ we denote terms regular at $\zeta\to z$. Similarly, from (51) we find that

$$T(\zeta)\mathcal{O}(z)=\underbrace{\dots+\frac{L_2\mathcal{O}(z)}{(\zeta-z)^4}+\frac{L_1\mathcal{O}(z)}{(\zeta-z)^3}+\frac{\Delta_{\mathcal{O}}\mathcal{O}(z)}{(\zeta-z)^2}+\frac{\partial\mathcal{O}(z)}{\zeta-z}}_{\text{finitely many simgular terms}}+\dots \tag{54}$$

Both relations (53) and (54) should be understood as (51).

Now, as we saw before, the conformal dimension of the field $\mathcal{O}$ differs from the conformal dimension of some primary field Φ by an integer positive number. It suggests that, may be, $\mathcal{O}$ can be obtained from Φ. To do so, we consider regular part of (53)

$$T(\zeta)\Phi(z)=\frac{\Delta\Phi(z)}{(\zeta-z)^2}+\frac{\partial\Phi(z)}{\zeta-z}+L_{-2}\Phi(z)+(\zeta-z)L_{-3}\Phi(z)+(\zeta-z)^2L_{-4}\Phi(z)+\dots, \tag{55}$$

where $L_{-k}\Phi(z)$ are, by definition, some new local fields (note that $L_{-1}\Phi(z)=\partial\Phi(z)$). Their existence can be justified by functional integral arguments and from (52). For example

$$L_{-2}\Phi(z)\approx T(z)\Phi(z),\quad L_{-3}\Phi(z)\approx T'(z)\Phi(z)\quad \text{etc},$$

where the symbol $\approx$ means some kind of regularization. We will make it simpler and just postulate, that (55) defines the new fields $L_{-k}\Phi(z)$, which will be called descendant fields (but not only them). It can be expressed as follows

$$L_{-k}\Phi(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} (\zeta - z)^{1-k} T(\zeta)\Phi(z) d\zeta \tag{56}$$

Using (52), one finds that

$$\langle L_{-k}\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} (\xi - z)^{1-k}\langle T(\xi)\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle =$$
$$= \hat{\mathcal{L}}_{-k}\langle\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle, \tag{57}$$

where the differential operator $\hat{\mathcal{L}}(z, z_k)$ is given by

$$\hat{\mathcal{L}}_{-k} = \sum_{j=1}^{N}\left[\frac{(k-1)\Delta_j}{(z_j - z)^k} - \frac{\partial_j}{(z_j - z)^{k-1}}\right].$$

The equality (57) can be obtained as follows. We use the conformal Ward identity (52)

$$\langle T(\xi)\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle =$$
$$= \left(\frac{\Delta}{(\xi - z)^2} + \frac{\partial}{\xi - z} + \sum_{k=1}^{N}\left(\frac{\Delta_k}{(\xi - z_k)^2} + \frac{\partial_k}{\xi - z_k}\right)\right)\langle\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle \tag{58}$$

where $\partial = \partial_z$, $\partial_k = \partial_{z_k}$. Then (57) can be obtained from the condition of absence of pole at infinity

$$\oint_{\mathcal{C}_z} + \sum_{j=1}^{N}\oint_{\mathcal{C}_{z_j}} = 0. \tag{59}$$

We note that (59) holds for $k > 1$ provided that the correlation function $\langle T(\xi)\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle$ falls at infinity. For $k = 1$ we have additionally to require

$$\langle T(\xi)\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle \sim \frac{1}{\xi^2} \overset{(58)}{\Longleftrightarrow} \left(\partial + \sum_{k=1}^{N}\partial_k\right)\langle\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle = 0, \tag{60}$$

which is nothing else, but the condition of translation invariance of correlation function.

Now we have to derive conformal transformation properties for the field $T(z)$ itself. Consider the product

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{\Lambda(w)}{(z-w)^3} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \ldots \tag{61}$$

Few comments are in order. First, the conformal dimension of $T(z)$ is 2. Since it is a conserved current its conformal dimension does not acquire quantum corrections. It can be easily seen by comparing scaling properties of both sides of (54). Second, the most singular term in (61) is proportional to the field of dimension 0. We assume that there is only one such field, namely the identity operator, and hence c in (61) is just a number. More singular terms are forbidden because of our assumption $\Delta \geq 0$. The field $\Lambda = L_1 T$ should have dimension 1. Since the product $T(z)T(w)$ is symmetric we also have

$$T(z)T(w) = \frac{c}{2(w-z)^4} + \frac{\Lambda(z)}{(w-z)^3} + \frac{2T(z)}{(w-z)^2} + \frac{T'(z)}{w-z} + \dots \tag{62}$$

Comparing (62) with (61) we find that $\Lambda = 0$. Therefore, under our assumptions we have

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots, \tag{63}$$

which is equivalent to the infinitesimal conformal transformation

$$\delta_\epsilon T(z) = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \frac{c}{12}\epsilon'''(z). \tag{64}$$

This infinitesimal transformation can be "exponentiated" to

$$T(z) \to \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w, z\}, \tag{65}$$

where $\{w, z\}$ is the Schwarzian derivative

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \underset{w=z+\epsilon}{=} \epsilon''' + \dots$$

In order to validate (65) we have to check the group property. It follows from the following property of the Schwarzian derivative

$$\{w, z\} = \left(\frac{d\zeta}{dz}\right)^2 \{w, \zeta\} + \{\zeta, z\}.$$

Moreover, one can check that $\{f, z\}$ vanishes on functions $f(z) = \frac{az+b}{cz+d}$, which correspond to global conformal transformations. The fields with transformation laws like (64), i.e. which behave as primary fields under Möbius transformations, are called quasi-primary or conformal.

Problems:

1. Derive basic properties of the Schwarzian derivative (see Wiki)

Lecture 4: Conformal families, Virasoro algebra

In the last lecture we have defined descendant fields (56) and have shown that correlation functions with one such field and arbitrary number of primary fields can be expressed through the correlation function with primary fields only by some differential operator (57). In order to compute more general correlation functions with multiple insertions of descendant fields

$$\langle L_{-k_1}\Phi(z_1)L_{-k_2}\Phi_2(z_2)\Phi_3(z_3)\dots\Phi_N(z_N)\rangle,$$

we have to use Ward identities with multiple T insertions

$$\begin{aligned}\langle T(\zeta)T(\eta)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle &= \\ = \left[\sum_{j=1}^{N}\left(\frac{\Delta_k}{(\zeta-z_k)^2}+\frac{\partial_k}{\zeta-z_k}\right)+\left(\frac{2}{(\zeta-\eta)^2}+\frac{\partial_\eta}{\zeta-\eta}\right)\right]&\langle T(\eta)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle+ \\ &+\frac{c}{2(\zeta-\eta)^4}\langle\Phi_1(z_1)\dots\Phi_n(z_n)\rangle,\end{aligned}\tag{66}$$

which follow from the OPE of T with itself (63). In principle, using the multipoint analog of (66), we can compute arbitrary correlation function of the form

$$\langle L_{-k_1}\Phi_1(z_1)L_{-k_2}\Phi_2(z_2)L_{-k_3}\Phi_3(z_3)\dots L_{-k_N}\Phi_N(z_N)\rangle=\mathcal{D}\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle.$$

It is given by some "hard to find", but explicit, differential operator $\mathcal{D}$, applied to the correlation function involving primary fields only.

It is useful to find conformal transformation properties of the field $L_{-k}\Phi(z)$. In a very general form it is

$$T(\zeta)L_{-k}\Phi(z)=\dots+\frac{(L_2L_{-k}\Phi(z))}{(\zeta-z)^4}+\frac{(L_1L_{-k}\Phi(z))}{(\zeta-z)^3}+\frac{(L_0L_{-k}\Phi(z))}{(\zeta-z)^2}+\frac{(L_{-1}L_{-k}\Phi(z))}{\zeta-z}+\dots\tag{67}$$

We remind that this formula is equivalent to the following transformation

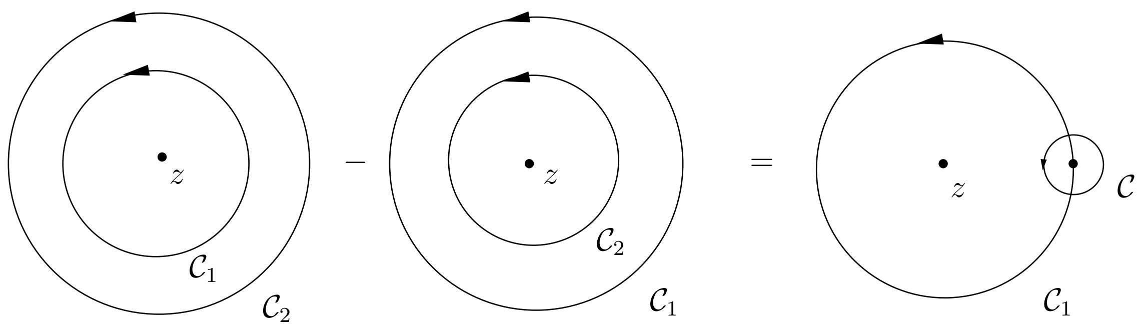
$$\delta_\epsilon L_{-k}\Phi=\epsilon L_{-1}L_{-k}\Phi+\epsilon' L_0L_{-k}\Phi+\frac{\epsilon''}{2}L_1L_{-k}\Phi+\dots$$

The fields appearing in the singular part of (67) are, as we will see shortly, not new fields. The fields from the regular part are new and will be denoted by $L_{-l}L_{-k}\Phi(z)$. From Ward identity we have

$$\begin{aligned}L_{-l}L_{-k}\Phi(z)&=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\eta-z)^{1-l}T(\eta)L_{-k}\Phi(z)d\eta= \\ &=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\eta-z)^{1-l}T(\eta)\left(\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\zeta-z)^{1-k}T(\zeta)\Phi(z)d\zeta\right)d\eta.\end{aligned}$$

This procedure can be repeated, producing an infinite tower of descendant fields

$$L_{-k_1}\dots L_{-k_n}\Phi(z).\tag{68}$$



The descendant fields (68) are not all linearly independent. To see this consider the commutator

$$
\begin{aligned}
&[L_m, L_n]\mathcal{O}(z) = L_m L_n \mathcal{O}(z) - L_n L_m \mathcal{O}(z) = \\
&\qquad = \frac{1}{2\pi i}\oint_{\mathcal{C}_2} (\eta - z)^{1+m} T(\eta) \left(\frac{1}{2\pi i} \oint_{\mathcal{C}_1} (\zeta - z)^{1+n} T(\zeta)\mathcal{O}(z) d\zeta \right) d\eta - \\
&\qquad\qquad - \frac{1}{2\pi i}\oint_{\mathcal{C}_1} (\zeta - z)^{1+n} T(\zeta) \left(\frac{1}{2\pi i} \oint_{\mathcal{C}_2} (\eta - z)^{1+m} T(\eta)\mathcal{O}(z) d\eta \right) d\zeta.
\end{aligned}
$$

Two integrals above look the same. The only difference is the order of contours $\mathcal{C}_1$ and $\mathcal{C}_2$. In the first integral the contour $\mathcal{C}_1$ goes first around z and then the contour $\mathcal{C}_2$ encircles both the point z and the contour $\mathcal{C}_1$. In the second integral the role of $\mathcal{C}_1$ and $\mathcal{C}_2$ is exchanged. Transforming both contours as shown on the picture we find

$$
\begin{aligned}
&[L_m, L_n]\mathcal{O}(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}_1} (\zeta - z)^{1+n} \left(\frac{1}{2\pi i} \oint_{\mathcal{C}} (\eta - z)^{1+m} T(\eta) T(\zeta) \mathcal{O}(z) d\eta \right) d\zeta = \\
&\quad = \frac{1}{2\pi i}\oint_{\mathcal{C}_1} (\zeta - z)^{1+n} \left(\frac{1}{2\pi i} \oint_{\mathcal{C}} (\eta - z)^{1+m} \left(\frac{c}{2(\eta-\zeta)^4} + \frac{2T(\zeta)}{(\eta-\zeta)^2} + \frac{T'(\zeta)}{\eta - \zeta} + \dots \right) \mathcal{O}(z) d\eta \right) d\zeta = \\
&\quad = \frac{1}{2\pi i}\oint_{\mathcal{C}_1} (\zeta - z)^{1+n} \left(\frac{c}{12}(m^3 - m)(\zeta - z)^{-2+m} + 2(m+1)(\zeta - z)^m T(\zeta) + (\zeta - z)^{1+m} T'(\zeta) \right) d\eta. \qquad (69)
\end{aligned}
$$

In the second line we used the conformal Ward identity for the field T itself. Evaluating the first integral and integrating by part the third one in (69), one arrives to the commutation relations

$$
[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \qquad (70)
$$

known as Virasoro algebra. Since the relations (70) are valid when applied to any field $\mathcal{O}$, we simply erased $\mathcal{O}$ in (70).

From (70) we see that

$$
\begin{gathered}
L_0 L_{-k}\Phi(z) = (\Delta + k) L_{-k}\Phi(z), \quad L_n L_{-k}\Phi(z) = (n + k) L_{n-k}\Phi(z) \quad \text{for} \quad n = 1, \dots, k-1, \\
L_k L_{-k}\Phi(z) = \left(2k\Delta + \frac{c}{12}(k^3 - k) \right) \Phi(z).
\end{gathered}
$$

We come to important conclusion. The conformal transformation properties of the descendant field $L_{-k}\Phi(z)$ involve *only* descendant fields build out of the same primary field Φ. The same is true for generic field (68). This fact leads us to the notion of the conformal family $[\Phi]$, i.e. the set (infinite) of all descendants fields (68). It is clear that because of the relations (70) there are linear relations among descendant fields (68). The conformal family $[\Phi]$ consists of

$$[\Phi] = \mathrm{Span}\left(L_{-k_1}L_{-k_2}\dots L_{-k_n}\Phi(z)\right| k_1 \geq k_2 \geq k_3 \geq \dots)$$

Since generic descendant can be obtained from the primary field by successive applications of (56), correlation functions involving descendants can be expressed from correlation function of primary fields only by means of some differential operators. Correlation functions of primaries are further constrained by the so called projective Ward identities. They follow from the fact that any correlation function involving $T(z)$ should fall at infinity as[7]

$$\langle T(z)\dots\rangle \sim \frac{1}{z^4} \quad \text{at} \quad z\to\infty. \tag{71}$$

Writing (71), we assumed that no field has been placed at $z=\infty$. Then $z=\infty$ should be regular point, as all other points. If we introduce local coordinate $z=\frac{1}{w}$, we have

$$\langle T(z)\dots\rangle = \left(\frac{dw}{dz}\right)^2\langle T(w)\dots\rangle = w^4\langle T(w)\dots\rangle \sim \frac{1}{z^4} \quad \text{at} \quad z\to\infty.$$

In the second equality we used transformation law for $T(z)$ derived before (65). We note that the anomalous term c does not contribute for inversion $z=1/w$. Now, let us apply (71) to the Ward identity (52). Terms of order $1/\zeta$, $1/\zeta^2$ and $1/\zeta^3$ in the right hand side in (52) should vanish

$$\begin{aligned}
&\sum_{k=1}^{N}\partial_k\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0,\\
&\sum_{k=1}^{N}\left(\Delta_k + z_k\partial_k\right)\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0,\\
&\sum_{k=1}^{N}\left(2z_k\Delta_k + z_k^2\partial_k\right)\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0
\end{aligned} \tag{72}$$

Let us study the consequences of these equations (here we do not write $\bar{z}$ dependence of correlation functions for simplicity). The one-point function vanishes unless $\Delta = 0$

$$\langle\Phi(z)\rangle \sim \delta_{\Delta,0}.$$

In that case it is a constant. Remember, that due to our assumption, there a unique primary field with $\Delta = 0$, the identity operator.

Now, let us study the two-point function $\langle\Phi_1(z_1)\Phi_2(z_2)\rangle$. First equation in (72) forces it to depend on the difference of variables only

$$\langle\Phi_1(z_1)\Phi_2(z_2)\rangle = F(z_1-z_2),$$

second equation implies

$$F(z_1-z_2) = \frac{\Lambda(\Delta_1,\Delta_2)}{(z_1-z_2)^{\Delta_1+\Delta_2}},$$

[7] We have already used weaker version of (71) in (60).

while the third one gives $\Lambda(\Delta_1, \Delta_2) = \mathcal{N}^2(\Delta_1)\delta_{\Delta_1,\Delta_2}$. We note that the factor $\mathcal{N}^2(\Delta_1)$ can always be set equal to one by changing normalizations of the fields. Similar analysis applies to the antiholomorphic part of correlation function. Thus, we have

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_1)\rangle = \frac{\delta_{\Delta_1,\Delta_2}\delta_{\bar{\Delta}_1,\bar{\Delta}_2}}{(z_1 - z_2)^{2\Delta_1}(\bar{z}_1 - \bar{z}_2)^{2\bar{\Delta}_1}}.$$

We call this canonical normalization of the two-point correlation function.

The three-point correlation function is given by

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\Phi_3(z_3, \bar{z}_3)\rangle = C(\Delta_1, \bar{\Delta}_1, \Delta_2, \bar{\Delta}_2, \Delta_3, \bar{\Delta}_3)\prod_{i<j}(z_i - z_j)^{-\Delta_{ij}}(\bar{z}_i - \bar{z}_j)^{-\bar{\Delta}_{ij}},$$

where $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$ etc and $C(\Delta_1, \bar{\Delta}_1, \Delta_2, \bar{\Delta}_2, \Delta_3, \bar{\Delta}_3)$ is some constant. In fact, this constant is the first "dynamical" quantity we wish to compute. It contains actual information about the theory, explicit Lagrangian for example. We will return to the problem of computation of $C(\Delta_1, \Delta_2, \Delta_3)$ later in this course. Going further, we consider four-point function. One can show that generic solution has the form

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_1)\Phi_3(z_3, \bar{z}_1)\Phi_4(z_4, \bar{z}_1)\rangle = \prod_{i<j}(z_i - z_j)^{\gamma_{ij}}(\bar{z}_i - \bar{z}_j)^{\bar{\gamma}_{ij}}F(z, \bar{z}),$$

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}, \quad \bar{z} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_1 - \bar{z}_4)(\bar{z}_3 - \bar{z}_2)}, \quad \sum_j \gamma_{ij} = -2\Delta_i, \quad \sum_j \bar{\gamma}_{ij} = -2\bar{\Delta}_i.$$

In general

$$\langle \Phi_1(z_1, \bar{z}_1)\dots\Phi_N(z_N, \bar{z}_N)\rangle = \prod_{i<j}(z_i - z_j)^{\gamma_{ij}}(\bar{z}_i - \bar{z}_j)^{\bar{\gamma}_{ij}}F(\boldsymbol{z}, \bar{\boldsymbol{z}}),$$

where

$$\sum_j \gamma_{ij} = -2\Delta_i, \quad \sum_j \bar{\gamma}_{ij} = -2\bar{\Delta}_i,$$

and $F(\boldsymbol{z}, \bar{\boldsymbol{z}})$ is some function of $N-3$ cross ratios z and $\bar{z}$.

We will use the projective invariance to set the positions of three points to 0, 1 and ∞. For 4−point correlation function of spinless primary fields (that is $\Delta_k = \bar{\Delta}_k$) one has

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\Phi_3(z_3, \bar{z}_3)\Phi_4(z_4, \bar{z}_4)\rangle = \prod_{i<j}|z_i - z_j|^{2\gamma_{ij}}F(z, \bar{z}),$$

where

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}, \quad \sum_j \gamma_{ij} = -2\Delta_i.$$

The choice of γ_{ij}'s is not unique, which is related to the obvious freedom

$$\prod_{i<j}|z_i - z_j|^{2\gamma_{ij}} \to \prod_{i<j}|z_i - z_j|^{2\gamma_{ij}}|z|^{2A}|1 - z|^{2B},$$

which certainly does not spoil the condition $\sum_j \gamma_{ij} = -2\Delta_i$. For example we fix this freedom by demanding that the prefactor does not change the behaviour of correlation function at $z_1 \to z_2$ and $z_1 \to z_3$. In other words we choose $\gamma_{12} = \gamma_{13} = 0$

$$\prod_{i<j}|z_i - z_j|^{2\gamma_{ij}} = |z_1 - z_4|^{-2\Delta_1}|z_2 - z_3|^{2(\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3)}|z_2 - z_4|^{2(\Delta_1 + \Delta_3 - \Delta_2 - \Delta_4)}|z_3 - z_4|^{2(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)}.$$

In this case the function $F(z,\bar{z})$ can be expressed through the limit

$$F(z,\bar{z}) = \lim_{\zeta\to\infty} |\zeta|^{4\Delta_4} \langle \Phi_1(z,\bar{z})\Phi_2(0)\Phi_3(1)\Phi_4(\zeta,\bar{\zeta})\rangle.$$

Combining altogether we obtain

$$\begin{aligned}
&\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\Phi_3(z_3,\bar{z}_3)\Phi_4(z_4,\bar{z}_4)\rangle = \\
&\quad = |z_1 - z_4|^{-2\Delta_1} |z_2 - z_3|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)} |z_2 - z_4|^{2(\Delta_1+\Delta_3-\Delta_2-\Delta_4)} |z_3 - z_4|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)} \times \\
&\qquad \times \lim_{\zeta\to\infty} |\zeta|^{4\Delta_4} \langle \Phi_1(z,\bar{z})\Phi_2(0)\Phi_3(1)\Phi_4(\zeta,\bar{\zeta})\rangle \quad \text{where} \quad z = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.
\end{aligned} \tag{73}$$

There is an instructive way to derive projective Ward identities as follows. We remind the variation formula for correlation function of generic fields, not necessarily primary ones,

$$\delta_\epsilon \langle \mathcal{O}_1(z_1)\dots\mathcal{O}_N(z_N)\rangle = \frac{1}{2\pi i}\sum_{k=1}^{N} \oint_{\mathcal{C}_{z_k}} \epsilon(\zeta)\langle T(\zeta)\mathcal{O}_1(z_1)\dots\mathcal{O}_N(z_N)\rangle d\boldsymbol{\zeta}. \tag{74}$$

Here $\epsilon = \epsilon(z)$ is an infinitesimal holomorphic function. We saw that the only globally defined holomorphic functions are

$$f(z) = \frac{az+b}{cz+d}. \tag{75}$$

Any further conformal transformations must have singularities and can not be one-to-one. So, let us assume that $f(z)$ has a singularity at $z = z_0$. While deriving (74) we implicitly assumed that this singular point is one of the positions z_k of the field insertions in (74). We saw, that singular conformal transformations produce descendant fields. Note, that we can also assume that $z_0 \neq z_k$ and treat this point as a place where the trivial operator I is inserted. Thus, after a singular conformal transformation one can produce a non-trivial descendant field from "nothing", like $T(z) = L_{-2}I(z)$ for example. It means, that the formula (74) should be understood in this generalized sense: where is some number of identity fields in the set of fields $\mathcal{O}_k$. After this remark, we note that the right hand side of (74) can be written as

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_\infty} \epsilon(\zeta)\langle T(\zeta)\mathcal{O}_1(z_1)\dots\mathcal{O}_N(z_N)\rangle d\boldsymbol{\zeta},$$

which vanishes for all functions $\epsilon(z) = \alpha + \beta z + \gamma z^2$. This function corresponds to the infinitesimal form of global conformal transformation (75) with $a = 1 + \beta/2$, $b = \alpha$, $c = -\gamma$ and $d = 1 - \beta/2$. We note that infinitesimal conformal transformations $\epsilon(z) = \alpha + \beta z + \gamma z^2$ correspond to $SL(2)$ subalgebra of Virasoro algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm}, \quad [L_1, L_{-1}] = 2L_0.$$

Consider the notion of quasiprimary fields

$$L_0\phi(z) = \Delta\phi(z), \qquad L_1\phi(z) = 0. \tag{76}$$

The multipoint correlation function involving only such fields should also satisfy the projective Ward identities (72). We note that the condition (76) is less restrictive than the condition for primary field, where we require $L_n\Phi(z) = 0$ for all $n > 0$. In particular $T(z)$ is a quasiprimary field since $L_1T(z) = 0$, but not a primary one $L_2T(z) = c/2 \neq 0$. The projective Ward identities (72) correspond to the following finite identity between correlation functions

$$\langle\phi_1(z_1)\dots\phi_N(z_N)\rangle = \prod_{k=1}^{N}(cz_k+d)^{-2\Delta_k}\Big\langle\phi_1\left(\frac{az_1+b}{cz_1+d}\right)\dots\phi_N\left(\frac{az_N+b}{cz_N+d}\right)\Big\rangle.$$

Now, we come to the distinguished feature of $2D$ conformal symmetry: the infinite dimensional algebra of local conformal transformations. As we saw they correspond to holomorphic functions, which are not necessarily globally defined. For example, we can consider the class of meromorphic functions with some number of poles. Typical holomorphic infinitesimal transformation of this class will have the form

$$z \to z + \alpha(z-w)^{n+1} \quad \alpha \ll 1, \tag{77}$$

centered at some point w. We note that transformations (77) with $n < -2$ are singular at w. The generator of this transformation is

$$l_n = -(z-w)^{n+1}\partial_z.$$

Similarly, we can define the generators $\bar{l}_n$. Let us compute the algebra of l_n's and $\bar{l}_n$'s. Short computation gives the algebra

$$[l_m, l_n] = (m-n)l_{n+m}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{n+m}, \quad [\bar{l}_m, l_n] = 0,$$

which is known as Witt algebra. We note that it contains finite subalgebra spanned by

$$\{l_0, l_1, l_{-1}\} \quad \text{and} \quad \{\bar{l}_0, \bar{l}_1, \bar{l}_{-1}\}.$$

This is exactly the subalgebra associated with global conformal transformations. Namely, the pair $(l_{-1}, \bar{l}_{-1})$ generates translations, the pair $(l_0, \bar{l}_0)$ generates dilations and rotations and $(l_1, \bar{l}_1)$ is responsible for special conformal transformations.

Problems:

1. Find explicitly differential operator $\mathcal{D}_{\boldsymbol{\lambda}}$ defined by

   $$\langle L_{-\boldsymbol{\lambda}}\Phi(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = \mathcal{D}_{\boldsymbol{\lambda}}\langle\Phi(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle,$$

   where $L_{-\boldsymbol{\lambda}}\Phi = L_{-\lambda_1}L_{-\lambda_2}\dots\Phi(z)$.

2. We define central extension of the Witt algebra

   $$[L_m, L_n] = (m-n)L_{n+m} + \lambda_{m,n}, \tag{78}$$

   where $\lambda_{m,n}$ is a $\mathbb{C}$ number. Show that the condition that (78) defines a Lie algebra implies

   $$\lambda_{m,n} = \frac{c}{12}(m^3-m)\delta_{m,-n}.$$

   Hint: it is allowed to shift generators $L_n \to L_n + q_n$, where $q_n \in \mathbb{C}$.

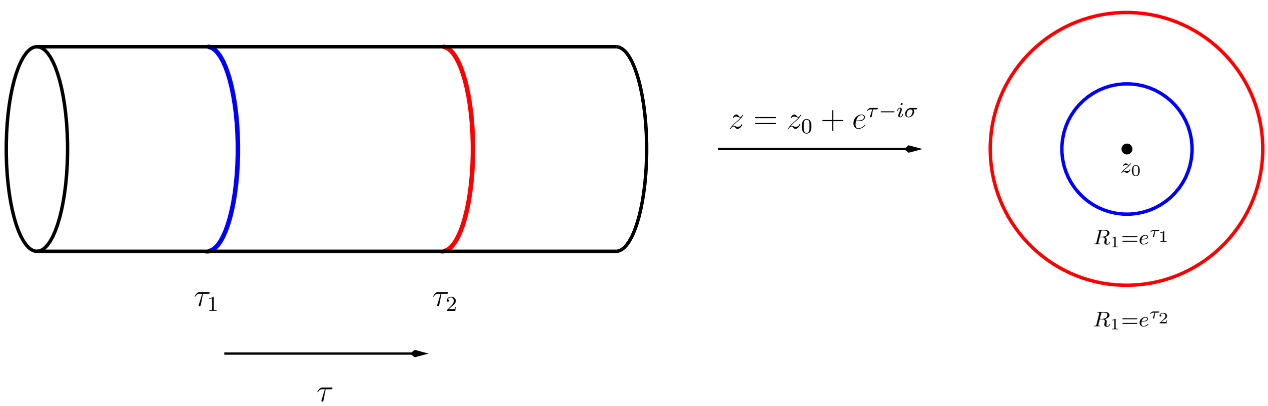
Lecture 5: Hamiltonian formalism in CFT, rep. theory of Virasoro algebra, null-vectors

In this lecture we consider general properties of representations of Virasoro algebra. It will be convenient to work in Hamiltonian formalism. In Euclidean space it is somewhat arbitrary: one can choose one of the Cartesian coordinates, say y to be Euclidean time, while the other x to be the space coordinate. There are many other choices related to the previous one by rotations.

In CFT one usually considers another choice of "space" and "time" – the formalism of the so called radial quantization. In that case equal time slices correspond to concentric circles centered at some point z_0, while the time "runs" in the radial direction. To make this picture more natural, we consider the theory living on a cylinder $\mathbb{R} \times S^1$, described by the coordinates $\tau \in [-\infty, \infty]$ and $\sigma \in [0, 2\pi]$. We can map this cylinder to the complex plane with market points z_0 and ∞ by the exponential map

$$z - z_0 = e^{-iu}, \quad u = \sigma + i\tau \implies ds^2 = dzd\bar{z} = e^{2\tau}\left(d\tau^2 + d\sigma^2\right). \tag{79}$$

We see that the map (79) is a conformal one, but not globally defined. It has two singular points $z = z_0$ and $z = \infty$, that correspond to $\tau = -\infty$ and $\tau = \infty$



The point z_0 can be taken arbitrary and the result for correlation functions (Green functions) should be independent on that choice. It is convenient to choose

$$z_0 = 0.$$

In the Hamiltonial formalism one studies Green functions – matrix elements between the vacuum states. The dictionary between the path integral and Hamiltonian approaches reads as follows. For each local field $\mathcal{O}(z,\bar{z})$ one associates Heisenberg operator

$$\mathcal{O}(z,\bar{z}) \to \widehat{\mathcal{O}}(z,\bar{z}), \tag{80}$$

while correlation functions correspond to Green functions

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_N(z_N,\bar{z}_N)\rangle = \langle 0|\mathcal{T}\left[\widehat{\mathcal{O}}_1(z_1,\bar{z}_1)\dots\widehat{\mathcal{O}}_N(z_N,\bar{z}_N)\right]|0\rangle,$$

where $\mathcal{T}$ stands for chronological ordering (radial ordering)

$$\mathcal{T}\left[\widehat{\mathcal{O}}_1(z_1,\bar{z}_1)\widehat{\mathcal{O}}_2(z_2,\bar{z}_2)\right] = \begin{cases}\widehat{\mathcal{O}}_1(z_1,\bar{z}_1)\widehat{\mathcal{O}}_1(z_2,\bar{z}_2) & \text{for} \quad |z_1| > |z_2|,\\ \widehat{\mathcal{O}}_2(z_2,\bar{z}_2)\widehat{\mathcal{O}}_1(z_1,\bar{z}_1) & \text{for} \quad |z_1| < |z_2|.\end{cases}$$

and $|0\rangle$ stands for the vacuum state.

In conformal field theory, any conformal field $\mathcal{O}(z,\bar{z})$ with conformal dimensions $(\Delta,\bar{\Delta})$ corresponds by (80) to the operator $\widehat{\mathcal{O}}(z,\bar{z})$, that might be expanded in modes on the plane

$$\widehat{\mathcal{O}}(z,\bar{z}) = \sum_{n,\bar{n}\in\mathbb{Z}} \frac{\widehat{\mathcal{O}}_{n,\bar{n}}}{z^{n+\Delta}\,\bar{z}^{\bar{n}+\bar{\Delta}}},$$

or on the cylinder

$$\widehat{\mathcal{O}}(u,\bar{u}) = \sum_{n,\bar{n}\in\mathbb{Z}} \widehat{\mathcal{O}}^{(\mathrm{cyl})}_{n,\bar{n}} e^{-inu} e^{i\bar{n}\bar{u}}.$$

We note that in general the modes $\widehat{\mathcal{O}}_{m,\bar{m}}$ and $\widehat{\mathcal{O}}^{(\mathrm{cyl})}_{m,\bar{m}}$ are nontrivially related. For primary operators one has

$$\widehat{\Phi}(z,\bar{z}) = \left(\frac{dw}{dz}\right)^{\Delta} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}} \widehat{\Phi}(w,\bar{w}).$$

Taking $w = u = i\log z$ (see (79)) one finds

$$\widehat{\Phi}(u,\bar{u}) = i^{\Delta}(-i)^{\bar{\Delta}} z^{\Delta}\bar{z}^{\bar{\Delta}}\widehat{\Phi}(z,\bar{z}) \implies \widehat{\Phi}^{\mathrm{cyl}}_{n,\bar{n}} = i^{\Delta}(-i)^{\bar{\Delta}}\widehat{\Phi}_{n,\bar{n}}.$$

However for descendant fields the relation is more complicated. For example, transformation law for the stress-energy operator

$$\widehat{T}(z) = \sum_{n\in\mathbb{Z}} \frac{\widehat{L}_n}{z^{n+2}}, \quad \widehat{T}(u) = \sum_{n\in\mathbb{Z}} \hat{L}^{\mathrm{cyl}}_n e^{-inu},$$

has the form (see (65))

$$\widehat{T}(z) = \left(\frac{dw}{dz}\right)^2 \widehat{T}(w) + \frac{c}{12}\{w,z\}.$$

Taking $w = u = i\log z$ and using

$$\{i\log z, z\} = \frac{1}{2z^2},$$

one finds

$$\widehat{T}(z) = \left(-\frac{1}{z^2}\right)\widehat{T}(u) + \frac{c}{24z^2} \implies \hat{L}^{\mathrm{cyl}}_n = \frac{c}{24}\delta_{n,0} - \widehat{L}_n.$$

In the following we will simplify our notations and drop $\widehat{\ }$ symbol for operators and their modes. Sometimes, it might lead to confusions. For example, by the field – operator correspondence (80) the descendant field $L_{-k}\mathcal{O}(z,\bar{z})$ corresponds to the operator

$$L_{-k}\mathcal{O}(z,\bar{z}) \to \widehat{L_{-k}\mathcal{O}}(z,\bar{z}).$$

The relation of the operator $\widehat{L_{-k}\mathcal{O}}(z,\bar{z})$ to $\widehat{\mathcal{O}}(z,\bar{z})$ and $\widehat{L}_n$'s is the following. By definition we have

$$L_{-k}\mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} (\xi - z)^{1-k} T(\xi)\mathcal{O}(z,\bar{z})d\xi =$$
$$= \frac{1}{2\pi i}\left(\int_{|\xi|>|z|} (\xi - z)^{1-k} T(\xi)\mathcal{O}(z,\bar{z})d\xi - \int_{|\xi|<|z|} (\xi - z)^{1-k} T(\xi)\mathcal{O}(z,\bar{z})d\xi\right),$$

that can be translated to the Hamiltonian language as

$$\widehat{L_{-k}\mathcal{O}}(z,\bar{z}) = \frac{1}{2\pi i}\left(\int_{|\xi|>|z|}(\xi-z)^{1-k}\widehat{T}(\xi)\widehat{\mathcal{O}}(z,\bar{z})d\xi - \int_{|\xi|<|z|}(\xi-z)^{1-k}\widehat{\mathcal{O}}(z,\bar{z})\widehat{T}(\xi)d\xi\right).$$

We note that for $|\xi|>|z|$ one can expand

$$(\xi-z)^{1-k} = \xi^{1-k} + (k-1)z\xi^{-k} + \dots$$

and hence

$$\frac{1}{2\pi i}\int_{|\xi|>|z|}(\xi-z)^{1-k}\widehat{T}(\xi)\widehat{\mathcal{O}}(z,\bar{z})d\xi = \widehat{L}_{-k}\widehat{\mathcal{O}}(z,\bar{z}) + (k-1)z\widehat{L}_{-(k+1)}\widehat{\mathcal{O}}(z,\bar{z}) + \dots$$

On the other hand for $|\xi|<|z|$ we expand

$$(\xi-z)^{1-k} = (-z)^{1-k} - (k-1)\xi(-z)^{-k} + \dots$$

that implies

$$\frac{1}{2\pi i}\int_{|\xi|<|z|}(\xi-z)^{1-k}\widehat{\mathcal{O}}(z,\bar{z})\widehat{T}(\xi)d\xi = (-z)^{1-k}\widehat{\mathcal{O}}(z,\bar{z})\widehat{L}_{-1} - (k-1)(-z)^{k}\widehat{\mathcal{O}}(z,\bar{z})\widehat{L}_0 + \dots$$

It is instructive also to compute the commutator

$$\begin{aligned}[\widehat{L}_n,\widehat{\mathcal{O}}(z)] &\overset{\text{def}}{=} \frac{1}{2\pi i}\left(\int_{|\xi|>|z|}\xi^{1+n}\widehat{T}(\xi)\widehat{\mathcal{O}}(z)d\xi - \int_{|\xi|<|z|}\xi^{1+n}\widehat{\mathcal{O}}(z)\widehat{T}(\xi)d\xi\right) = \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_z}\xi^{1+n}\mathcal{T}\left[\widehat{T}(\xi)\widehat{\mathcal{O}}(z)\right]d\xi = \frac{1}{2\pi i}\int_{\mathcal{C}_z}\xi^{1+n}\left(\dots + \frac{\widehat{L_1\mathcal{O}}(z)}{(\xi-z)^3} + \frac{\Delta\widehat{\mathcal{O}}(z)}{(\xi-z)^2} + \frac{\partial\widehat{\mathcal{O}}(z)}{\xi-z} + \dots\right)d\xi = \\ &= z^{1+n}\partial\widehat{O}(z) + (n+1)\Delta z^n\widehat{O}(z) + \frac{n(n+1)}{2}z^{n-1}\widehat{L_1 O}(z) + \dots \end{aligned} \tag{81}$$

We note that (81) become particularly simple for primary field

$$[\widehat{L}_n,\widehat{\Phi}(z)] = \left(z^{1+n}\partial + \Delta(n+1)z^n\right)\widehat{\Phi}(z)$$

Having in mind such issues, we will simply erase $\widehat{\ }$ symbol.

The Hamiltonian H has the form

$$H = \frac{1}{2\pi}\int_0^{2\pi}T_{\tau\tau}d\sigma = L_0 + \bar{L}_0 - \frac{c}{12}, \tag{82}$$

where the constant shift comes from the Schwarzian in transformation law for $T(z)$. The vacuum state $|0\rangle$ is an eigenstate of Hamiltonian (82). The vacuum state should be invariant under global conformal transformations and hence

$$L_n|0\rangle = \bar{L}_n|0\rangle = 0 \quad \text{for} \quad n\geq -1.$$

Similarly, we have

$$\langle 0|L_n = \langle 0|\bar{L}_n = 0 \quad \text{for} \quad n\leq 1.$$

The primary field with dimension Δ generates bra-ket states according to the rule

$$|\Delta\rangle \stackrel{\text{def}}{=} \Phi_\Delta(0)|0\rangle \qquad \langle\Delta| \stackrel{\text{def}}{=} \langle 0|\Phi(\infty) = \lim_{z\to\infty}\langle 0|\Phi(z)z^{2L_0}.$$

From the definition of primary fields these states satisfy the conditions

$$L_n|\Delta\rangle = 0, \qquad \langle\Delta|L_{-n} = 0 \quad \text{for} \quad n>0.$$

We define the representation $\mathcal{V}_\Delta$, which is known as Verma module

$$L_{-\boldsymbol{\lambda}}|\Delta\rangle \stackrel{\text{def}}{=} L_{-\lambda_1}\dots L_{-\lambda_n}|\Delta\rangle: \quad L_n|\Delta\rangle = 0 \quad \text{for} \quad n>0, \quad L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad \lambda_1\geq\lambda_2\geq\dots$$

is decomposed into the direct sum of finite dimensional subspaces (here $|\boldsymbol{\lambda}| = \lambda_1+\lambda_2+\dots$)

$$\mathcal{V}_{\Delta,N} = \text{span}\{L_{-\boldsymbol{\lambda}}|\Delta\rangle : |\boldsymbol{\lambda}| = N\},$$

which are eigenspaces of the operator L_0:

$$L_0L_{-\boldsymbol{\lambda}}|\Delta\rangle = (\Delta+|\boldsymbol{\lambda}|)L_{-\boldsymbol{\lambda}}|\Delta\rangle$$

On first few levels one has

$$\begin{aligned}
&|\Delta\rangle \quad \text{for} \quad N=0,\\
&L_{-1}|\Delta\rangle \quad \text{for} \quad N=1,\\
&L_{-2}|\Delta\rangle \quad \text{and} \quad L_{-1}^2|\Delta\rangle \quad \text{for} \quad N=2,\\
&L_{-3}|\Delta\rangle,\ L_{-2}L_{-1}|\Delta\rangle \quad \text{and} \quad L_{-1}^3|\Delta\rangle \quad \text{for} \quad N=3,\\
&L_{-4}|\Delta\rangle,\ L_{-3}L_{-1}|\Delta\rangle,\ L_{-2}^2|\Delta\rangle,\ L_{-2}L_{-1}^2|\Delta\rangle \quad \text{and} \quad L_{-1}^4|\Delta\rangle \quad \text{for} \quad N=4.
\end{aligned}$$

In general there are $p(N)$ states in $\mathcal{V}_{\Delta,N}$, where $p(N)$ is the number of partitions of N. It is convenient to define the character (holomorphic block of the partition function)

$$\chi_\Delta(q) \stackrel{\text{def}}{=} \text{Tr}\left(q^{L_0-\frac{c}{24}}\right)\Big|_{\mathcal{V}_\Delta}.$$

Then we have

$$\chi_\Delta(q) = q^{\Delta-\frac{c}{24}}\sum_{N=0}^\infty p(N)q^N = \frac{q^{\Delta-\frac{c}{24}}}{\prod_{k=1}^\infty(1-q^k)}$$

So far, we assumed that the values of the conformal dimension Δ and of the central charge c are generic. In this case the Verma module $\mathcal{V}_\Delta$ is irreducible. However, interesting things happen for quantized values of Δ. Remember, that we have postulated that $\Phi_{\Delta=0} = I$ is an identity operator and hence

$$\partial I = L_{-1}I = 0,$$

as it should be for coordinate independent field. But does that consistent with the conformal symmetry? Evidently, we have to check that

$$L_nL_{-1}|\Delta\rangle = 0 \quad \text{for} \quad n>0. \tag{83}$$

Well, in our case $\Delta = 0$, but we leave it arbitrary in order to see how does that happen. Actually, the condition (83) is satisfied for all $n>1$ identically. We only have to demand it for $n=1$

$$0 = L_1L_{-1}|\Delta\rangle = 2\Delta|\Delta\rangle.$$

We see that $\Delta = 0$ is necessary condition for the vector $L_{-1}|\Delta\rangle$ to vanish. But not sufficient or course. We can claim that for $\Delta = 0$ one can remove the state $L_{-1}|\Delta\rangle$, as well as all its descendants

$$L_{-\boldsymbol{k}}L_{-1}|\Delta\rangle,$$

from our Hilbert space without violating the conformal symmetry. We call such a state a null-vector. The fact that the null-vector vanishes leads us to the trivial conclusion that any correlation function involving the identity operator should satisfy

$$\partial_z\langle I(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0.$$

Now we try to generalize this. On level 2 we have two states $L_{-1}^2|\Delta\rangle$ and $L_{-2}|\Delta\rangle$. Probably, we can find their linear combination which vanishes, or, at least, can be safely removed from $\mathcal{V}_\Delta$. We have to require

$$L_n\left(L_{-1}^2 + \lambda L_{-2}\right)|\Delta\rangle = 0 \quad \text{for} \quad n > 0. \tag{84}$$

We note that here we have to impose two conditions (84) with $n = 1$ and $n = 2$. For $n = 1$ we have

$$(4\Delta + 2 + 3\lambda)L_{-1}|\Delta\rangle = 0 \qquad \Longrightarrow \qquad \lambda = -\frac{2(2\Delta+1)}{3}.$$

For $n = 2$ we have

$$6\Delta - \frac{2(2\Delta+1)}{3}\left(4\Delta + \frac{c}{2}\right) = 0 \qquad \Longrightarrow \qquad \Delta = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right). \tag{85}$$

Going further, we consider a descendant on third level

$$|\chi\rangle = \left(\lambda_1 L_{-1}^3 + \lambda_2 L_{-2}L_{-1} + \lambda_3 L_{-3}\right)|\Delta\rangle.$$

If it is a null-vector it has to obey $L_1|\chi\rangle = L_2|\chi\rangle = L_3|\chi\rangle = 0$, but since $L_3 = [L_2, L_1]$ it is enough to impose only first two conditions. Simple algebra gives

$$\begin{aligned}
L_1|\chi\rangle &= \left(6(\Delta+1)\lambda_1 + 3\lambda_2\right)L_{-1}^2|\Delta\rangle + \left(2\Delta\lambda_2 + 4\lambda_3\right)L_{-2}|\Delta\rangle,\\
L_2|\chi\rangle &= \left(6(3\Delta+1)\lambda_1 + \left(4\Delta + \frac{c}{2} + 4\right)\lambda_2 + 5\lambda_3\right)L_{-1}|\Delta\rangle.
\end{aligned}$$

We have three linear equations

$$\begin{aligned}
&6(\Delta+1)\lambda_1 + 3\lambda_2 = 0,\\
&2\Delta\lambda_2 + 4\lambda_3 = 0,\\
&6(3\Delta+1)\lambda_1 + \left(4\Delta + \frac{c}{2} + 4\right)\lambda_2 + 5\lambda_3 = 0
\end{aligned}$$

for three unknowns $(\lambda_1, \lambda_2, \lambda_3)$. So, the determinant should vanish

$$12\left(3(\Delta+1)^2 + (c-13)(\Delta+1) + 12\right) = 0,$$

which has two solutions

$$\Delta = \frac{1}{6}\left(7 - c \pm \sqrt{(c-1)(c-25)}\right). \tag{86}$$

We see that the expressions for null-vectors (85) and (86) look very similar. One can simplify them by introducing Liouville like parametrization of the central charge and of conformal dimension

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad \Delta = \Delta(\alpha) = \alpha(Q - \alpha). \tag{87}$$

Then the singular vectors appear at the values

$$\alpha = -\frac{b}{2}, \qquad \alpha = -\frac{b^{-1}}{2}$$

on level 2 and

$$\alpha = -b, \qquad \alpha = -b^{-1}$$

on level 3. Corresponding null-vectors have the form

$$\big(L_{-1}^2 + b^2 L_{-2}\big)\,|\Delta\rangle \quad \text{and} \quad \big(L_{-1}^3 + 4b^2 L_{-2}L_{-1} + 2b^2(2b^2+1)L_{-3}\big)\,|\Delta\rangle,$$

and similar expressions for $b \to b^{-1}$. One can compute null-vectors on higher levels in a similar manner. General result states that at level N, for any two positive integers m and n such that $N = mn$, there exist a null vector

$$|\chi_{m,n}\rangle = D_{m,n}|\Delta_{m,n}\rangle \tag{88}$$

with (this result is known as Kac-Feigin-Fuks theorem)

$$\Delta = \Delta_{m,n} = \Delta(\alpha_{m,n}), \qquad \alpha_{m,n} = -\frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2}. \tag{89}$$

The operator $D_{m,n}$ in (88) is known as *null vector creation operator*. It is convenient to adopt the following normalization

$$D_{m,n} = L_{-1}^{mn} + c_1(b)L_{-2}L_{-1}^{mn-2} + c_2(b)L_{-3}L_{-1}^{mn-3} + \dots. \tag{90}$$

The coefficients $c_k(b)$ in (90) can be recursively found

$$c_1(b) = \frac{mn}{6}\left((m^2-1)b^2 + (n^2-1)b^{-2}\right),$$

$$c_2(b) = \frac{m^2n^2}{12}\left((m^2-1)b^2 + (n^2-1)b^{-2}\right) + \frac{mn}{30}\Big((m^2-1)\big((m^2-4)b^4 - 5b^2 - 5\big) + \\ + (n^2-1)\big((n^2-4)b^{-4} - 5b^{-2} - 5\big)\Big) + \frac{mn(m^2n^2-1)}{6}$$

For generic values of the central charge c, $|\chi_{m,n}\rangle$ is the only singular vector in the Verma module $\mathcal{V}_{\Delta_{m,n}}$ with $\Delta = \Delta_{m,n} + mn = \Delta_{m,-n}$. We can define the factor space

$$\mathcal{V}_{\Delta_{m,n}}/\mathcal{V}_{\Delta_{m,-n}}$$

without violating the conformal symmetry. The character of the corresponding factor space is

$$\chi'_{m,n}(q) = \frac{q^{\Delta_{m,n}-\frac{c}{24}}(1-q^{mn})}{\prod_{k=1}^{\infty}(1-q^k)},$$

etc.

It is convenient to think about representation theory of Virasoro algebra with the help of Shapovalov form, that is Hermitian form defined by

$$\langle\Delta|\Delta\rangle = 1, \qquad (L_n)^+ = L_{-n}.$$

We introduce Gram matrix

$$G_{\boldsymbol{\lambda},\boldsymbol{\mu}} \stackrel{\text{def}}{=} \langle\Delta|L_{\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\Delta\rangle \tag{91}$$

Clearly, it is block diagonal matrix $G = \{G_0, G_1, G_2 \dots\}$ with block sizes $p(N)\times p(N)$. The degeneracies of this matrix are closely related to the reducibility of the corresponding Verma module. For example, one has

$$G_1 = 2\Delta,$$

and hence the determinant $\det G_1$ vanishes for degenerate dimension $\Delta = \Delta_{1,1}$. In general, it is clear that any descendant of a singular vector is orthogonal to everything else in Verma module

$$\langle\Delta_{m,n}|L_{\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\chi_{m,n}\rangle = 0 \quad \text{for all} \quad \boldsymbol{\lambda} \text{ and } \boldsymbol{\mu}.$$

That is we have $p(N-mn)$ singular vectors on level N of the form $L_{-\boldsymbol{\lambda}}|\chi_{m,n}\rangle$ with $|\boldsymbol{\lambda}| = N$, which implies that the determinant of the Shapovalov form on level N vanishes as

$$\det G_N \sim (\Delta - \Delta_{m,n})^{p(N-mn)} \tag{92}$$

In fact it can be shown that taking the product of all factors (92) with $mn \leq N$ exhausts Kac determinant completely

$$\det G_N = C_N \prod_{mn\leq N} (\Delta - \Delta_{m,n})^{p(N-mn)}, \tag{93}$$

for some numerical coefficient C_N. Formula (93) is equivalent to Kac-Feigin-Fuks theorem. We will not prove it in full generality. However, it is easy to show that (93) provides correct degree in Δ. Indeed, it is easy to show that the degree of the matrix element (91) is non-greater than $l(\boldsymbol{\lambda})$ and $l(\boldsymbol{\mu})$, where $l(\boldsymbol{\lambda})$ is the length of partition $\boldsymbol{\lambda}$. Thus the degree of Kac determinant is non greater than

$$\sum_{|\boldsymbol{\lambda}|=N} l(\boldsymbol{\lambda})$$

There is a simple combinatorial fact that[8]

$$\sum_{|\boldsymbol{\lambda}|=N} l(\boldsymbol{\lambda}) = \sum_{mn\leq N} p(N-mn).$$

[8] I can be proven as follows (here $l_n(\boldsymbol{\lambda})$ is the number of parts equal to n in partition $\boldsymbol{\lambda}$)

$$\sum_{N=0}^{\infty} q^N \sum_{mn\leq N} P(N-mn) = \sum_{m,n>0} q^{mn} \sum_{N=0}^{\infty} q^N p(N) = \sum_{n>0} \frac{q^n}{1-q^n} \prod_{k>0} \frac{1}{1-q^k} = \sum_{n>0} \frac{q^n}{(1-q^n)^2} \prod_{k>0,k\neq n} \frac{1}{1-q^k} =$$
$$= \sum_{n>0} \frac{1}{1-q} \cdots \frac{1}{1-q^{n-1}} \frac{\partial}{\partial q^n}\left(\frac{1}{1-q^n}\right) \frac{1}{1-q^{n+1}} \cdots = \sum_{n>0}\sum_{\boldsymbol{\lambda}} l_n(\boldsymbol{\lambda}) q^{|\boldsymbol{\lambda}|} = \sum_{\boldsymbol{\lambda}} l(\boldsymbol{\lambda}) q^{|\boldsymbol{\lambda}|}.$$

Let us assume that we have a field $\Phi(z)$ with $\Delta = \Delta_{2,1}$

$$(L_{-1}^2 + b^2 L_{-2})|\Delta_{2,1}\rangle,$$

and consider the following correlation function

$$\Psi(z|z_1,\ldots,z_N) \stackrel{\text{def}}{=} \langle\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle.$$

This function satisfies partial differential equation

$$\left[\partial_z^2 + b^2 \sum_{k=1}^{N} \left(\frac{\Delta_k}{(z-z_k)^2} + \frac{\partial_k}{z-z_k}\right)\right]\Psi(z|z_1,\ldots,z_N) = 0.$$

In the case of $N=3$ this partial differential equation actually becomes an ordinary differential equation. Indeed, in this case the projective Ward identities allow one to express derivatives ∂_k through ∂. As a result we have a hypergeometric equation for correlation function

$$\left[\frac{d^2}{dz^2} + b^2\left(\sum_{k=1}^{3}\left(\frac{\Delta_k}{(z-z_k)^2} - \frac{1}{z-z_k}\frac{d}{dz}\right) - \sum_{i<j}\frac{\Delta_{1,2}+\Delta_{ij}}{(z-z_i)(z-z_j)}\right)\right]\Psi(z|z_1,z_2,z_3) = 0. \tag{94}$$

It is convenient to change $\Psi(z|z_1,z_2,z_3) = \prod_{k=1}^{3}(z-z_k)^{-\frac{b^2}{2}}\tilde{\Psi}(z|z_1,z_2,z_3)$ in such a way that the term with first derivative vanishes. Then (94) reduces to

$$\left(\frac{d^2}{dz^2} + \mathcal{T}(z)\right)\tilde{\Psi}(z|z_1,z_2,z_3) = 0, \qquad \mathcal{T}(z) = \sum_{k=1}^{3}\left(\frac{\delta_k}{(z-z_k)^2} + \frac{c_k}{z-z_k}\right).$$

The parameters δ_k are given by

$$\delta_k = b^2\left(\Delta_k - \frac{1}{2}\right) - \frac{b^4}{4},$$

and three "accessory" parameters c_k are subject to three linear equations following from condition of vanishing of singularity at infinity

$$\mathcal{T}(z) = \frac{1}{z^4} \quad \text{at} \quad z \to \infty,$$

and hence are uniquely determined. Let us look for the solution to this equation in the form

$$\tilde{\Psi}(z|z_1,z_2,z_3) = (z-z_1)^{\lambda}\left(1 + a_1(z-z_1) + \ldots\right) \quad \text{at} \quad z \to z_1.$$

In the leading order we obtain two solutions for λ

$$\lambda = b\alpha_1 - \frac{b^2}{2} \quad \text{and} \quad \lambda = 1 - b\alpha_1 + \frac{b^2}{2}.$$

These two exponents correspond to the following behavior of correlation function

$$\Psi(z|z_1,z_2,z_3) = (z-z_1)^{\Delta(\alpha_1 \pm \frac{b}{2}) - \Delta(\alpha_1) - \Delta(-\frac{b}{2})}\left(C_0(z_1,z_2,z_3) + \ldots\right) \quad \text{at} \quad z \to z_1$$

We will interpret this as a fact that the degenerate field $\Phi_{-\frac{b}{2}}$ "fusses" with general field as

$$[\Phi_{-\frac{b}{2}}][\Phi_\alpha] = [\Phi_{\alpha+\frac{b}{2}}] + [\Phi_{\alpha-\frac{b}{2}}]. \tag{95}$$

We note that (96) is similar to the formula of tensor products of representations of $\mathfrak{sl}_2$

$$\pi_{\frac{1}{2}} \otimes \pi_s = \pi_{s+\frac{1}{2}} \oplus \pi_{s-\frac{1}{2}}.$$

Problems:

1. Compute singular vectors on level 4.
2. Let $|\chi\rangle$ be the null-vector at level N. How many equations provide the constraints $L_1|\chi\rangle = L_2|\chi\rangle = 0$? Count the number of equations on level 5 and explain that the excess equations are algebraically dependent from non-excess ones.

Lecture 6: Free bosonic CFT I: path integral approach

Let us start with the theory of free massless bosonic field

$$S[\varphi] = \frac{1}{8\pi} \int \left(\partial_\mu \varphi(\boldsymbol{x})\right)^2 d^2\boldsymbol{x}. \tag{96}$$

First of all, we notice that this is our "patient": the theory is conformally invariant (at least classically). This follows from the identity

$$\int \partial_\mu \varphi(\boldsymbol{x}) \partial_\mu \varphi(\boldsymbol{x}) d^2\boldsymbol{x} = -2 \int \partial\varphi(z,\bar{z})\bar{\partial}\varphi(z,\bar{z}) dz d\bar{z}, \quad z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2.$$

In this complex form it is obvious, that the action is invariant under conformal transformations

$$z = f(\zeta), \qquad \bar{z} = f^*(\bar{\zeta}).$$

The stress-energy tensor

$$T_{\mu\nu} \overset{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta_{\mu\nu}\mathcal{L} = \frac{1}{4\pi}\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\delta_{\mu\nu}\left(\partial_\mu\varphi\right)^2\right),$$

is indeed traceless $T_{\mu\mu} = 0$ and hence the components

$$T = -\frac{\pi}{2}\left(T_{11} - T_{22} - 2iT_{12}\right) = -\frac{1}{2}(\partial\varphi)^2, \quad \bar{T} = -\frac{\pi}{2}\left(T_{11} - T_{22} + 2iT_{12}\right) = -\frac{1}{2}(\bar{\partial}\varphi)^2,$$

obey

$$\bar{\partial} T = \partial \bar{T} = 0.$$

on-shell.

Now let us study the theory (97) quantum mechanically. It is easy, since the theory is Gaussian. There are however some subtleties. Consider the partition function

$$Z = \int [\mathcal{D}\varphi] e^{-S}.$$

This integral diverges since the action does not contain the zero mode φ_0 of the field φ: $Z \sim \int d\varphi_0$. We define the measure $[\mathcal{D}\varphi]'$ simply as an integral over all non-zero modes of the field φ.

Moreover, anticipating that we will have to deal with infrared divergencies, we will consider our theory in a finite volume. That is we impose the periodic conditions $\varphi(x_1, x_2 + 2\pi R) = \varphi(x_1, x_2)$. Let us compute the two-point function in this theory

$$G(\boldsymbol{x} - \boldsymbol{y}) \overset{\text{def}}{=} \langle \varphi(\boldsymbol{x})\varphi(\boldsymbol{y})\rangle = \frac{1}{Z}\int [\mathcal{D}\varphi] \varphi(\boldsymbol{x})\varphi(\boldsymbol{y}) e^{-S}.$$

As usual in Gaussian theory, one has to invert the quadratic form

$$-\Delta G(\boldsymbol{x}) = 4\pi \delta_R^2(\boldsymbol{x}) \quad \text{where} \quad \delta_R^2(\boldsymbol{x}) = \delta(x_1) \sum_{n=-\infty}^{\infty} \delta(x_2 + 2\pi n R),$$

Clearly

$$G(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} K\big(|z - 2i\pi nR|\big) \quad \text{where} \quad -\Delta K(|z|) = 4\pi\delta^2(\boldsymbol{x}).$$

Using $\Delta = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\varphi^2$ and integrating last equation over the disk of radius r, we obtain[9]

$$-rK'(r) = 2 \qquad \Longrightarrow \qquad K(r) = -2\log r + \text{const} = -\log|z|^2 + \text{const},$$

which implies

$$G(\boldsymbol{x}) = -\log\left(4\sinh\frac{z}{2R}\sinh\frac{\bar{z}}{2R}\right) = -\log\frac{z\bar{z}}{R^2} + O\left(\frac{1}{R^2}\right) \quad \text{at} \quad R \to \infty. \tag{97}$$

We treat R as an infrared cut-off: it is assumed to be infinite, but we keep it large in the intermediate calculations and then send $R \to \infty$ in the final answer.

Multipoint correlation functions are computed by the Wick rules:

$$\begin{aligned}
&\langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\rangle = \\
&\quad = \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\rangle\langle\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\rangle + \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_3)\rangle\langle\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_4)\rangle + \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_4)\rangle\langle\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_3)\rangle = \\
&\qquad = G(\boldsymbol{x}_1 - \boldsymbol{x}_2)G(\boldsymbol{x}_3 - \boldsymbol{x}_4) + G(\boldsymbol{x}_1 - \boldsymbol{x}_3)G(\boldsymbol{x}_2 - \boldsymbol{x}_4) + G(\boldsymbol{x}_1 - \boldsymbol{x}_4)G(\boldsymbol{x}_2 - \boldsymbol{x}_3) \quad \text{etc}
\end{aligned}$$

We note that the field φ does not look like a conformal field, its correlation functions behave logarithmically rather than power-like. Conformal fields in the theory (97) are represented by the exponential fields

$$e^{i\alpha\varphi(\boldsymbol{x})} \qquad \alpha \in \mathbb{R}. \tag{98}$$

We are interested in multipoint correlation functions

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)} \dots e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle$$

One can compute these correlation functions by expanding exponents in series, then using the Wick theorem and then resuming again. But it is better and much easier to use the following general fact, that for any Φ functional linear in fundamental field φ: $\Phi = \int J(\boldsymbol{x})\varphi(\boldsymbol{x})d^2\boldsymbol{x}$ we have

$$\langle e^{\Phi}\rangle = e^{\frac{1}{2}\langle\Phi^2\rangle}. \tag{99}$$

In our case

$$\Phi = i\sum_{k=1}^{n}\alpha_k\varphi(\boldsymbol{x}_k) = \int J(\boldsymbol{x})\varphi(\boldsymbol{x})d^2\boldsymbol{x} \quad \text{where} \quad J(\boldsymbol{x}) = i\sum_{k=1}^{n}\alpha_k\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_k).$$

Then we have

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)} \dots e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\langle\varphi(\boldsymbol{x}_k)\varphi(\boldsymbol{x}_k)\rangle - \sum_{i<j}\alpha_i\alpha_j\langle\varphi(\boldsymbol{x}_i)\varphi(\boldsymbol{x}_j)\rangle\right).$$

At this point we have a UV problem, since

$$\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = G(0) = \infty.$$

[9] Add here how we integrate!!!

A standard way to deal with it is to introduce the UV cut-off. It s not universal. There many ways to do it, or as one says, there are many regularization schemes. In renormalizable QFT physically observable quantities must be independent on regularization scheme used for their computation. We define the scheme as follows

$$\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = -\log\frac{r_0^2}{R^2},$$

where $r_0 \ll 1$. Then, according to (100), the correlation function has the form

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)}\dots e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle = \frac{r_0^{\sum\alpha_k^2}}{R^{(\sum\alpha_k)^2}}\prod_{i<j}|z_i-z_j|^{2\alpha_i\alpha_j}. \tag{100}$$

Observables should be independent on the UV cut-off. We define the new field

$$V_\alpha \stackrel{\text{def}}{=} r_0^{-\alpha^2}e^{i\alpha\varphi} = z_0^{-\frac{\alpha^2}{2}}\bar{z}_0^{-\frac{\alpha^2}{2}}e^{i\alpha\varphi}$$

We note that the operator V_α depends explicitly on a scale and hence has an anomalous conformal dimension $\Delta(\alpha) = \bar{\Delta}(\alpha) = \frac{\alpha^2}{2}$. Even for renormalized operators we see that the correlation function (101) vanishes in the limit $R\to\infty$ unless the neutrality condition

$$\sum_{k=1}^n\alpha_k = 0 \tag{101}$$

is satisfied.

It is instructive to derive the anomalous dimension of the operator V_α in different, but equivalent way. We expand

$$e^{a\varphi} = \sum_{k=0}^\infty\frac{a^k}{k!}\varphi^k, \tag{102}$$

and express it in terms of Wick ordered quantities and then resum back. The field φ^k is not Wick ordered. Namely, consider the correlation function

$$\langle\varphi(\boldsymbol{x})^k\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle.$$

If one knows how to compute these correlation functions for any n, one knows (in principle) how to compute everything, like correlation functions of exponential operators (99) for example. While computing (104) one meets two types of contractions: either $\varphi(\boldsymbol{x})$'s are contracted among themselves, or with some of the $\varphi(\boldsymbol{x}_j)$'s. For example,

$$\begin{aligned}\langle\varphi(\boldsymbol{x})^2\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle &= \langle\varphi(\boldsymbol{x})^2\rangle\langle\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle+\\ &+\sum_{i\neq j}\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_i)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_j)\rangle\langle\varphi(\boldsymbol{x}_1)\dots\cancel{\varphi(\boldsymbol{x}_i)}\dots\cancel{\varphi(\boldsymbol{x}_j)}\dots\varphi(\boldsymbol{x}_n)\rangle,\end{aligned}$$

or

$$\begin{aligned}\langle\varphi(\boldsymbol{x})^4\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle &= 3\langle\varphi(\boldsymbol{x})^2\rangle\langle\varphi(\boldsymbol{x})^2\rangle\langle\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle+\\ &+6\langle\varphi(\boldsymbol{x})^2\rangle\sum_{i\neq j}\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_i)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_j)\rangle\langle\varphi(\boldsymbol{x}_1)\dots\cancel{\varphi(\boldsymbol{x}_i)}\dots\cancel{\varphi(\boldsymbol{x}_j)}\dots\varphi(\boldsymbol{x}_n)\rangle+\\ &+\sum_{i\neq j\neq k\neq l}\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_i)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_j)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_k)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_l)\rangle\times\\ &\times\langle\varphi(\boldsymbol{x}_1)\dots\cancel{\varphi(\boldsymbol{x}_i)}\dots\cancel{\varphi(\boldsymbol{x}_j)}\dots\cancel{\varphi(\boldsymbol{x}_k)}\dots\cancel{\varphi(\boldsymbol{x}_l)}\dots\varphi(\boldsymbol{x}_n)\rangle.\end{aligned}$$

According to the formulae above one can define what is called Wick ordered fields (with our choice of UV regularization scheme)

$$\varphi(\boldsymbol{x})^2 =: \varphi(\boldsymbol{x})^2 : + G_0, \quad \varphi(\boldsymbol{x})^4 =: \varphi(\boldsymbol{x})^4 : + 6G_0 : \varphi(\boldsymbol{x})^2 : + 3G_0^2 \quad \text{where} \quad G_0 = -\log \frac{r_0^2}{R^2}.$$

The field $:\mathcal{O}:$ is ordered, meaning that in correlation function it is contracted only with other fields, not with the one entering the symbol of $\mathcal{O}$. In general, it is clear that

$$\varphi^k(\boldsymbol{x}) = \sum_{l=0}^{[k/2]} \frac{k!}{l!(k-2l)!} \frac{G_0^l}{2^l} : \varphi^{k-2l}(\boldsymbol{x}) : .$$

Substituting this in (104), we have

$$\sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{a^k}{k!} \frac{k!}{l!(k-2l)!} \frac{G_0^l}{2^l} : \varphi^{k-2l}(\boldsymbol{x}) := \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^{n+2l}}{l!n!} \frac{G_0^l}{2^l} : \varphi^n(\boldsymbol{x}) := e^{\frac{a^2 G_0}{2}} : e^{a\varphi(\boldsymbol{x})} : .$$

Applying this to the field $e^{i\alpha\varphi(\boldsymbol{x})}$ we obtain

$$e^{i\alpha\varphi(\boldsymbol{x})} = \left(\frac{r_0^2}{R^2}\right)^{\frac{\alpha^2}{2}} : e^{i\alpha\varphi(\boldsymbol{x})} : \quad \text{or} \quad V_\alpha(\boldsymbol{x}) = \frac{1}{R^{\alpha^2}} : e^{i\alpha\varphi(\boldsymbol{x})} : \tag{103}$$

Now, let us check the conformal Ward identities and find that $V_\alpha(\boldsymbol{x})$ is actually a primary field. First we note that while $\varphi(\boldsymbol{x})$ itself is not a conformal field, its derivative is. The two-point functions have the form

$$\langle \partial\varphi(z)\varphi(w,\bar{w})\rangle = -\frac{1}{(z-w)}, \quad \langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{(z-w)^2}.$$

In multipoint correlation functions we can use

$$\partial\varphi(z)\partial\varphi(w) = -\frac{1}{(z-w)^2} + : \partial\varphi(z)\partial\varphi(w) :=$$
$$= -\frac{1}{(z-w)^2} + : (\partial\varphi(w))^2 : + : \partial^2\varphi(w)\partial\phi(w) : (z-w) + \frac{1}{2} : \partial^3\varphi(w)\partial\phi(w) : (z-w)^2 + \ldots,$$

where we expanded the right hand side at $z \to w$.

Now, let us compute the OPE of $T(\zeta)$:

$$T(\zeta) = -\frac{1}{2} : (\partial\varphi(\zeta))^2 : .$$

with $V_\alpha(z)$ (we hide the dependence on $\bar{z}$)

$$T(\zeta)V_\alpha(z) = \frac{\frac{\alpha^2}{2} V_\alpha(z)}{(\zeta-z)^2} + \frac{i\alpha : \partial\varphi(\zeta)V_\alpha(z) :}{\zeta - z} + : T(\zeta)V_\alpha(z) := \frac{\frac{\alpha^2}{2} V_\alpha(z)}{(\zeta-z)^2} + \frac{\partial V_\alpha(z)}{\zeta - z} + \ldots \tag{104}$$

We see that $V_\alpha(z)$ is a primary field with the conformal dimension $\Delta(\alpha) = \frac{\alpha^2}{2}$. As we learned, this should imply the conformal Ward identity

$$\langle T(\zeta)V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n)\rangle = \sum_{k=1}^{n} \left(\frac{\Delta(\alpha_k)}{(\zeta - z_k)^2} + \frac{\partial_k}{\zeta - z_k}\right) \langle V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n)\rangle, \tag{105}$$

where we assumed that the neutrality condition (103) is fulfilled. Similarly one can show that

$$T(\zeta)T(z) = \frac{1}{2(\zeta-z)^4} - \frac{:\partial\varphi(\zeta)\partial\varphi(z):}{(\zeta-z)^2} + :T(\zeta)T(z): = \frac{1}{2(\zeta-z)^4} + \frac{2T(z)}{(\zeta-z)^2} + \frac{\partial T(z)}{\zeta-z} + \dots, \tag{106}$$

and hence the stress-energy tensor $T(z) = -\frac{1}{2} : (\partial\varphi(z))^2 :$ defines the Virasoro algebra with the central charge

$$c = 1.$$

As we learned before, two identities (106) and (108) are enough to express any correlation function of descendant fields through the correlation function of the primary fields only. In the free theory the last one is pretty trivial

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_n}(z_n,\bar{z}_n)\rangle = \begin{cases} \prod_{i<j}|z_i-z_j|^{2\alpha_i\alpha_j} \quad \text{if} \quad \sum_{k=1}^n \alpha_k = 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

We note however, that the Ward identities or the OPE's are universal. They do not depend on an actual theory, being just the constraints imposed by the conformal symmetry.

We mention here the following important point. What we constructed in this lecture is the map from the Verma module to the free fields, usually referred as bosonization. Namely, we have a Verma module

$$\mathcal{V}_\Delta = \{\Phi_\Delta, L_{-1}\Phi_\Delta, L_{-1}^2\Phi_\Delta, L_{-2}\Phi_\Delta, L_{-1}^3\Phi_\Delta, L_{-1}L_{-2}\Phi_\Delta, L_{-3}\Phi_\Delta, \dots\}$$

and a *Fock* module (here all fields are assumed to be Wick ordered)

$$\mathcal{F}_\alpha = \{V_\alpha, (\partial\varphi)V_\alpha, (\partial\varphi)^2V_\alpha, (\partial^2\varphi)V_\alpha, (\partial\varphi)^3V_\alpha, (\partial\varphi)(\partial^2\varphi)V_\alpha, (\partial^3\varphi)V_\alpha, \dots\}$$

The map $\pi : \mathcal{V}_\Delta \xrightarrow{\pi} \mathcal{F}_\alpha$ goes as follows ($\Delta = \frac{\alpha^2}{2}$)

$$\begin{gathered} \Phi_\Delta \xrightarrow{\pi} V_\alpha, \quad L_{-1}\Phi_\Delta \xrightarrow{\pi} i\alpha(\partial\varphi)V_\alpha, \\ L_{-1}^2\Phi_\Delta \xrightarrow{\pi} \big(i\alpha\partial^2\varphi - \alpha^2(\partial\varphi)^2\big)V_\alpha, \quad L_{-2}\Phi_\Delta \xrightarrow{\pi} \big(i\alpha\partial^2\varphi - \frac{1}{2}(\partial\varphi)^2\big)V_\alpha, \quad \dots \end{gathered} \tag{107}$$

For generic values of $\Delta = \frac{\alpha^2}{2}$ this map provides an isomorphism between the two modules. However, for special values of α it has a kernel. For example, for $\alpha = 0$ all fields of the form

$$L_{-\boldsymbol{\lambda}}L_{-1}\Phi_\Delta$$

are mapped to zero. We interpret this as a fact that the field V_0 is a degenerate field with $\Delta = \Delta_{1,1}$: it has a null-vector at the first level. In the language of bosonization it implies that this field together with all its descendants vanishes identically. Next example of the kernel occurs at the level 2. There are two fields

$$L_{-1}^2\Phi_\Delta \sim \big(i\alpha\partial^2\varphi - \alpha^2(\partial\varphi)^2\big)V_\alpha \quad \text{and} \quad L_{-2}\Phi_\Delta \sim \big(i\alpha\partial^2\varphi - \frac{1}{2}(\partial\varphi)^2\big)V_\alpha.$$

They are linearly dependent provided that either $\alpha = 0$ or $\alpha^2 = \frac{1}{2}$. First possibility corresponds to the one we already know, the descendant of the null-vector for the degenerate field $\Phi_{1,1}$. Second possibility corresponds to the degenerate field $\Phi_{2,1}$ or $\Phi_{1,2}$ for the special value of the central charge that we have

$c = 1$. We note that it follows from (85) that $\Delta_{2,1} = \Delta_{1,2}$ for $c = 1$. This condition can be relaxed if one consider the bosonization map for improved stress-energy tensor (see exercise 2).

Last, consider the product $V_\alpha(z,\bar{z})V_\beta(w,\bar{w})$. Let us bring it to the Wick ordered form. Using

$$:\varphi(z,\bar{z})^k :: e^{i\beta\varphi(w,\bar{w})} := \sum_{l=0}^{k} \frac{(i\beta)^l k! G^l(z-w)}{(k-l)!\, l!} :\varphi(z,\bar{z})^{l-k} e^{i\beta\varphi(w,\bar{w})}:,$$

we find that

$$:e^{i\alpha\varphi(z,\bar{z})} :: e^{i\beta\varphi(w,\bar{w})} := \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(i\alpha)^k}{k!}\frac{(i\beta)^l k! G^l(z-w)}{(k-l)!\, l!} :\varphi(z,\bar{z})^{l-k} e^{i\beta\varphi(w,\bar{w})} :=$$

$$= \frac{|z-w|^{2\alpha\beta}}{R^{2\alpha\beta}} :e^{i\alpha\varphi(z,\bar{z})} e^{i\beta\varphi(w,\bar{w})}:. \quad (108)$$

Using the relation (105) we can rewrite this in the form of OPE

$$V_\alpha(z,\bar{z})V_\beta(w,\bar{w}) = |z-w|^{2\alpha\beta}\left(V_{\alpha+\beta}(w,\bar{w}) + (z-w)\frac{\alpha}{\alpha+\beta}L_{-1}V_{\alpha+\beta}(w,\bar{w}) + \dots\right) \quad \text{at} \quad z \to w. \quad (109)$$

We note that the degree $2\alpha\beta$ in (111) has a natural interpretation

$$\alpha\beta = \Delta(\alpha+\beta) - \Delta(\alpha) - \Delta(\beta),$$

which follows from dimensional analysis of both sides of the OPE (111).

Problems:

1. Show by explicit free-field calculations that (107) is satisfied.

2. Consider the bosonization map π from Verma module realized by the improved stress-energy tensor
$$T(z) = -\frac{1}{2}(\partial\varphi)^2 + \frac{Q}{\sqrt{2}}\partial^2\varphi \quad \text{where} \quad Q = b + \frac{1}{b}.$$
Find the values of α at which this map has a kernel at levels 1, 2 and 3.

Lecture 7: Free bosonic CFT II: Hamiltonian approach

It is instructive to rederive the results obtained in the previous lecture starting from the $U(1)$ current algebra. Namely, we have a current

$$J(z) = i\partial\varphi(z), \tag{110}$$

which satisfies an OPE

$$J(z)J(w) = \frac{1}{(z-w)^2} + \text{reg} \tag{111}$$

We define the mode of the current $J(z)$ applied to the local field $\mathcal{O}$ by

$$a_n\mathcal{O}(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\mathcal{C}_z} (\xi - z)^n J(\xi)\mathcal{O}(z)d\xi. \tag{112}$$

Repeating the same calculations which lead us to the Virasoro algebra (70) we obtain commutation relation for the modes (114)

$$[a_m, a_n] = m\delta_{m,-n}, \tag{113}$$

known as Heisenberg algebra. Among other fields there are $U(1)$-primary ones, which have the simplest OPE with $J(\xi)$

$$J(\xi)V_\alpha(z) = \frac{\alpha V_\alpha(z)}{(\xi - z)} + \ldots \quad \text{at} \quad \xi \to z,$$

which implies the following Ward identity

$$\langle J(\xi)V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\rangle = \sum_{k=1}^{N} \frac{\alpha_k}{\xi - z_k}\langle V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\rangle. \tag{114}$$

Now we define the stress-energy tensor via Sugawara formula

$$T(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\mathcal{C}_z} \frac{J(\xi)J(z)}{2(\xi - z)}d\xi, \tag{115}$$

where the components of $T(z)$ satisfy (108). Then in terms of modes Sugawara formula (117) takes the form

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k a_{n-k}, \qquad L_0 = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_{-k}a_k. \tag{116}$$

Then one can easily see that the field $J(z)$ is primary field with $\Delta = 1$

$$T(\xi)J(z) = \frac{J(z)}{(\xi - z)^2} + \frac{J'(z)}{\xi - z} + \ldots$$

It implies that

$$J(\xi) \sim \frac{1}{\xi^2} \quad \text{at} \quad \xi \to \infty.$$

Substituting this asymptotic into the $U(1)$ Ward identity (116), one obtains a $U(1)$ global Ward identity

$$\left(\sum_{k=1}^{N} \alpha_k\right)\langle V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\rangle = 0.$$

Now we notice that the Sugawara construction (117) of $T(\xi)$ leads to additional differential equations on correlation functions. We have

$$\partial V_\alpha = L_{-1}V_\alpha = a_{-1}a_0V_\alpha = \frac{\alpha}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{-1}J(\xi)V_\alpha(z)d\xi.$$

Applying this to correlation function, we get

$$\left(\partial_k + \sum_{j\neq k}\frac{\alpha_j\alpha_k}{z_j-z_k}\right)\langle V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_N)\rangle = 0.$$

This implies

$$\langle V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_N)\rangle \sim \prod_{i<j}(z_i-z_j)^{\alpha_i\alpha_j}.$$

In Hamiltonian approach an exponential fields V_α corresponds to the highest weight state $|\alpha\rangle$

$$a_n|\alpha\rangle = 0 \quad \text{for} \quad n>0, \qquad a_0|\alpha\rangle = \alpha|\alpha\rangle,$$

which generates a representation of the Heisenberg algebra (115) known as Fock module

$$\mathcal{F}_P = \text{span}\Big(a_{\boldsymbol{\lambda}}|P\rangle \overset{\text{def}}{=} a_{-\lambda_1}a_{-\lambda_2}\dots|P\rangle\Big)$$

Then according to Sugawara formula (118) one can define an action of the Virasoro algebra with $c=1$ on $\mathcal{F}_P$.

To be more precise, in radial quantization picture our bosonic field $\varphi(z,\bar{z})$ admits the following mode expansion[10]

$$\varphi(z,\bar{z}) = -i\hat{q} - i\hat{p}\log\left(\frac{z\bar{z}}{R^2}\right) - i\sum_{k\neq 0}\left(\frac{a_k}{k}z^{-k} + \frac{\bar{a}_k}{k}\bar{z}^{-k}\right), \tag{117}$$

where the modes satisfy the commutation relations

$$[\hat{p},\hat{q}] = 1, \quad [a_m,a_n] = [\bar{a}_m,\bar{a}_n] = m\delta_{m,-n}, \quad [a_m,\bar{a}_n] = 0.$$

The absolute vacuum state $|0\rangle$ is defined as follows

$$\hat{p}|0\rangle = 0, \quad a_n|0\rangle = \bar{a}_n|0\rangle = 0 \quad \text{for} \quad n>0.$$

One can also define excited vacuum $|\alpha\rangle$

$$|\alpha\rangle \overset{\text{def}}{=} \lim_{z\to 0} :e^{i\alpha\varphi(z,\bar{z})}:|0\rangle = e^{\alpha\hat{q}}|0\rangle \implies \hat{p}|\alpha\rangle = \alpha|\alpha\rangle, \quad a_n|\alpha\rangle = \bar{a}_n|\alpha\rangle = 0 \quad \text{for} \quad n>0,$$

where the normal ordered exponent called the *verter operator* has the form

$$:e^{i\alpha\varphi(z,\bar{z})}:= e^{\alpha\hat{q}}\left(\frac{z\bar{z}}{R^2}\right)^{\alpha\hat{p}}\exp\left(-\alpha\sum_{k>0}\left(\frac{a_{-k}}{k}z^k + \frac{\bar{a}_{-k}}{k}\bar{z}^k\right)\right)\exp\left(\alpha\sum_{k>0}\left(\frac{a_k}{k}z^{-k}+\frac{\bar{a}_k}{k}\bar{z}^{-k}\right)\right),$$

[10] We note that (119) corresponds to the most general central symmetric solution of the Laplace equation.

i.e. we placed all the creation operators to the left of annihilation ones. We also define the Hermitian conjugation by $a_n^+ = a_{-n}$ and $\bar{a}_n^+ = \bar{a}_{-n}$ and hence the conjugated vacuum satisfies

$$\langle 0|a_n = \langle 0|\bar{a}_n = 0 \quad \text{for} \quad n < 0.$$

Let us compute the two-point Green function (we assume that $|z| > |w|$)

$$\begin{aligned}
\langle\varphi(z,\bar{z})\varphi(w,\bar{w})\rangle &= \langle 0|\varphi(z,\bar{z})\varphi(w,\bar{w})|0\rangle = -\langle 0|\hat{q}^2|0\rangle - \langle 0|\hat{q}\hat{p}\log\left(\frac{w\bar{w}}{R^2}\right) + \hat{p}\hat{q}\log\left(\frac{z\bar{z}}{R^2}\right)|0\rangle + \\
&+ \sum_{k>0}\frac{1}{k^2}\Big\langle 0\Big| a_k a_{-k}\left(\frac{w}{z}\right)^k + \bar{a}_k\bar{a}_{-k}\left(\frac{\bar{w}}{\bar{z}}\right)^k\Big|0\Big\rangle = -\langle 0|\hat{q}^2|0\rangle - \log\left(\frac{z\bar{z}}{R^2}\right) + \sum_{k>0}\frac{1}{k}\Big(\left(\frac{w}{z}\right)^k + \left(\frac{\bar{w}}{\bar{z}}\right)^k\Big) = \\
&= -\langle 0|\hat{q}^2|0\rangle - \log\left(\frac{z\bar{z}}{R^2}\right) - \log\left(1-\frac{w}{z}\right)\left(1-\frac{\bar{w}}{\bar{z}}\right) = -\langle 0|\hat{q}^2|0\rangle - \log\frac{|z-w|^2}{R^2}.
\end{aligned}$$

Comparing to (98) we find that the average $\langle 0|\hat{q}^2|0\rangle$ can be identified to the IR cut-off R as

$$\langle 0|\hat{q}^2|0\rangle = 0. \tag{118}$$

Now consider the product of two vertex operators (here $|z| > |w|$)

$$\begin{aligned}
:e^{i\alpha\varphi(z,\bar{z})}::e^{i\beta\varphi(w,\bar{w})}: &= e^{\alpha\hat{q}}\left(\frac{z\bar{z}}{R^2}\right)^{\alpha\hat{p}}\exp\left(-\alpha\sum_{k>0}\left(\frac{a_{-k}}{k}z^k + \frac{\bar{a}_{-k}}{k}\bar{z}^k\right)\right)\exp\left(\alpha\sum_{k>0}\left(\frac{a_k}{k}z^{-k} + \frac{\bar{a}_k}{k}\bar{z}^{-k}\right)\right)\times \\
&\times e^{\beta\hat{q}}\left(\frac{w\bar{w}}{R^2}\right)^{\beta\hat{p}}\exp\left(-\beta\sum_{k>0}\left(\frac{a_{-k}}{k}w^k + \frac{\bar{a}_{-k}}{k}\bar{w}^k\right)\right)\exp\left(\beta\sum_{k>0}\left(\frac{a_k}{k}w^{-k} + \frac{\bar{a}_k}{k}\bar{w}^{-k}\right)\right).
\end{aligned}$$

We note that this expression is not normal ordered. To make it normal ordered, one has to flip red terms with red and blue ones with blue. Using Baker-Campbell-Hausdorff formula we know that if the commutator of two operators $[A,B]$ is a c number, then one has

$$e^A e^B = e^{[A,B]}e^B e^A.$$

In our case we have

$$A = \alpha\hat{p}\log\frac{z\bar{z}}{R^2} + \alpha\sum_{k>0}\left(\frac{a_k}{k}z^{-k} + \frac{\bar{a}_k}{k}\bar{z}^{-k}\right), \quad B = \beta\hat{q} - \beta\sum_{k>0}\left(\frac{a_{-k}}{k}w^k + \frac{\bar{a}_{-k}}{k}\bar{w}^k\right)$$

and hence[11]

$$[A,B] = \alpha\beta\left(\log\left(\frac{z\bar{z}}{R^2}\right) - \underbrace{\sum_{k>0}\frac{1}{k}\left(\left(\frac{w}{z}\right)^k + \left(\frac{\bar{w}}{\bar{z}}\right)^k\right)}_{-\log\left(1-\frac{w}{z}\right)\left(1-\frac{\bar{w}}{\bar{z}}\right)}\right) = \log\left(\frac{|z-w|^2}{R^2}\right)^{\alpha\beta}. \tag{119}$$

Exponentiating this results we obtain already familiar expression (110)

$$:e^{i\alpha\varphi(z,\bar{z})}::e^{i\beta\varphi(w,\bar{w})}: = \frac{|z-w|^{2\alpha\beta}}{R^{2\alpha\beta}}:e^{i\alpha\varphi(z,\bar{z})}e^{i\beta\varphi(w,\bar{w})}:. \tag{120}$$

[11]We note that the sum in (121) converges for $|z| > |w|$.

Using (122) one can show that (here $|z_1| > |z_2| > \cdots > |z_n|$)

$$\langle 0| : e^{i\alpha_1\varphi(z_1,\bar{z_1})} : \cdots : e^{i\alpha_n\varphi(z_n,\bar{z_n})} : |0\rangle = \prod_{i<j} \frac{|z_i - z_j|^{2\alpha_i\alpha_j}}{R^{2\alpha_i\alpha_j}} \langle 0|e^{\sum \alpha_k \hat{q}}|0\rangle = \prod_{i<j} \frac{|z_i - z_j|^{2\alpha_i\alpha_j}}{R^{2\alpha_i\alpha_j}} \tag{121}$$

where in the last equality we have used (120). We note that the average in (123) is non-zero in the IR limit $R \to \infty$ only if the neutrality condition $\sum_k \alpha_k = 0$ holds. In fact it is convenient to set $R = 1$ from the very beginning holding in mind the neutrality condition.

We note that formally one can consider holomorphic bosonic field

$$\varphi(z) \overset{\text{def}}{=} -i\hat{q} - i\hat{p}\log(z) - i\sum_{k\neq 0} \frac{a_k}{k} z^{-k},$$

which is intrinsically non-local. This non-locality manifests itself as

$$\langle 0|\varphi(z)\varphi(w)|0\rangle = -\log(z-w)$$

and for $|z| > |w|$

$$: e^{i\alpha\varphi(z)} :: e^{i\beta\varphi(w)} := (z-w)^{\alpha\beta} : e^{i\alpha\varphi(z)} e^{i\beta\varphi(w)} : . \tag{122}$$

On the other hand for $|z| < |w|$ one has

$$: e^{i\beta\varphi(w)} :: e^{i\alpha\varphi(z)} := (w-z)^{\alpha\beta} : e^{i\alpha\varphi(z)} e^{i\beta\varphi(w)} := e^{i\pi\alpha\beta}(z-w)^{\alpha\beta} : e^{i\alpha\varphi(z)} e^{i\beta\varphi(w)} : . \tag{123}$$

Comparing (124) and (125), one finds that the operators $: e^{i\alpha\varphi(z)} :$ and $: e^{i\beta\varphi(w)}$ obey fractional commutation relations (for simplicity we have hidden normal ordering signs ::)

$$\mathcal{T}\left[e^{i\alpha\varphi(z)} e^{i\beta\varphi(w)}\right] = \begin{cases} e^{i\alpha\varphi(z)} e^{i\beta\varphi(w)} & \text{for} \quad |z| > |w|, \\ e^{i\pi\alpha\beta}\, e^{i\beta\varphi(w)} e^{i\alpha\varphi(z)} & \text{for} \quad |z| < |w|. \end{cases} \tag{124}$$

It is interesting to study the bosonization map (109) in Hamiltonian approach. We discuss it here for generic values of the central charge c. We will use the following parametrization (compare to (87))

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad \Delta = \Delta(\alpha) = \alpha(Q - \alpha).$$

Moreover, here (and only here) it will be convenient to generators of Heisenberg algebra

$$a_n \to a_n\sqrt{2} \implies [a_m, a_n] = \frac{m}{2}\delta_{m,-n}$$

Every state from Verma module $\mathcal{V}_\Delta$ with

$$\Delta = \alpha(Q - \alpha),$$

is mapped to the state from Fock module $\mathcal{F}_\alpha$ according to the formula

$$L_{-\boldsymbol{\lambda}}|\Delta\rangle \xrightarrow{L_n \to \frac{1}{2}\sum\limits_{k\neq 0,n} a_k a_{n-k} + \alpha a_{-n}} \sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|} C_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}(\alpha) a_{-\boldsymbol{\mu}}|\alpha\rangle \quad .$$

Problems:

1.

Lecture 8: Free fermionic CFT, boson-fermion correspondence, $\beta\gamma$ system

We consider Euclidean theory of massless Dirac fermions

$$S=\frac{1}{8\pi}\int\Psi^{+}\gamma_1\partial\!\!\!/\Psi d^2z=\frac{1}{4\pi}\int\left(\bar{\psi}^*\partial\bar{\psi}+\psi^*\bar{\partial}\psi\right)d^2z, \tag{125}$$

where

$$\gamma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad\gamma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad\Psi=\begin{pmatrix}\psi\\\bar{\psi}\end{pmatrix},\qquad\Psi^{+}=\begin{pmatrix}\psi^*&\bar{\psi}^*\end{pmatrix}$$

Classical equations of motion following from the action (127) are

$$\partial\bar{\psi}=\partial\bar{\psi}^*=0,\qquad\bar{\partial}\psi=\bar{\partial}\psi^*=0.$$

The non-trivial two-point functions are

$$\langle\psi^*(z)\psi(w)\rangle=\langle\psi(z)\psi^*(w)\rangle=\frac{1}{z-w},\qquad\langle\bar{\psi}^*(\bar{z})\bar{\psi}(\bar{w})\rangle=\langle\bar{\psi}(\bar{z})\bar{\psi}^*(\bar{w})\rangle=\frac{1}{\bar{z}-\bar{w}}$$

Since the theory (127) is Gaussian its correlation functions are computed via Wick rules. For example (here the minus sign between the two terms is due to the Grasmanian nature of the field Ψ)

$$\langle\psi^*(z_1)\psi(w_1)\psi^*(z_2)\psi(w_2)\rangle=\frac{1}{z_1-w_1}\frac{1}{z_2-w_2}-\frac{1}{z_1-w_2}\frac{1}{z_2-w_1}$$

In general, the correlation function is given by Cauchy determinant

$$\langle\psi^*(z_1)\psi(w_1)\dots\psi^*(z_n)\psi(w_n)\rangle=\det\left(\frac{1}{z_k-w_l}\right). \tag{126}$$

We will treat the theory (127) as a representation of fermionic algebra. We have two holomorphic current $\psi(z)$ and $\psi^*(z)$ (and two antiholomorphic), which satisfy the OPE

$$\psi(z)\psi^*(w)=\frac{1}{z-w}+\dots,\qquad\psi(z)\psi(w)=\text{reg},\qquad\psi^*(z)\psi^*(w)=\text{reg} \tag{127}$$

One can defined their modes as

$$\psi_r\mathcal{O}(z)\overset{\text{def}}{=}\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{r-\frac{1}{2}}\psi(\xi)\mathcal{O}(z)d\xi,\qquad\psi_r^*\mathcal{O}(z)\overset{\text{def}}{=}\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{r-\frac{1}{2}}\psi^*(\xi)\mathcal{O}(z)d\xi.$$

Then one can computes their commutation relations. The only non-trivial one is

$$\{\psi_r,\psi_s^*\}\mathcal{O}(z)\overset{\text{def}}{=}(\psi_r\psi_s^*+\psi_s^*\psi_r)\mathcal{O}(z)=\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_z}\oint_{\mathcal{C}_\xi}(\xi-z)^{r-\frac{1}{2}}(\eta-z)^{s-\frac{1}{2}}\psi^*(\eta)\psi(\xi)\mathcal{O}(z)d\xi d\eta=$$
$$=\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_z}\oint_{\mathcal{C}_\xi}(\xi-z)^{r-\frac{1}{2}}(\eta-z)^{s-\frac{1}{2}}\left(\frac{1}{\eta-\xi}+\dots\right)\mathcal{O}(z)d\xi d\eta=\delta_{r,-s}\mathcal{O}(z).$$

We have

$$\{\psi_r,\psi_s\}=\{\psi_r^*,\psi_s^*\}=0,\qquad\{\psi_r,\psi_s^*\}=\delta_{r,-s} \tag{128}$$

We note that if one requires the locality of an operator $\mathcal{O}$ with respect to currents $\psi(z)$ and $\psi^*(z)$, the indexes in (130) must be half integer $r,s\in\mathbb{Z}+\frac{1}{2}$. We call such fields Neveu-Schwarz fields (NS). One can also argue that it is natural to assume the existence of the fields wich are semi-local with respect to the currents $\psi(z)$ and $\psi^*(z)$. In this case the modes of ψ_r and ψ_r^* take integer values.

We define the fermionic Wick ordering as

$$:\psi(z)\psi^*(w):=\psi(z)\psi^*(w)-\frac{1}{z-w}.$$

Then one can define the $U(1)$ current algebra inside the fermionic algebra

$$J(z)\overset{\text{def}}{=}:\psi^*(z)\psi(z):. \tag{129}$$

Using (129) one finds

$$J(z)J(w)=\frac{1}{(z-w)^2}+\dots \tag{130}$$

which coincides with (113). Regular term in (132) can be associated with the stress-energy tensor. Explicitly, one has

$$T=\frac{1}{2}:\big(\partial\psi^*\psi-\psi^*\partial\psi\big):,$$

which defines the Virasoro algebra with the central charge $c=1$, as it should be. One can also check the following OPE's

$$T(\xi)\psi(z)=\frac{\frac{1}{2}\psi(z)}{(\xi-z)^2}+\frac{\partial\psi(z)}{\xi-z}+\dots,\qquad T(\xi)\psi^*(z)=\frac{\frac{1}{2}\psi^*(z)}{(\xi-z)^2}+\frac{\partial\psi^*(z)}{\xi-z}+\dots,$$

which means that the fields $\psi(z)$ and $\psi^*(z)$ are both primary fields with conformal dimensions $\Delta_\psi=\Delta_{\psi^*}=\frac{1}{2}$.

We have two realizations of the same $U(1)$ current algebra (112) and (131), which implies

$$i\partial\varphi=:\psi^*(z)\psi(z): \tag{131}$$

Formula (133) is known as bosonization. It terms of the modes, it reads

$$a_n=\sum_{r\in\mathbb{Z}+\frac{1}{2}}:\psi_r^*\psi_{-r+n}:$$

The bosonization formula (133) can be inverted

$$\psi(z)\bar{\psi}(z)=:e^{i\varphi(z,\bar{z})},\qquad \psi^*(z)\bar{\psi}^*(z)=:e^{-i\varphi(z,\bar{z})}$$

Actually, it will be more convenient to work in terms of holomorphic bosonic field $\varphi(z)$ and holomorphic vertex operators $:e^{i\alpha\varphi}:$ with commutation relations (126). We note that (126) implies that for $\alpha\beta\in2\mathbb{Z}$ these fields commute, while for $\alpha\beta\in\mathbb{Z}$ they anti-commute i.e. behave as fermions. We identify

$$\psi=:e^{i\varphi(z)}:,\qquad \psi^*=:e^{-i\varphi(z)}:, \tag{132}$$

which has correct OPE (129). In terms of correlation functions, the relation (134) is equivalent to Cauchy determinant identity

$$\det\left(\frac{1}{z_k-w_l}\right)=(-1)^{\frac{n(n-1)}{2}}\frac{\prod_{i<j}(z_i-z_j)(w_i-w_j)}{\prod_{k,l}(z_k-w_l)}. \tag{133}$$

We note that the bosonization map identifies more vertex operators $:e^{ik\varphi(z)}:$ with $k\in\mathbb{Z}$ with fermionic operators. In particular,

$$:e^{2i\varphi(z)}:=:\partial\psi\psi:,\ :e^{-2i\varphi(z)}:=:\partial\psi^*\psi^*:,\ :e^{3i\varphi(z)}:=\frac{1}{2}:\partial^2\psi\partial\psi\psi:,\ :e^{-3i\varphi(z)}:=\frac{1}{2}:\partial^2\psi^*\partial\psi^*\psi^*:\quad \text{etc}$$

Consider highest weight representation $\mathfrak{F}$ of the fermion algebra (130). It is defined by the highest weight state $|0\rangle$ (an image of an identity operator)

$$\psi_r|0\rangle=\psi_r^*|0\rangle=0\quad\text{for}\quad r>0.$$

Then the module $\mathfrak{F}$ is spanned by the vectors of the form

$$\psi_{-\boldsymbol{r}}\psi^*_{-\boldsymbol{s}}|0\rangle\overset{\text{def}}{=}\big(\psi_{-r_1}\psi_{-r_2}\dots\big)\big(\psi^*_{-s_1}\psi^*_{-s_2}\dots\big)|0\rangle,\tag{134}$$

where $\boldsymbol{r}=\{r_1>r_2>\dots\}$ and $\boldsymbol{s}=\{s_1>s_2>\dots\}$ are two strictly decreasing sequences.

It is convenient to think about representation $\mathfrak{F}$ in terms of particles/holes and Dirac sees. Namely, we introduce absolute vacua state $|\varnothing\rangle$ by

$$\psi_r^*|\varnothing\rangle=0\quad\text{for all}\quad r.$$

Then the vacua state $|0\rangle$ corresponds to the semi-infinite product state

$$|0\rangle=\psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\psi_{\frac{5}{2}}\dots|\varnothing\rangle.$$

It can be interpreted as follows. We have an infinite line $\mathbb{R}$, where $\mathbb{R}^+$ is all filled by particles at positions $\frac{1}{2}$, $\frac{3}{2}$ etc and $\mathbb{R}^-$ is empty (or filled by holes). Then ψ_{-r} creates a particle at position $-r$ and ψ^*_{-s} deletes a particle at position s (creates a hole at position s). The corresponding sequence of particles and holes is usually referred as Maya diagram. We also define dual absolute vacua state $\langle\varnothing|$ by

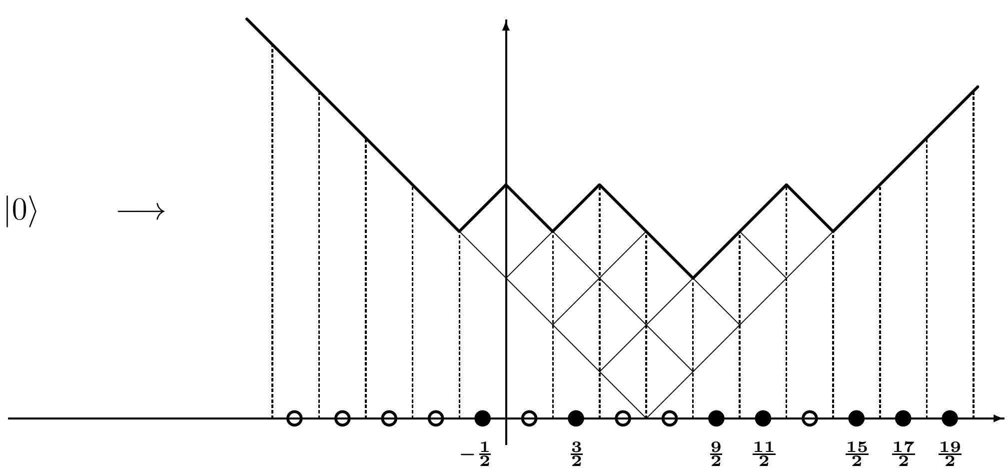
$$\langle\varnothing|\psi_r=0\quad\text{for all}\quad r\implies\langle0|=\langle\varnothing|\dots\psi^*_{\frac{5}{2}}\psi^*_{\frac{3}{2}}\psi^*_{\frac{1}{2}}.$$

Then the state conjugated to (136) can be represented as

$$\langle0|\psi_{\boldsymbol{s}}\psi^*_{\boldsymbol{r}}\overset{\text{def}}{=}\langle0|\big(\dots\psi_{s_2}\psi_{s_1}\big)\big(\dots\psi^*_{r_2}\psi^*_{r_1}\big).$$

There is a nice bijection between Maya diagrams and charged partitions, which can be explained by the following picture

$$\psi_{-\frac{1}{2}}\psi^*_{-\frac{1}{2}}\psi^*_{-\frac{5}{2}}\psi^*_{-\frac{7}{2}}\psi^*_{-\frac{13}{2}}|0\rangle\qquad\longrightarrow$$



where the "charge" of Maya diagram is a distance between the origin and the bottom corner of the Young diagram. Any Maya diagram has its charge given by an eigenvalue of the operator

$$\hat{c} = a_0 = \sum_{r\in\mathbb{Z}+\frac{1}{2}} :\psi_r^*\psi_{-r}: \Longrightarrow [\hat{c},\psi_r] = -\psi_r, \qquad [\hat{c},\psi_r^*] = -\psi_r^*,$$

and an energy given by an eigenvalue of the operator

$$L_0 = \sum_{r\in\mathbb{Z}+\frac{1}{2}} r:\psi_{-r}\psi_r^*: \Longrightarrow [L_0,\psi_r] = -r\psi_r, \qquad [L_0,\psi_r^*] = -r\psi_r^*.$$

It is interesting to compute the character of the fermionic module $\mathfrak{F}$ (the partition function)

$$Z(q,t) \stackrel{\text{def}}{=} \operatorname{tr}\left(q^{L_0}t^{\hat{c}}\right)\Big|_{\mathfrak{F}}.$$

According to boson/fermion correspondence, there are two ways to compute this character.

1. Bosonic way: at any value of c we have the bosonic Fock module $\mathcal{F}_c$, which implies the character formula
$$Z(q,t) = \sum_{c=-\infty}^{\infty} t^c \sum_{\boldsymbol{\lambda}} q^{\frac{c^2}{2}+|\boldsymbol{\lambda}|} = \left(\prod_{k=1}^{\infty}\frac{1}{1-q^k}\right)\sum_{c=-\infty}^{\infty} q^{\frac{c^2}{2}}t^c. \tag{135}$$
is the Jacobi theta function.

2. Fermionic way. At position $(k+\frac{1}{2})$ for $k\geq 0$ we have two options: a hole with weight 1 and a particle with weight $t^{-1}q^{k+\frac{1}{2}}$. Similarly, at position $-(k+\frac{1}{2})$ for $k\geq 0$ we could have a hole with the weight $tq^{k+\frac{1}{2}}$ and a particle with the weight 1. Since all the positions are independent we have for partition function
$$Z(q,t) = \prod_{k=1}^{\infty}\left(1+tq^{k-\frac{1}{2}}\right)\left(1+t^{-1}q^{k-\frac{1}{2}}\right). \tag{136}$$

Comparing (137) and (138) we arrive to the Jacobi triple product identity

$$\prod_{k=1}^{\infty}\left(1+tq^{k-\frac{1}{2}}\right)\left(1+t^{-1}q^{k-\frac{1}{2}}\right)(1-q^k) = \sum_{c=-\infty}^{\infty} q^{\frac{c^2}{2}}t^c. \tag{137}$$

Another system, that we consider is the $\beta-\gamma$ system

$$S = \frac{1}{4\pi}\int\left(\bar{\beta}\partial\bar{\gamma}+\beta\bar{\partial}\gamma\right)d^2z,$$

where now, compared to (127), fundamental fields $(\beta,\gamma,\bar{\beta},\bar{\gamma})$ are considered as bosonic variables in the path integral formalism. We have two holomorphic current $\beta(z)$ and $\gamma(z)$, which satisfy the OPE

$$\gamma(z)\beta(w) = \frac{1}{z-w}+\ldots, \qquad \beta(z)\beta(w) = \text{reg}, \qquad \gamma(z)\gamma(w) = \text{reg} \tag{138}$$

One can defined their modes as (where β and γ have conformal weights 1 and 0 respectively)

$$\beta_r\mathcal{O}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^r\beta(\xi)\mathcal{O}(z)d\xi, \qquad \gamma_r\mathcal{O}(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{r-1}\gamma(\xi)\mathcal{O}(z)d\xi.$$

Then one can compute the commutation relations

$$[\beta_r, \beta_s] = [\gamma_r, \gamma_s] = 0, \qquad [\beta_r, \gamma_s] = \delta_{r,-s}$$

For the choice of weights 1 and 0 the stress-energy for $\beta\gamma$ system has the form

$$T = \beta \partial \gamma$$

and has the central charge 2.

Correlation functions of $\beta\gamma$ fields are computed by the Wick rules

$$\langle \beta(z_1)\gamma(w_1)\dots\beta(z_n)\gamma(w_n)\rangle = \frac{1}{z_1 - w_1}\frac{1}{z_2 - w_2}\dots\frac{1}{z_n - w_n} + \text{Perms} = \text{perm}\left(\frac{1}{z_i - w_j}\right) \tag{139}$$

We note that compared to fermionic case (128) there are no signs in (142) and this correlation function can not be written as a determinant but rather as permanent.

One can try to find a representation for the algebra (141) similar to the one in fermionic case (134). A naive attempt $\beta \sim e^{i\alpha\varphi}$, $\gamma \sim e^{i\beta\varphi}$ would fail since the $\beta - \gamma$ fields are bosonic and in particular

$$\gamma(z)\gamma(w) = \gamma^2(w) + \dots$$

This field can be bosonized by two holomorphic bosonic fields u and v as

$$\langle u(z)u(w)\rangle = \langle v(z)v(w)\rangle = -\log(z - w)$$

as (we note that here normal ordering is not needed)

$$\gamma = e^{-u-iv}.$$

Then the β field should be a level one descendant of e^{u+iv}. This follows from charge conservation and dimensional arguments. We have

$$\beta =: (\lambda_1 \partial u + \lambda_2 \partial v) e^{u+iv} : .$$

Computing the OPE one has

$$\beta(z)\beta(w) = -\frac{2\lambda_1(\lambda_1 + i\lambda_2)}{(z-w)^2} e^{u(z)+u(w)+iv(z)+iv(w)} + \text{reg}, \qquad \beta(z)\gamma(w) = \frac{\lambda_1 + i\lambda_2}{z - w} + \text{reg}$$

i.e. we have $\lambda_1 = 0$, $\lambda_2 = i$, which leads to Friedan, Matrinec and Shenker bosonization of $\beta\gamma$ system [2]

$$\beta = i : \partial v e^{u+iv} :, \qquad \gamma = e^{-u-iv}. \tag{140}$$

Using the bosonization formula (143) one can compute correlation functions. The result should be the same as the one coming from $\beta\gamma-$system Wick rules

$$\begin{aligned} &\langle \beta(z_1)\gamma(w_1)\dots\beta(z_n)\gamma(w_n)\rangle = \\ &\quad = \langle e^{u(z_1)}\dots e^{u(z_n)} e^{-u(w_1)}\dots e^{-u(w_n)}\rangle\, \partial_{z_1}\dots\partial_{z_n} \langle e^{iv(z_1)}\dots e^{iv(z_n)} e^{-iv(w_1)}\dots e^{-iv(w_n)}\rangle = \\ &\qquad = \frac{\prod(z_i - w_j)}{\prod(z_i - z_j)(w_i - w_j)}\, \partial_{z_1}\dots\partial_{z_n} \frac{\prod(z_i - z_j)(w_i - w_j)}{\prod(z_i - w_j)}. \end{aligned} \tag{141}$$

The equality of both representations (142) and (144) can be viewed as a bosonic version of the Cauchy determinant identity (135). It is basically equivalent to Borchardt's identity

$$\det\left(\frac{1}{z_i - w_j}\right) \operatorname{perm}\left(\frac{1}{z_i - w_j}\right) = \det\left(\frac{1}{(z_i - w_j)^2}\right).$$

Problems:

1. Consider generalized stress-energy tensor

$$T(z) = \lambda_1 : \partial\psi^*\psi : + \lambda_2 : \partial\psi\psi^* : .$$

   Find the conformal dimensions of ψ and ψ^* under this $T(z)$. Compute the central charge.

Lecture 9: Operator algebra in CFT, conformal blocks

We introduce the notion of operator algebra. Suppose we have a theory and a complete set of fields $\{\mathcal{O}\} = \{\mathcal{O}_1, \mathcal{O}_2, \dots\}$. Inspired by the intuition learned from free field CFT, we formulate the hypothesis of the operator algebra. Namely, we assume that the product of fields can be expanded in neighboring points

$$\mathcal{O}_i(\boldsymbol{x})\mathcal{O}_j(\boldsymbol{y}) = \sum_k C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})\mathcal{O}_k(\boldsymbol{y}) \quad \text{at} \quad \boldsymbol{x} \to \boldsymbol{y}, \tag{142}$$

with $C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})$ known as structure constants of operator algebra, which are the functions depending only on difference of points (due to translation invariance). As we already saw many times, the relation (145) should be understood as a series of relations on correlation functions

$$\langle\mathcal{O}_i(\boldsymbol{x})\mathcal{O}_j(\boldsymbol{y})X\rangle = \sum_k C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})\langle\mathcal{O}_k(\boldsymbol{y})X\rangle \quad \text{where} \quad X = \mathcal{O}_{j_1}(\boldsymbol{x}_1)\dots\mathcal{O}_{j_n}(\boldsymbol{x}_n).$$

By performing OPE, any N-point correlation can be reduced to the sum of two-point ones, which are universal in CFT and considered as known quantities. The set of structure constants satisfy the condition of associativity

$$\sum_\sigma C_{i_1 i_2}^{i_\sigma}(\boldsymbol{x}_1-\boldsymbol{x}_2)C_{i_\sigma i_3}^{i_4}(\boldsymbol{x}_2-\boldsymbol{x}_3) = \sum_\tau C_{i_1 i_\tau}^{i_4}(\boldsymbol{x}_1-\boldsymbol{x}_3)C_{i_2 i_3}^{i_\tau}(\boldsymbol{x}_2-\boldsymbol{x}_3), \tag{143}$$

also known as *bootstrap* equations in CFT. It can be thought as an infinite system of quadratic equations. In general, this is the task which is hardly believed to be accomplished. However, as we will see in 2D CFT the conformal symmetry puts strong constraints of the coefficients $C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})$. So strong, that in some cases (146) can be solved.

Let us consider 2D CFT. As we learned it is enough to study the correlation functions of primary fields. Consider the OPE of two primary fields

$$\Phi_1(z,\bar{z})\Phi_2(w,\bar{w}) = \sum_{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}} C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(z-w)^{\Delta_k-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}(\bar{z}-\bar{w})^{\bar{\Delta}_k-\bar{\Delta}_1-\bar{\Delta}_2+|\bar{\boldsymbol{\lambda}}|}\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w}), \tag{144}$$

here $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ are the partitions

$$\boldsymbol{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \dots\}, \qquad \bar{\boldsymbol{\lambda}} = \{\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \dots\}, \quad |\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \dots, \quad |\bar{\boldsymbol{\lambda}}| = \bar{\lambda_1} + \bar{\lambda_2} + \dots$$

and

$$\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w}) \overset{\text{def}}{=} (L_{-\lambda_1}L_{-\lambda_2}\dots)\left(\bar{L}_{-\bar{\lambda}_1}\bar{L}_{-\bar{\lambda}_2}\dots\right)\Phi_k(w,\bar{w})$$

is a descendant of the primary field $\Phi_k(w,\bar{w})$. The coordinate dependence of the OPE (147) is completely fixed by scaling properties. The coordinate independent coefficients $C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}$ is what is remained to be determined. In fact there are infinitely many constraints on them following from $2D$ conformal invariance.

Let us "act" on (147) by

$$\frac{1}{2\pi i}\oint_{\mathcal{C}}(\zeta-w)^{n+1}T(\zeta)d\zeta \quad n>0,$$

where the contour $\mathcal{C}$ encircles both z and w in counterclockwise direction: $\mathcal{C} = \mathcal{C}_z + \mathcal{C}_w$. On the left hand side of (147) it acts only on Φ_1

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\zeta - w)^{n+1}T(\zeta)\Phi_1(z)d\zeta = \frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\zeta - w)^{n+1}\left(\frac{\Delta_1\Phi_1(z)}{(\zeta - z)^2} + \frac{\partial_z\Phi_1(z)}{\zeta - z} + \dots\right)d\zeta = \mathcal{L}_n\Phi_1(z), \quad (145)$$

where

$$\mathcal{L}_n = \Big((z-w)^{n+1}\partial_z + (n+1)\Delta_1(z-w)^n\Big).$$

While on the right hand side it acts simply by

$$\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w}) \to L_n\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w}) = \sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|-n}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}}\Phi_k^{\boldsymbol{\mu},\bar{\boldsymbol{\lambda}}}(w,\bar{w}). \quad (146)$$

Applying (148) and (149) (and similar antiholomorphic equations) to the OPE (147) one finds

$$\begin{aligned} C_{12}^{k,\boldsymbol{\mu},\bar{\boldsymbol{\lambda}}}\left(\Delta_k + n\Delta_1 - \Delta_2 + |\boldsymbol{\mu}|\right) &= \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|+n} C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}}, \\ C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\mu}}}\left(\bar{\Delta}_k + n\bar{\Delta}_1 - \bar{\Delta}_2 + |\bar{\boldsymbol{\mu}}|\right) &= \sum_{|\bar{\boldsymbol{\lambda}}|=|\bar{\boldsymbol{\mu}}|+n} C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\mu}}}\bar{\Lambda}_{\bar{\boldsymbol{\mu}}}^{\bar{\boldsymbol{\lambda}}} \end{aligned} \quad (147)$$

These relations are enough to find them uniquely (for generic values of the parameters). We note that because of commutation relations (here $m, n > 0$)

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [\mathcal{L}_m, \mathcal{L}_n] \overset{!}{=} -(m-n)\mathcal{L}_{m+n},$$

it is enough to impose (150) for $n = 1$ and $n = 2$ and the rest will follow.

We note that the partitions $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ enter (150) completely independent. It is clear that the solution can be represented in the form

$$C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}} = C_{12}^k\beta_{\boldsymbol{\lambda}}\beta_{\bar{\boldsymbol{\lambda}}} \quad \text{where by definition} \quad \beta_{\varnothing} = 1.$$

Here C_{12}^k is the structure constant which gives the contribution of the primary field Φ_k in the OPE of Φ_1 with Φ_2. The constants $\beta_{\boldsymbol{\lambda}}$ encode the relative contribution of the descendant fields. The structure constant C_{12}^k factors out of (150) and we have

$$\begin{aligned} \beta_{\boldsymbol{\mu}}\left(\Delta_k + n\Delta_1 - \Delta_2 + |\boldsymbol{\mu}|\right) &= \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|+n}\beta_{\boldsymbol{\lambda}}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}}, \qquad \beta_{\varnothing} = 1, \\ L_n\Phi_k^{\boldsymbol{\lambda}} &= \sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|-n}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}}\Phi_k^{\boldsymbol{\mu}}. \end{aligned} \quad (148)$$

As we already mentioned, it is enough to consider (151) for $n = 1$ and $n = 2$ only. It is convenient to imagine, that we are computing the following function

$$\begin{aligned} \boldsymbol{\Phi}_k(w) \overset{\text{def}}{=} \Phi_k(w) + (z-w)\beta_{\square}L_{-1}\Phi(w) + (z-w)^2\Big(\beta_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}L_{-1}^2\Phi_k(w) + \beta_{\square\square}L_{-2}\Phi_k(w)\Big) + \\ + (z-w)^2\left(\beta_{\square\square\square}L_{-1}^3\Phi_k(w) + \beta_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}L_{-2}L_{-1}\Phi_k(w) + \beta_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}L_{-3}\Phi_k(w)\right) + \dots \quad (149) \end{aligned}$$

All the coefficients $\beta_{\boldsymbol{\lambda}}$ in (152) are computed recursively by (151). Consider first examples

Level 1**:**

$$(\Delta_k + \Delta_1 - \Delta_2) = 2\Delta_k \beta_{\square} \implies \beta_{\square} = \frac{\Delta_k + \Delta_1 - \Delta_2}{2\Delta_k}, \tag{150}$$

provided that $\Delta_k \neq 0$, which we assume.

Level 2**:** We have two states

$$(\beta_{\substack{\square\\\square}} L_{-1}^2 \Phi_k + \beta_{\square\square} L_{-2} \Phi_k)$$

From level $\varnothing$ with $n = 2$ we obtain

$$(\Delta_k + 2\Delta_1 - \Delta_2) = \beta_{\substack{\square\\\square}} \Lambda_{\varnothing}^{\substack{\square\\\square}} + \beta_{\square\square} \Lambda_{\varnothing}^{\square\square} \quad \text{where} \quad \Lambda_{\varnothing}^{\substack{\square\\\square}} = 6\Delta_k, \quad \Lambda_{\varnothing}^{\square\square} = 4\Delta_k + \frac{c}{2}.$$

From level 1 with $n = 1$ we obtain

$$\beta_{\square}(\Delta_k + \Delta_1 - \Delta_2 + 1) = \beta_{\substack{\square\\\square}} \Lambda_{\square}^{\substack{\square\\\square}} + \beta_{\square\square} \Lambda_{\square}^{\square\square} \quad \text{where} \quad \Lambda_{\square}^{\substack{\square\\\square}} = 2(2\Delta_k + 1), \quad \Lambda_{\square}^{\square\square} = 3.$$

Altogether we have a system of equations

$$\begin{aligned} \frac{(\Delta_k + \Delta_1 - \Delta_2)(\Delta_k + \Delta_1 - \Delta_2 + 1)}{2\Delta_k} &= 2(2\Delta_k + 1)\beta_{\substack{\square\\\square}} + 3\beta_{\square\square} \\ (\Delta_k + 2\Delta_1 - \Delta_2) &= 6\Delta_k \beta_{\substack{\square\\\square}} + \left(4\Delta_k + \frac{c}{2}\right) \beta_{\square\square}. \end{aligned}$$

This system is non-degenerate, provided that the determinant

$$\det \begin{pmatrix} 2(2\Delta_k + 1) & 3 \\ 6\Delta_k & (4\Delta_k + \frac{c}{2}) \end{pmatrix} = 2\left(8\Delta_k^2 + (c - 5)\Delta_k + \frac{c}{2}\right)$$

does not vanish. We note, that the determinant actually vanishes at the values

$$\Delta_k = \frac{5 - c \pm \sqrt{(c-1)(c-25)}}{16},$$

which are exactly the values of conformal dimensions of the degenerate fields $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$. Similar phenomenon holds at level 1: the coefficient $\beta_{\square}$ has a pole at $\Delta = 0$, i.e. at $\Delta = \Delta_{1,1}$ (see (153)).

In general, at level N we have $p(N)$ constants $\beta_{\boldsymbol{\lambda}}$ with $|\boldsymbol{\lambda}| = N$ subject to $p(N-1) + p(N-2)$ relations, provided that the coefficients $\beta_{\boldsymbol{\mu}}$ with $|\boldsymbol{\mu}| = N - 1$ and $|\boldsymbol{\mu}| = N - 2$ are known. In fact

$$p(N) < p(N-1) + p(N-2), \tag{151}$$

so we have an over determined system of equations and it looks puzzled that we have a solution. The resolution of this puzzle is hidden in the fact that the equations followed from (151) for $n = 1$ and $n = 2$ are not all algebraically independent. First example of an algebraic relation is

$$[L_1, [L_1, [L_1, L_2]]] + 6[L_2, [L_1, L_2]] \equiv 0,$$

and hence we have to correct (154) by subtracting an auxiliary term

$$p(N) < p(N-1) + p(N-2) - p(N-5). \tag{152}$$

In fact, there are more algebraic relations. If we take them all into account, we will correct the inequalities (154), (155) to equality known as pentagonal number identity

$$p(N) = p(N-1) + p(N-2) - p(N-5) - p(N-7) + p(N-12) + p(N-15) + \dots. \tag{153}$$

It follows from the identity

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots \tag{154}$$

Indeed from (199) one has

$$1 = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)}\prod_{k=1}^{\infty}(1-q^k) = \left(\sum_{N=0}^{\infty} p(N) q^N\right)\left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots\right),$$

which implies (156). We note that the pentagonal number identity is a special case of Jacobi triple identity (139) with

$$q \to q^3, \qquad t \to -q^{\frac{1}{2}}.$$

The calculation above can be formalized with the help of a dual "basis". Consider a generic descendant

$$\Phi_k^{\boldsymbol{\lambda}} = L_{-\lambda_1} L_{-\lambda_2} \dots \Phi_k.$$

Suppose we found a generator $\chi_{\boldsymbol{\lambda}} \in \mathrm{Vir}_+$

$$\chi_{\boldsymbol{\lambda}} = \sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|} a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} L_{\boldsymbol{\mu}} \quad \text{where} \quad L_{\boldsymbol{\mu}} \overset{\text{def}}{=} L_{\mu_1} L_{\mu_2} \dots,$$

such that

$$\chi_{\boldsymbol{\lambda}} \Phi_k^{\boldsymbol{\lambda}} = \Phi_k, \qquad \chi_{\boldsymbol{\lambda}} \Phi_k^{\boldsymbol{\lambda}'} = 0 \quad \text{for all} \quad \boldsymbol{\lambda}' \neq \boldsymbol{\lambda} \quad |\boldsymbol{\lambda}'| = |\boldsymbol{\lambda}|.$$

We note that the matrix $a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}$ is nothing else, but the inverse Gram matrix

$$a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} = \left(\Gamma^{-1}\right)_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \quad \text{where} \quad \Gamma_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} = \frac{\langle\Delta|L_{\boldsymbol{\mu}} L_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta|\Delta\rangle}.$$

The examples of the vectors from dual basis are

$$\chi_{\square} = \frac{1}{2\Delta} L_1,$$

$$\chi_{\substack{\square\\\square}} = \frac{8\Delta_k + c}{8\Delta_k\left(8\Delta_k^2 + (c-5)\Delta_k + \frac{c}{2}\right)} L_{-1}^2 - \frac{3}{2\left(8\Delta_k^2 + (c-5)\Delta_k + \frac{c}{2}\right)} L_{-2},$$

$$\chi_{\square\square} = -\frac{3}{2\left(8\Delta_k^2 + (c-5)\Delta_k + \frac{c}{2}\right)} L_{-1}^2 + \frac{2\Delta_k + 1}{8\Delta_k^2 + (c-5)\Delta_k + \frac{c}{2}} L_{-2},$$

etc.

Given the dual basis constructed, it is easy to show that

$$\beta_{\boldsymbol{\lambda}} = \frac{1}{(z-w)^{\Delta_k - \Delta_1 - \Delta_2 + |\boldsymbol{\lambda}|}} \sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|} a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \mathcal{L}_{\boldsymbol{\mu}} \cdot (z-w)^{\Delta_k - \Delta_1 - \Delta_2} \quad \text{where} \quad \mathcal{L}_{\boldsymbol{\mu}} \overset{\text{def}}{=} \dots \mathcal{L}_{\mu_2} \mathcal{L}_{\mu_1}.$$

We note that there is another in a sense more transparent way to compute the same expansion. Namely, we can rearrange the states by derivatives of quasiprimary fields

$$\mathbf{\Phi}_k(w) = \Phi_k(w) + (z-w)\rho_1 L_{-1}\Phi_k + (z-w)^2\left(\rho_2 L_{-1}^2 + \nu_0\left(L_{-2} + \frac{3}{2(2\Delta+1)}L_{-1}^2\right)\right)\Phi_k + \dots \quad (155)$$

By definition, L_1 kills quasiprimary fields. It is clear, that acting by L_1 only one stays within given quasiprimary family. For example, for coefficients ρ_k in

$$\Phi_k(w) + (z-w)\rho_1 L_{-1}\Phi_k + (z-w)^2\rho_2 L_{-1}^2\Phi_k(w) + (z-w)^3\rho_3 L_{-1}^3\Phi_k(w) + \dots,$$

we have a recursive system

$$\begin{aligned}
2\Delta_k\rho_1 &= (\Delta_k + \Delta_1 - \Delta_2),\\
2(2\Delta_k+1)\rho_2 &= (\Delta_k + \Delta_1 - \Delta_2 + 1)\rho_1,\\
3(2\Delta_k+2)\rho_3 &= (\Delta_k + \Delta_1 - \Delta_2 + 2)\rho_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

which can be explicitly solved

$$\rho_N = \frac{1}{N!}\prod_{j=1}^{N}\frac{\Delta_k + \Delta_1 - \Delta_2 + j - 1}{2\Delta_k + j - 1}$$

We can proceed further and collect the derivatives of the next quasiprimary field in (158)

$$(z-w)^2\left(\nu_0\left(L_{-2} + \frac{3}{2(2\Delta+1)}L_{-1}^2\right)\Phi_k + (z-w)\nu_1 L_{-1}\left(L_{-2} + \frac{3}{2(2\Delta+1)}L_{-1}^2\right)\Phi_k + \dots\right).$$

Clearly, we have

$$\nu_N = \frac{\nu_0}{N!}\prod_{j=1}^{N}\frac{\Delta_k + 2 + \Delta_1 - \Delta_2 + j - 1}{2(\Delta_k+2) + j - 1}.$$

The coefficient ν_0 can not be determined from commutation relations with L_1 only, since the quasiprimary state $L_{-2} + \frac{3}{2(2\Delta_k+1)}L_{-1}^2$ belongs to its kernel. One has to use L_2 as well. In fact, we know that

$$\nu_0 = \beta_{\square\square},$$

which was found before.

We note here a strange singularities of the coefficients ρ_k of the form

$$\rho_2 \sim \frac{1}{2\Delta_k + 1}.$$

This fake singularity is cancelled by the term $\frac{3}{2(2\Delta_k+1)}L_{-1}^2$ in

$$\left(L_{-2} + \frac{3}{2(2\Delta_k+1)}L_{-1}^2\right)\Phi_k.$$

It can be shown that the only singularities of $\beta_{\boldsymbol{\lambda}}$'s are located at the values $\Delta = \Delta_{m,n}$.

Applying the OPE (147) to the 4-point correlation function, we obtain

$$\langle\Phi_1(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)\Phi_3(z_3,\bar z_3)\Phi_4(z_4,\bar z_4)\rangle=\sum_k C_{12}^k|z_1-z_2|^{2(\Delta_k-\Delta_1-\Delta_2)}\times$$
$$\times\sum_{\boldsymbol\lambda,\bar{\boldsymbol\lambda}}\beta_{\boldsymbol\lambda}\beta_{\bar{\boldsymbol\lambda}}(z_1-z_2)^{|\boldsymbol\lambda|}(\bar z_1-\bar z_2)^{|\bar{\boldsymbol\lambda}|}\langle\Big((L_{-\lambda_1}L_{-\lambda_2}\ldots)(\bar L_{-\bar\lambda_1}\bar L_{-\bar\lambda_2}\ldots)\Phi_k(z_2,\bar z_2)\Big)\Phi_3(z_3,\bar z_3)\Phi_4(z_4,\bar z_4)\rangle=$$
$$=\sum_k C_{12}^k C_{k34}\left|\sum_{\boldsymbol\lambda}(z_1-z_2)^{\Delta_k-\Delta_1-\Delta_2+|\boldsymbol\lambda|}\beta_{\boldsymbol\lambda}\Big(\hat{\mathcal L}_{-\boldsymbol\lambda}\cdot(z_2-z_3)^{\gamma_{23}}(z_2-z_4)^{\gamma_{24}}(z_3-z_4)^{\gamma_{34}}\Big)\right|^2=$$
$$=\sum_k C_{12}^k C_{k34}\left|\sum_{\boldsymbol\lambda,\boldsymbol\mu:|\boldsymbol\lambda|=|\boldsymbol\mu|}\left(\Gamma^{-1}\right)_{\boldsymbol\lambda}^{\boldsymbol\mu}\Big(\mathcal L_{\boldsymbol\mu}\cdot(z_1-z_2)^{\Delta_k-\Delta_1-\Delta_2}\Big)\Big(\hat{\mathcal L}_{-\boldsymbol\lambda}\cdot(z_2-z_3)^{\gamma_{23}}(z_2-z_4)^{\gamma_{24}}(z_3-z_4)^{\gamma_{34}}\Big)\right|^2,$$

where $\gamma_{23}=\Delta_4-\Delta_k-\Delta_3$, $\gamma_{24}=\Delta_3-\Delta_k-\Delta_4$ and $\gamma_{34}=\Delta_k-\Delta_3-\Delta_4$ and

$$\hat{\mathcal L}_{-\boldsymbol\lambda}\stackrel{\text{def}}{=}\hat{\mathcal L}_{-\lambda_1}\hat{\mathcal L}_{-\lambda_2}\ldots,\quad \hat{\mathcal L}_{-n}=\sum_{j=3,4}\left(\frac{(n-1)\Delta_j}{(z_j-z_2)^n}-\frac{\partial_j}{(z_j-z_2)^{n-1}}\right).$$

In the third line we used the explicit form of the three-point function. We see that the 4-point function has split into a sum of modulus squared of holomorphic functions

$$\langle\Phi_1(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)\Phi_3(z_3,\bar z_3)\Phi_4(z_4,\bar z_4)\rangle=\sum_k C_{12}^k C_{k34}\Big|\mathcal F_{\Delta_k}\left(\Delta_1,\Delta_2,\Delta_3,\Delta_4|z_1,z_2,z_3,z_4\right)\Big|^2,$$

where

$$\mathcal F_{\Delta_k}\left(\Delta_1,\Delta_2,\Delta_3,\Delta_4|z_1,z_2,z_3,z_4\right)\stackrel{\text{def}}{=}\sum_{\boldsymbol\lambda,\boldsymbol\mu:|\boldsymbol\lambda|=|\boldsymbol\mu|}a_{\boldsymbol\lambda}^{\boldsymbol\mu}\Big(\mathcal L_{\boldsymbol\mu}\cdot(z_1-z_2)^{\Delta_k-\Delta_1-\Delta_2}\Big)\times$$
$$\times\Big(\hat{\mathcal L}_{-\boldsymbol\lambda}\cdot(z_2-z_3)^{\Delta_4-\Delta_k-\Delta_3}(z_2-z_4)^{\Delta_3-\Delta_k-\Delta_4}(z_3-z_4)^{\gamma_{34}}\Big)\quad(156)$$

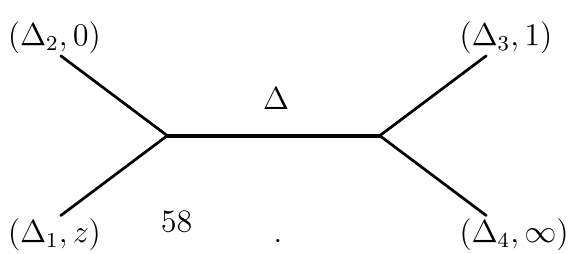
is known as a conformal block. It sums explicitly the contribution of entire conformal family. And this sum is universal in a sense that it does not depend on dynamics of the theory.

In fact, it is rather inconvenient to work with the definition (159). The problem is with the action of the operator $\hat{\mathcal L}_{-n}$ which produces many terms inconvenient for "logarithmization". The computation can be facilitated by remembering the projective invariance of correlation functions. Namely, the following formula (73)

$$\langle\Phi_1(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)\Phi_3(z_3,\bar z_3)\Phi_4(z_4,\bar z_4)\rangle=$$
$$=|z_1-z_4|^{-2\Delta_1}|z_2-z_3|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}|z_2-z_4|^{2(\Delta_1+\Delta_3-\Delta_2-\Delta_4)}|z_3-z_4|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}\times$$
$$\times\lim_{\zeta\to\infty}\zeta^{2\Delta_4}\langle\Phi_1(z,\bar z)\Phi_2(0)\Phi_3(1)\Phi_4(\zeta,\bar\zeta)\rangle\quad\text{where}\quad z=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.$$

It means that it is enough to find $z_4\to\infty$ limit of the conformal block

$$\mathfrak F_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right)=$$



.

which we define as follows

$$\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right) = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} a^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}} \Big(\mathcal{L}_{\boldsymbol{\mu}}\cdot z^{\Delta-\Delta_1-\Delta_2}\Big)\Big(\hat{\mathcal{L}}_{-\boldsymbol{\lambda}}\cdot x^{\Delta_4-\Delta-\Delta_3}\Big)\Bigg|_{x=1},$$

where

$$\mathcal{L}_{\boldsymbol{\mu}} = \ldots \mathcal{L}_{\mu_2}\mathcal{L}_{\mu_1}, \quad \mathcal{L}_n = z^{n+1}\partial_z + (n+1)\Delta_1 z^n,$$
$$\hat{\mathcal{L}}_{-\boldsymbol{\lambda}} = (-\mathcal{L}_{-\lambda_1})(-\mathcal{L}_{-\lambda_2})\ldots, \quad \mathcal{L}_{-n} = x^{-n+1}\partial_x + (-n+1)\Delta_3 x^{-n}.$$

All this can be formalized as follows. We introduce the matrix element

$$\langle\Delta'|L_{\boldsymbol{\mu}}\Phi_k(z)L_{-\boldsymbol{\lambda}}|\Delta\rangle \overset{\text{def}}{=} \lim_{\zeta\to\infty}|\zeta|^{2\Delta'}\langle\Phi^{\boldsymbol{\lambda}}_\Delta(0)\Phi_{\Delta_k}(z,\bar z)\Phi^{\boldsymbol{\mu}}_{\Delta'}(\zeta,\bar\zeta)\rangle,$$
$$L_{-\boldsymbol{\lambda}} = L_{-\lambda_1}L_{-\lambda_2}\ldots, \quad L_{\boldsymbol{\mu}} = \ldots L_{\mu_2}L_{\mu_1}.$$

This matrix element is computed with the help of Virasoro commutation relations and

$$[L_n,\Phi_k(z)] = \left(z^{n+1}\partial_z + \Delta_k(n+1)z^n\right)\Phi_k(z) = \mathcal{L}_n\cdot\Phi_k(z). \tag{157}$$

Note that

$$[L_m,[L_n,\Phi_k(z)]] \overset{!}{=} \mathcal{L}_n\cdot\mathcal{L}_m\cdot\Phi_k(z), \qquad \Phi_k(z)\sim z^{\Delta'-\Delta-\Delta_k}.$$

Using these commutation relations, one can compute any matrix element

$$\frac{\langle\Delta'|L_{\boldsymbol{\mu}}\Phi_k L_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta'|\Phi_k|\Delta\rangle} \overset{\text{def}}{=} \lim_{z\to1}\frac{\langle\Delta'|L_{\boldsymbol{\mu}}\Phi_k(z)L_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta'|\Phi_k(z)|\Delta\rangle},$$

which is some polynomial in Δ, Δ' and Δ_k.

In these terms the conformal block is given by

$$\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right) = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} z^{\Delta-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}\left(\Gamma^{-1}\right)^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}}\frac{\langle\Delta_4|\Phi_3 L_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta_4|\Phi_3|\Delta\rangle}\frac{\langle\Delta|L_{\boldsymbol{\mu}}\Phi_1|\Delta_2\rangle}{\langle\Delta|\Phi_1|\Delta_2\rangle}. \tag{158}$$

Explicitly, we have

$$\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right) = z^{\Delta-\Delta_1-\Delta_2}\left(1+\frac{(\Delta+\Delta_3-\Delta_4)(\Delta+\Delta_1-\Delta_2)}{2\Delta}z+\ldots\right)$$

In Hamiltonian language the expansion (161) can be viewed as an "insertion of a complete set of states"

$$\mathbf{1}\text{ " = " }\sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} z^{\Delta_k-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}\left(\Gamma^{-1}\right)^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}}\Big(L_{-\boldsymbol{\lambda}}|\Delta\rangle\langle\Delta|L_{\boldsymbol{\mu}}\Big),$$

and the z dependence in this formula is due to the fact that operators are taken at different time slices.

Using the conformal block decomposition, one can rewrite the associativity condition (146) as

$$\sum_k C^k_{12}C_{k34}\left|\mathfrak{F}_{\Delta_k}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right)\right|^2 = |z|^{-4\Delta_1}\sum_l C^l_{14}C_{l34}\left|\mathfrak{F}_{\Delta_l}\left(\begin{matrix}\Delta_4 & \Delta_3\\ \Delta_1 & \Delta_2\end{matrix}\middle| \frac{1}{z}\right)\right|^2 \tag{159}$$

Probs:

1. Solve (151) for level $N = 3$. Find the values of Δ_k for which the determinant of the corresponding matrix vanishes.

2. Consider the case $\Delta_1 = \Delta(\alpha)$, $\Delta_2 = \Delta(\beta)$ and $\Delta_k = \Delta(\alpha + \beta)$, where

$$\Delta(\alpha) = \alpha(Q - \alpha), \qquad c = 1 + 6Q^2,$$

and show by explicit calculations on first two levels that the OPE coefficients $\beta_{\boldsymbol{\lambda}}$'s coincide with those following form the free-field formula

$$: e^{\sqrt{2}\alpha\varphi(z,\bar{z})} :: e^{\sqrt{2}\beta\varphi(w,\bar{w})} := \frac{R^{4\alpha\beta}}{|z-w|^{4\alpha\beta}} : e^{\sqrt{2}\alpha\varphi(z,\bar{z})} e^{\sqrt{2}\beta\varphi(w,\bar{w})} : .$$

Lecture 10: Zamolodchikov recursion formula

There is very efficient algorithm for computing of the conformal block suggested by Zamolodchikov. we will study the pole structure of the four-point conformal block (161) following [3]. This function, being considered as a function of the intermediate dimension Δ has poles at Kac values $\Delta \to \Delta_{m,n}$ (see (89)). Clearly, the poles come from the inverse of the Shapovalov matrix (91). At $\Delta = \Delta_{m,n}$ there is a singular vector at the level mn

$$|\chi_{m,n}\rangle = D_{m,n}|\Delta_{m,n}\rangle \quad \text{where} \quad D_{m,n} = L_{-1}^{mn} + c_1(b)L_{-2}L_{-1}^{mn-2} + c_2(b)L_{-3}L_{-1}^{mn-3} + \dots, \tag{160}$$

with (see (90))

$$c_1 = \frac{mn}{6}\left((m^2-1)b^2 + (n^2-1)b^{-2}\right) \quad \text{etc.}$$

From the Kac determinant formula (93) we know that for any two partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ the following holds $|\boldsymbol{\lambda}| = |\boldsymbol{\nu}| + mn$

$$\langle\Delta|L_{\boldsymbol{\lambda}}L_{-\boldsymbol{\nu}}D_{m,n}|\Delta\rangle \sim (\Delta - \Delta_{m,n})$$

We can compute the conformal block (161) in any basis we like. We take the following one

$$\tilde{L}_{-\boldsymbol{\lambda}}|\Delta\rangle \overset{\text{def}}{=} \begin{cases} L_{-\boldsymbol{\lambda}}|\Delta\rangle & \text{if} \quad \boldsymbol{\lambda} = \{\dots, \underbrace{1, \dots, 1}_{k<mn}\}, \\ L_{-\boldsymbol{\nu}}D_{m,n}|\Delta\rangle & \text{otherwise} \end{cases}$$

In this basis one can write

$$\mathfrak{F}_{\Delta}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right) = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} z^{\Delta-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}\left(\tilde{\Gamma}^{-1}\right)_{\boldsymbol{\lambda}\boldsymbol{\mu}} \frac{\langle\Delta_4|\Phi_3\tilde{L}_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta_4|\Phi_3|\Delta\rangle}\frac{\langle\Delta|\tilde{L}_{\boldsymbol{\mu}}\Phi_1|\Delta_2\rangle}{\langle\Delta|\Phi_1|\Delta_2\rangle},$$

where

$$\tilde{\Gamma}_{\boldsymbol{\mu}\boldsymbol{\lambda}} \overset{\text{def}}{=} \langle\Delta|\tilde{L}_{\boldsymbol{\mu}}\tilde{L}_{-\boldsymbol{\lambda}}|\Delta\rangle.$$

Clearly, only the vectors $L_{-\boldsymbol{\nu}}D_{m,n}|\Delta\rangle$ lead to the singular behavior[12]. Moreover, one has

$$\langle\Delta|D^{+}_{m,n}L_{\boldsymbol{\nu}}L_{-\boldsymbol{\rho}}D_{m,n}|\Delta\rangle = \langle\Delta_{m,-n}|L_{\boldsymbol{\nu}}L_{-\boldsymbol{\rho}}|\Delta_{m,-n}\rangle\langle\Delta|D^{+}_{m,n}D_{m,n}|\Delta\rangle + O\big((\Delta-\Delta_{m,n})^2\big),$$

where we have used the relation

$$\Delta_{m,n} + mn = \Delta_{m,-n}.$$

[12] It follows from the formula for determinant of the block matrix (provided that D is non-degenerate)

$$\det\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \det(D)\det\big(A - BD^{-1}C\big) = \det(D)\det\big(A\big) + \dots$$

Then any entry of the inverse matrix is of the form $(-1)^{\pm}\det\begin{pmatrix}A' & B'\\ C' & D'\end{pmatrix}\Big/\det\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ where (A', B', C', D') were obtained from (A, B, C, D) by erasing one row and one column. Clearly, only the elements with $D' = D$ lead to the singular behavior.

Collecting all this one arrives to

$$\operatorname{Res}\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right)\Big|_{\Delta=\Delta_{m,n}} = (r_{m,n})^{-1}\times$$
$$\times\sum_{|\boldsymbol{\nu}|=|\boldsymbol{\rho}|} z^{\Delta_{m,-n}-\Delta_1-\Delta_2+|\boldsymbol{\nu}|}\left(\Gamma^{-1}_{\Delta_{m,-n}}\right)_{\boldsymbol{\nu}\boldsymbol{\rho}}\frac{\langle\Delta_4|\Phi_3 L_{-\boldsymbol{\nu}} D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle}\frac{\langle\Delta_{m,n}|D^+_{m,n}L_{\boldsymbol{\rho}}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle},$$

where

$$r_{m,n} = \lim_{\Delta\to\Delta_{m,n}}\frac{\langle\Delta|D^+_{m,n}D_{m,n}|\Delta\rangle}{\Delta-\Delta_{m,n}} \tag{161}$$

Moreover, we have

$$\frac{\langle\Delta_4|\Phi_3 L_{-\boldsymbol{\nu}} D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle} = \frac{\langle\Delta_4|\Phi_3 L_{-\boldsymbol{\nu}}|\Delta_{m,-n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,-n}\rangle}\frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle},$$
$$\frac{\langle\Delta_{m,n}|D^+_{m,n}L_{\boldsymbol{\rho}}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle} = \frac{\langle\Delta_{m,-n}|L_{\boldsymbol{\rho}}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,-n}|\Phi_1|\Delta_2\rangle}\frac{\langle\Delta_{m,n}|D^+_{m,n}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle}.$$

Collecting altogether one has

$$\operatorname{Res}\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right)\Big|_{\Delta=\Delta_{m,n}} = \frac{R_{m,n}}{r_{m,n}}\;\mathfrak{F}_{\Delta_{m,-n}}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right), \tag{162}$$

where

$$R_{m,n} = \frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle}\frac{\langle\Delta_{m,n}|D^+_{m,n}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle}.$$

Our next goal is to compute the factors $r_{m,n}$ and $R_{m,n}$. The last one is relatively easy. Consider first examples

$$\frac{\langle\Delta_4|\Phi_3 D_{1,1}|\Delta_{1,1}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{1,1}\rangle} = -\partial_z\cdot z^{\Delta_4-\Delta_{1,1}-\Delta_3}\Big|_{z=1} = \big(\Delta(\alpha_3)-\Delta(\alpha_4)\big),$$

and

$$\frac{\langle\Delta_4|\Phi_3 D_{2,1}|\Delta_{2,1}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{2,1}\rangle} = \Big(\partial_z^2 - b^2\left(z^{-1}\partial_z-\Delta_3 z^{-2}\right)\Big)z^{\Delta_4-\Delta_{2,1}-\Delta_3}\Big|_{z=1} =$$
$$= (\Delta_4-\Delta_{2,1}-\Delta_3)(\Delta_4-\Delta_{2,1}-\Delta_3-1) - b^2(\Delta_4-\Delta_{2,1}-2\Delta_3) =$$
$$= \left(\Delta(\alpha_3)-\Delta\Big(\alpha_4+\frac{b}{2}\Big)\right)\left(\Delta(\alpha_3)-\Delta\Big(\alpha_4-\frac{b}{2}\Big)\right),$$

and

$$\frac{\langle\Delta_4|\Phi_3 D_{3,1}|\Delta_{3,1}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{3,1}\rangle} = \cdots = \big(\Delta(\alpha_3)-\Delta(\alpha_4)\big)\big(\Delta(\alpha_3)-\Delta(\alpha_4+b)\big)\big(\Delta(\alpha_3)-\Delta(\alpha_4-b)\big). \tag{163}$$

It suggests the following generic formula

$$\frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle} = \prod_{r,s}\left(\Delta(\alpha_3)-\Delta\Big(\alpha_4+\frac{rb}{2}+\frac{sb^{-1}}{2}\Big)\right), \tag{164}$$

where the product goes over the sets

$$r = \{m-1, m-3, \ldots, 3-m, 1-m\} \quad \text{and} \quad s = \{n-1, n-3, \ldots, 3-n, 1-n\}. \tag{165}$$

The explicit formula (167) is a manifestation of the fusion rules for degenerate fields. We have already seen an example of such fusion (96)

$$\Phi_{2,1}\Phi_\alpha = [\Phi_{\alpha+\frac{b}{2}}] + [\Phi_{\alpha-\frac{b}{2}}], \qquad \Phi_{1,2}\Phi_\alpha = [\Phi_{\alpha+\frac{b^{-1}}{2}}] + [\Phi_{\alpha-\frac{b^{-1}}{2}}].$$

In particular it implies that

$$\Phi_{2,1}\Phi_{m,n} = [\Phi_{m+1,n}] + [\Phi_{m-1,n}], \qquad \Phi_{1,2}\Phi_{m,n} = [\Phi_{m,n+1}] + [\Phi_{m,n-1}].$$

Both these fusion rules can be interpreted as $\mathfrak{sl}(2)$ fusion rules. Namely, the product of 2-dimensional and m-dimensional (or n-dimensional) representations of $\mathfrak{sl}(2)$ is the sum of $m+1$-dimensional and $m-1$-dimensional representations ($n+1$-dimensional and $n-1$-dimensional). Then using associativity of the OPE, one finds that

$$\Phi_{m,n}\Phi_\alpha = \sum_{r,s}\Big[\Phi_{\alpha+\frac{rb}{2}+\frac{sb^{-1}}{2}}\Big], \tag{166}$$

where the sum goes over the set (188). Now consider the matrix element (168). The state $D_{m,n}|\Delta_{m,n}\rangle = |\chi_{m,n}\rangle$ is a singular vector which one can consistently set to zero

$$\frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle} = \mathcal{D}_{m,n} z^{\Delta_4-\Delta_{m,n}-\Delta_3} = 0, \tag{167}$$

where $\mathcal{D}_{m,n}$ is a differential operator obtained from $D_{m,n}$ according to the rules (160). Comparing (171) with (170) and taking into account large Δ_3 asymptotic, one arrives to the r.h.s. of (168).

Similarly, one has

$$\frac{\langle\Delta_{m,n}|D^+_{m,n}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle} = \prod_{r,s}\left(\Delta(\alpha_1) - \Delta\Big(\alpha_2 + \frac{rb}{2} + \frac{sb^{-1}}{2}\Big)\right),$$

which implies

$$R_{m,n} = \prod_{r,s}\left(\Delta(\alpha_1) - \Delta\Big(\alpha_2 + \frac{rb}{2} + \frac{sb^{-1}}{2}\Big)\right)\left(\Delta(\alpha_3) - \Delta\Big(\alpha_4 + \frac{rb}{2} + \frac{sb^{-1}}{2}\Big)\right).$$

The constant $r_{m,n}$ is more complicated. From its definition (165) it is clear that $r_{m,n}$ is a polynomial in b and b^{-1}. Explicit calculations on first levels give

$$r_{1,1} = 2, \quad r_{2,1} = 4b^2\left(b+b^{-1}\right)\left(b-b^{-1}\right), \quad r_{3,1} = 24b^4\left(2b+b^{-1}\right)\left(2b-b^{-1}\right)\left(b+b^{-1}\right)\left(b-b^{-1}\right) \quad \ldots \tag{168}$$

Zamolodchikov computed more terms and conjectured the generic formula [3]

$$r_{m,n} = \frac{2}{mb+nb^{-1}} \prod_{\substack{1-m\le i\le m \\ 1-n\le j\le n}}^{\prime} (ib+jb^{-1}),$$

where $\prod'$ means that the term with $i = j = 0$ is absent.

Using (166) it is tempting to write

$$\mathfrak{F}_\Delta\begin{pmatrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{pmatrix}\!\Big|z\Big) = \sum_{m,n} \frac{R_{mn}}{r_{m,n}(\Delta - \Delta_{m,n})}\,\mathfrak{F}_{\Delta_{m,-n}}\begin{pmatrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{pmatrix}\!\Big|z\Big) + \dots . \tag{169}$$

However such an expansion does not catch the asymptotic of the conformal block a at $\Delta \to \infty$. Alexey Zamolodchikov has shown in [3] that the conformal block admits the semiclassical behavior

$$\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right) = \mathfrak{f}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right)\left(1 + O\left(\frac{1}{\Delta}\right)\right),$$

where

$$\mathfrak{f}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right) = (16q)^{\Delta - \frac{Q^2}{4}} z^{\frac{Q^2}{4} - \Delta_1 - \Delta_2} (1-z)^{\frac{Q^2}{4} - \Delta_1 - \Delta_3} \theta_3(q)^{3Q^2 - 4\sum_k \Delta_k},$$

with[13]

$$\theta_3(q) = \sum_{k\in\mathbb{Z}} q^{k^2}, \quad q = e^{i\pi\tau}, \quad \tau = i\frac{K(1-z)}{K(z)} \quad \text{where} \quad K(x) = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t(1-t)(1-xt)}}.$$

Motivating by this formula we define elliptic conformal block

$$\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right) = (16q)^{\Delta - \frac{Q^2}{4}} z^{\frac{Q^2}{4} - \Delta_1 - \Delta_2} (1-z)^{\frac{Q^2}{4} - \Delta_1 - \Delta_3} \theta_3(q)^{3Q^2 - 4\sum_k \Delta_k} \mathfrak{H}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|q\right).$$

The elliptic conformal block converges to 1 at $\Delta \to \infty$. This allows to write the elliptic recurrence formula [3]

$$\mathfrak{H}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|q\right) = 1 + \sum_{m,n} \frac{(16q)^{mn} R_{m,n}}{r_{m,n}(\Delta - \Delta_{m,n})} \mathfrak{H}_{\Delta_{m,-n}}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|q\right) \tag{170}$$

The formula (174) is very efficient. Taking

$$\mathfrak{H}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|q\right) = 1 + \sum_{N=1}^{\infty} q^N \mathfrak{H}_N(\Delta),$$

we obtain

$$\mathfrak{H}_N(\Delta) = \sum_{mn\le N} \frac{(16)^{mn} R_{m,n}}{r_{m,n}(\Delta - \Delta_{m,n})} \mathfrak{H}_{N-mn}(\Delta_{m,-n}).$$

One can also change the point of view and study analytic structure of the conformal block as a function of c. Namely, equation $\Delta = \Delta_{m,n}(c)$ can also be solved as

$$c = c_{m,n}(\Delta).$$

[13] The elliptic parameter q has the following expansion in terms of z

$$16q = z + \frac{z^2}{2} + \frac{21z^3}{64} + \frac{31z^3}{128} + \dots$$

Then using (173) one finds

$$\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right) = \sum_{m,n} \frac{R_{mn}\left(\frac{\partial \Delta_{m,n}(c)}{\partial c}\right)^{-1}}{r_{m,n}(c-c_{m,n})}\,\mathfrak{F}_{\Delta_{m,-n}}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right) + \lim_{c\to\infty}\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right).$$

In the limit $c \to \infty$ only the states $L_{-1}^N|\Delta\rangle$ contribute and their effect can be easily summed up

$$\lim_{c\to\infty}\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\middle| z\right) = \sum_{N=0}^{\infty} z^{\Delta-\Delta_1-\Delta_2+N}\frac{1}{\langle\Delta|L_1^N L_{-1}^N|\Delta\rangle}\frac{\langle\Delta_4|\Phi_3 L_{-1}^N|\Delta\rangle}{\langle\Delta_4|\Phi_3|\Delta\rangle}\frac{\langle\Delta|L_1^N\Phi_1|\Delta_2\rangle}{\langle\Delta|\Phi_1|\Delta_2\rangle} =$$

$$= z^{\Delta-\Delta_1-\Delta_2}F\left(\begin{matrix}\Delta+\Delta_1-\Delta_2 \;\; \Delta+\Delta_3-\Delta_4\\ 2\Delta\end{matrix}\middle| z\right), \quad (171)$$

where $F\left(\begin{matrix}A\;B\\ C\end{matrix}\middle| z\right)$ is the hypergeometric function.

Probs:

1. Show that the coefficient $r_{2,1}$ is given by (172).
2. Show that the large c limit of the conformal block is given by (175).

Lecture 11: BPZ differential equation and three-point function

In this lecture we will study the associativity condition (163) for the special case with one of the fields being degenerate. We will consider the case of $\Phi_{2,1}$ field. Consider 4-point correlation function

$$\Psi(z,\bar{z}) \overset{\text{def}}{=} \langle \Phi_{-\frac{b}{2}}(z,\bar{z})\Phi_{\alpha_1}(z_1,\bar{z}_1)\Phi_{\alpha_2}(z_2,\bar{z}_2)\Phi_{\alpha_3}(z_3,\bar{z}_3)\rangle.$$

which satisfies BPZ differential equation (94) and similar anti-holomorphic equation. Using the projective invariance, one can set $z_1 = 0$, $z_2 = \infty$ and $z_3 = 1$

$$\left[z(1-z)\partial^2 + b^2\left((2z-1)\partial + \frac{\Delta_1}{z} + \frac{\Delta_3}{1-z} + \Delta_{2,1} - \Delta_2\right)\right]\Psi(z,\bar{z}) = 0. \tag{172}$$

It can be brought to the conventional hypergeometric form by the following change of variables

$$\Psi(z,\bar{z}) = z^{b\alpha_1}(1-z)^{b\alpha_3}f(z).$$

We obtain

$$\left[z(1-z)\partial^2 + \left(C-(A+B+1)z\right)\partial - AB\right]f(z) = 0, \tag{173}$$

where

$$A = \frac{1}{2} + b(\alpha_1+\alpha_3-Q) + b\left(\alpha_2 - \frac{Q}{2}\right), \quad B = \frac{1}{2} + b(\alpha_1+\alpha_3-Q) - b\left(\alpha_2 - \frac{Q}{2}\right), \\ C = 1 + b\left(2\alpha_1 - Q\right). \tag{174}$$

This equation has two solutions with diagonal monodromy around $z=0$ expressed through hypergeometric function

$$F\left(\begin{smallmatrix} A\, B \\ C \end{smallmatrix}\middle| z\right) \qquad \text{and} \qquad z^{1-C}F\left(\begin{smallmatrix} 1+A-C\; 1+B-C \\ 2-C \end{smallmatrix}\middle| z\right),$$

where the second one is obtained from the first one by substitution $\alpha_1 \to Q-\alpha_1$. For the original equation (176) we have

$$\mathcal{F}^s_-(z) = z^{b\alpha_1}(1-z)^{b\alpha_3}F\left(\begin{smallmatrix} A\, B \\ C \end{smallmatrix}\middle| z\right), \qquad \mathcal{F}^s_+(z) = z^{b(Q-\alpha_1)}(1-z)^{b\alpha_3}F\left(\begin{smallmatrix} 1+A-C\; 1+B-C \\ 2-C \end{smallmatrix}\middle| z\right). \tag{175}$$

We note that these solutions correspond to $s-$channel conformal blocks

$$\mathcal{F}^s_\pm(z) = \mathfrak{F}_{\Delta_\pm}\left(\begin{matrix} \Delta_1 & \Delta_2 \\ \Delta_{2,1} & \Delta_3 \end{matrix}\middle| z\right),$$

so that they have simple monodromic properties at $z=0$.

There is another basis of solutions to (177) with diagonal monodromy around $z=\infty$

$$z^{-A}F\left(\begin{smallmatrix} A\;\; 1+A-C \\ 1+A-B \end{smallmatrix}\middle| \frac{1}{z}\right), \qquad z^{-B}F\left(\begin{smallmatrix} B\;\; 1+B-C \\ 1+B-A \end{smallmatrix}\middle| \frac{1}{z}\right).$$

which correspond to $t-$channel conformal blocks.

$$\mathcal{F}^t_-(z)=z^{-2\Delta_{2,1}}\left(\frac{1}{z}\right)^{b\alpha_2}\left(1-\frac{1}{z}\right)^{b\alpha_3}F\left(\begin{smallmatrix}A\ 1+A-C\\1+A-B\end{smallmatrix}\Big|\frac{1}{z}\right),$$
$$\mathcal{F}^t_+(z)=z^{-2\Delta_{2,1}}\left(\frac{1}{z}\right)^{b(Q-\alpha_2)}\left(1-\frac{1}{z}\right)^{b\alpha_3}F\left(\begin{smallmatrix}B\ 1+B-C\\1+B-A\end{smallmatrix}\Big|\frac{1}{z}\right).$$

Of course these two bases of solutions are linearly dependent. To see this we consider Mellin-Barnes representation for hypergeometric function

$$\frac{\Gamma(A)\Gamma(B)}{\Gamma(C)}F\left(\begin{smallmatrix}A\ B\\C\end{smallmatrix}\Big|z\right)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma(A+s)\Gamma(B+s)\Gamma(-s)}{\Gamma(C+s)}(-z)^s ds,$$

where $|z|<1$ and the contour $\mathcal{C}$ encircles the poles of $\Gamma(-s)$ in counterclockwise direction. For $|z|>1$ the contour $\mathcal{C}$ rather picks the poles of $\Gamma(A+s)$ and $\Gamma(B+s)$ and hence we have

$$\frac{\Gamma(A)\Gamma(B)}{\Gamma(C)}F\left(\begin{smallmatrix}A\ B\\C\end{smallmatrix}\Big|z\right)=\frac{\Gamma(A)\Gamma(B-A)}{\Gamma(C-A)}(-z)^{-A}F\left(\begin{smallmatrix}A\ 1+A-C\\1+A-B\end{smallmatrix}\Big|\frac{1}{z}\right)+$$
$$+\frac{\Gamma(B)\Gamma(A-B)}{\Gamma(C-B)}(-z)^{-B}F\left(\begin{smallmatrix}B\ 1+B-C\\1+B-A\end{smallmatrix}\Big|\frac{1}{z}\right).$$

Similar transformation law we have for another solution. In terms of $s-$ and $t-$ channel conformal blocks the relation can be written as

$$\begin{aligned}\mathcal{F}^s_+&=\frac{\Gamma(A-B)\Gamma(2-C)}{\Gamma(1-B)\Gamma(1+A-C)}\mathcal{F}^t_++\frac{\Gamma(B-A)\Gamma(2-C)}{\Gamma(1-A)\Gamma(1+B-C)}\mathcal{F}^t_+,\\ \mathcal{F}^s_-&=\frac{\Gamma(A-B)\Gamma(C)}{\Gamma(A)\Gamma(C-B)}\mathcal{F}^t_++\frac{\Gamma(B-A)\Gamma(C)}{\Gamma(B)\Gamma(C-A)}\mathcal{F}^t_+.\end{aligned}\tag{176}$$

We have only two conformal blocks appearing in the $s-$channel decomposition

$$\langle\Phi_{-\frac{b}{2}}(z,\bar{z})\Phi_{\alpha_1}(0)\Phi_{\alpha_2}(\infty)\Phi_{\alpha_3}(1)\rangle=C^{\alpha_1+\frac{b}{2}}_{-\frac{b}{2},\alpha_1}C\Big(\alpha_1+\frac{b}{2},\alpha_2,\alpha_3\Big)|\mathcal{F}^s_+(z)|^2+C^{\alpha_1-\frac{b}{2}}_{-\frac{b}{2},\alpha_1}C\Big(\alpha_1+\frac{b}{2},\alpha_2,\alpha_3\Big)|\mathcal{F}^s_-(z)|^2.\tag{177}$$

At the same the $t-$channel decomposition should also hold

$$\langle\Phi_{-\frac{b}{2}}(z,\bar{z})\Phi_{\alpha_1}(0)\Phi_{\alpha_2}(\infty)\Phi_{\alpha_3}(1)\rangle=C^{\alpha_2+\frac{b}{2}}_{-\frac{b}{2},\alpha_2}C\Big(\alpha_1,\alpha_2+\frac{b}{2},\alpha_3\Big)|\mathcal{F}^t_+(z)|^2+C^{\alpha_2+\frac{b}{2}}_{-\frac{b}{2},\alpha_2}C\Big(\alpha_1,\alpha_2-\frac{b}{2},\alpha_3\Big)|\mathcal{F}^t_-(z)|^2.\tag{178}$$

The validity of both decompositions (181) and (182) guaranties the the correlation function is single-valued on the thrice punctured sphere. We note however, that applying (180) to (181) we will have unwanted terms like

$$\mathcal{F}^t_+(z)\mathcal{F}^t_-(\bar{z}),$$

which will destroy this property. The condition that unwanted terms cancel leads us to

$$\frac{C^{\alpha_1-\frac{b}{2}}_{-\frac{b}{2},\alpha_1}\,C\Big(\alpha_1-\frac{b}{2},\alpha_2,\alpha_3\Big)}{C^{\alpha_1+\frac{b}{2}}_{-\frac{b}{2},\alpha_1}\,C\Big(\alpha_1+\frac{b}{2},\alpha_2,\alpha_3\Big)}=\frac{\gamma(A)\gamma(B)\gamma(C-A)\gamma(C-B)}{\gamma(C)\gamma(C-1)},\qquad\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}.\tag{179}$$

It is convenient to rewrite the relation (183) in the form

$$\frac{C\left(\alpha_1+b,\alpha_2,\alpha_3\right)}{C(\alpha_1,\alpha_2,\alpha_3)}\sim\frac{\gamma(2b\alpha_1)\gamma(2b\alpha_1-1)\gamma\big(b(\alpha_3+\alpha_2-\alpha_1)-b^2\big)}{\gamma\big(b(\alpha_1+\alpha_2-\alpha_3)\big)\gamma\big(b(\alpha_1+\alpha_3-\alpha_2)\big)\gamma\big(b(\alpha_1+\alpha_2+\alpha_3-Q)\big)},\tag{180}$$

where $\sim$ means up to a factor depending only on α_1. Similar relation should hold with b being replaced with b^{-1}

$$\frac{C\left(\alpha_1+b^{-1},\alpha_2,\alpha_3\right)}{C(\alpha_1,\alpha_2,\alpha_3)}\sim\frac{\gamma(2b^{-1}\alpha_1)\gamma(2b^{-1}\alpha_1+b^{-2})\gamma\big(b^{-1}(\alpha_3+\alpha_2-\alpha_1)-b^{-2}\big)}{\gamma\big(b^{-1}(\alpha_1+\alpha_2-\alpha_3)\big)\gamma\big(b^{-1}(\alpha_1+\alpha_3-\alpha_2)\big)\gamma\big(b^{-1}(\alpha_1+\alpha_2+\alpha_3-Q)\big)}.\tag{181}$$

In order to solve (184) and (185) it is desirable to have a function $\Upsilon(x)$, which is self dual with respect to $b\leftrightarrow b^{-1}$ and satisfies

$$\Upsilon(x+b)=b^{1-2bx}\gamma(bx)\Upsilon(x),\quad \Upsilon\left(x+\frac{1}{b}\right)=b^{\frac{2x}{b}-1}\gamma\left(\frac{x}{b}\right)\Upsilon(x).\tag{182}$$

We note that this condition is consistent with the requirement

$$\frac{\Upsilon\left(x+b+\frac{1}{b}\right)}{\Upsilon\left(x+b\right)}\frac{\Upsilon\left(x+b\right)}{\Upsilon\left(x\right)}=\frac{\Upsilon\left(x+b+\frac{1}{b}\right)}{\Upsilon\left(x+\frac{1}{b}\right)}\frac{\Upsilon\left(x+\frac{1}{b}\right)}{\Upsilon\left(x\right)}.$$

It is clear, that if the function which obeys (186) exists, it should be unique up to a constant provided that $b^2\neq p/q$. One can easily check that $\Upsilon(Q-x)$ satisfies exactly the same properties (186), which suggests

$$\Upsilon(x)=\Upsilon(Q-x),\qquad \Upsilon\left(\frac{Q}{2}\right)=1.$$

There is an integral representation

$$\log\Upsilon(x)=\int_0^\infty\frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2e^{-2t}-\frac{\sinh^2\left(\left(\frac{Q}{2}-x\right)t\right)}{\sinh(bt)\sinh\left(\frac{t}{b}\right)}\right],$$

which is valid in the domain $0<x<Q$. The function $\Upsilon(x)$ does not have poles, only zeroes

$$x=-mb-\frac{n}{b},\qquad x=Q+mb+\frac{n}{b},\qquad m,n\geq 0.$$

Having the $\Upsilon(x)$ function defined, one can write the solution to (184) and (185) as

$$C(\alpha_1,\alpha_2,\alpha_3)=\frac{\mathcal{N}(\alpha_1)\mathcal{N}(\alpha_2)\mathcal{N}(\alpha_3)}{\Upsilon(\alpha-Q)\prod_{k=1}^3\Upsilon(\alpha-2\alpha_k)},\quad\text{where}\quad \alpha=\alpha_1+\alpha_2+\alpha_3,$$

where the factors $\mathcal{N}(\alpha_k)$ correspond to unknown normalization factors for primary fields.

Lecture 12: Minimal models I

Minimal models were introduced by Belavin, Polyakov and Zamolodchikov in their seminal paper [4].

We have seen many times a simplification happening for correlation functions involving degenerate fields. For example the fusion (96)

$$\Phi_{2,1}\Phi_\alpha = [\Phi_{\alpha+\frac{b}{2}}] + [\Phi_{\alpha-\frac{b}{2}}], \qquad \Phi_{1,2}\Phi_\alpha = [\Phi_{\alpha+\frac{b^{-1}}{2}}] + [\Phi_{\alpha-\frac{b^{-1}}{2}}].$$

In particular it implies that

$$\Phi_{2,1}\Phi_{m,n} = [\Phi_{m+1,n}] + [\Phi_{m-1,n}], \qquad \Phi_{1,2}\Phi_{m,n} = [\Phi_{m,n+1}] + [\Phi_{m,n-1}].$$

Both these fusion rules can be interpreted as $\mathfrak{sl}(2)$ fusion rules. Namely, the product of 2-dimensional and m-dimensional (or n-dimensional) representations of $\mathfrak{sl}(2)$ is the sum of $m+1$-dimensional and $m-1$-dimensional representations ($n+1$-dimensional and $n-1$-dimensional). Then using associativity of the OPE, one finds that

$$\Phi_{m,n}\Phi_\alpha = \sum_{r,s}\Big[\Phi_{\alpha+\frac{rb}{2}+\frac{sb^{-1}}{2}}\Big], \tag{183}$$

where the sum goes over the set

$$r = \{m-1, m-3, \ldots, 3-m, 1-m\} \quad \text{and} \quad s = \{n-1, n-3, \ldots, 3-n, 1-n\}. \tag{184}$$

But what if correlation function consists of degenerate fields only? Consider the OPE (187) were the sum goes over the set (188) and suppose that $\alpha = \alpha_{m',n'}$. Then there are two ways to rewrite (187)

$$\Phi_{m,n}\Phi_{m',n'} = \sum_{r,s}\left[\Phi_{r,s}\right], \qquad \Phi_{m,n}\Phi_{m',n'} = \sum_{r',s'}\left[\Phi_{r',s'}\right],$$

where the sums go over the sets

$$\begin{aligned} r &\in (m'-m+1, \ldots, m'+m-1), \qquad & s &\in (n'-n+1, \ldots, n'+n-1), \\ r' &\in (m-m'+1, \ldots, m'+m-1), \qquad & s' &\in (n-n'+1, \ldots, n'+n-1). \end{aligned} \tag{185}$$

The compatibility condition for validity of both expansions requires the sum go over the intersection of two sets (189). That is

$$\Phi_{m,n}\Phi_{m',n'} = \sum_{r,s}\left[\Phi_{r,s}\right], \qquad r \in \big(|m'-m|+1, \ldots, m'+m-1\big),\ s \in \big(|n'-n|+1, \ldots, n'+n-1\big), \tag{186}$$

Since negative numbers do not appear in the r.h.s. of (190), we conclude that the sum goes over the degenerate fields only. In other words the OPE is closed on degenerate fields. So, we might try to construct a CFT which will consists of degenerate fields $\Phi_{m,n}$ only, where (m,n) belong to some set. Actually, this time we will be more cautious about unitarity issues. In particular, we will require

$$\Delta_{m,n} = \frac{(b+b^{-1})^2}{4} - \frac{(mb+nb^{-1})^2}{4} \geq 0. \tag{187}$$

We see that (191) does not hold for $b \in \mathbb{R}$, for $b = e^{i\theta}$ it is in general complex, while the only hope is for $b = i\beta$ with $\beta \in \mathbb{R}$. In this case one can still find (m,n), such that $|m\beta + n\beta^{-1}| \ll 1$ and so $\Delta_{m,n} < 0$.

So, the only hope to construct an unitary CFT with degenerate fields only, would be the case where set of possible (m,n) is restricted.

This is where doubly-degenerate fields come into a play. Namely, suppose that

$$b^2 = -\frac{p}{q}, \tag{188}$$

where p and q are coprime positive integers $q > p$. In this case all the fields

$$\Phi_{m+kq,n+kp} \quad \text{and} \quad \Phi_{q-m+kq,p-n+kp} \tag{189}$$

have the same conformal dimensions. In new parametrization (192) one has

$$c = 1 - \frac{6(p-q)^2}{pq} \tag{190}$$

Consider the notion of the Kac table, that is the set $0 < m < q$, $0 < n < p$. Take basic degenerate field $\Phi_{m,n}$ with $0 < m < q$, $0 < n < p$ and its nearest partner $\Phi_{m',n'} = \Phi_{q-m,p-n}$ (also with $0 < m' < q$, $0 < n' < p$) and consider the OPE

$$\Phi_{m,n}\Phi_\alpha = \sum_{i,j}\Big[\Phi_{\alpha-\frac{(m-1)b}{2}-\frac{(n-1)b^{-1}}{2}+ib+jb^{-1}}\Big] = \sum_{i',j'}\Big[\Phi_{\alpha-\frac{(m'-1)b}{2}-\frac{(n'-1)b^{-1}}{2}+i'b+j'b^{-1}}\Big], \tag{191}$$

where $i = 0,1,\dots,m-1$, $j = 0,1,\dots,n-1$ etc. In the r.h.s. of (195) we should either have

$$\alpha - \frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2} + ib + jb^{-1} = \alpha - \frac{(m'-1)b}{2} - \frac{(n'-1)b^{-1}}{2} + i'b + j'b^{-1},$$

or

$$\alpha - \frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2} + ib + jb^{-1} = Q - \alpha + \frac{(m'-1)b}{2} + \frac{(n'-1)b^{-1}}{2} - i'b - j'b^{-1}.$$

First possibility leads to the condition

$$p(m + i' - i) = q(n + j' - j),$$

which never holds, while the second one gives

$$\alpha = \alpha_{m'',n''} \quad \text{with} \quad m'' = i + i' + 1, \quad n'' = j + j' + 1.$$

We note that $0 < m'' < q$, $0 < n'' < p$. That is the necessary condition for the field Φ_α to have a non-trivial OPE with degenerate field $\Phi_{m,n}$ from Kac table $0 < m < q$, $0 < n < p$, is to be a degenerate field from the Kac table also. Then the associativity implies that the OPE is closed on degenerate fields from the Kac table only.

The fact noticed above opens the possibility to have a CFT for quantized values of the parameter (192), which has only finitely many degenerate fields from the Kac table with

$$\Delta_{m,n} = \frac{\big(mp - nq\big)^2 - (p-q)^2}{4pq}. \tag{192}$$

Such CFT's are known as minimal models $\mathcal{M}_{p,q}$. One can show that there are always negative numbers in (196) except the case $q = p + 1$ where all the values

$$\Delta_{m,n} = \frac{\big(mp - n(p+1)\big)^2 - 1}{4p(p+1)} \tag{193}$$

are obligatory non-negative. The models $\mathcal{M}_{p,p+1}$ are known as unitary series of minimal models. Moreover, there is Friedan Qiu and Shenker theorem [5] which states that the Verma module $\mathcal{V}_\Delta$ does not have vectors of negative norm only in two cases:

- For $\Delta \geq 0$ and $c \geq 1$
- For unitary minimal model

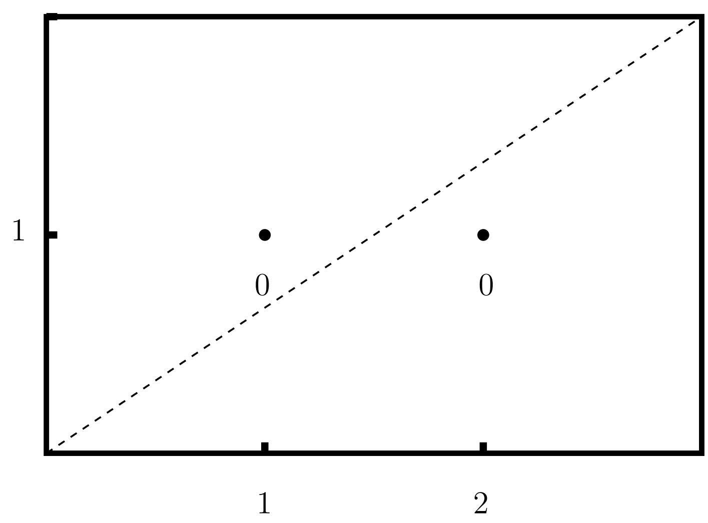
$$c = 1 - \frac{6}{p(p+1)}, \quad \Delta_{m,n} = \frac{(mp - n(p+1))^2 - 1}{4p(p+1)} \quad \text{for} \quad 0 < n < m < p + 1.$$

We will provide the details of the proof of this theorem later in this course.

There might be a problem since according to (193) for any field from the Kac table we have infinitely many null vectors. The situation is very interesting. Consider the simplest minimal model $\mathcal{M}_{2,3}$, which corresponds to the value

$$b = i\sqrt{\frac{2}{3}} \implies c = 0.$$

This theory has the following Kac table



In fact the Hilbert space of this theory consists of just one vacuum state such that

$$L_n|0\rangle = 0 \quad n \in \mathbb{Z}.$$

The character of this module is just 1. However, from the point of view of Verma modules the situation is more subtle. Kac dimensions for $c = 0$ model ($b = i\sqrt{\frac{2}{3}}$) have the form

$$\Delta_{m,n} = \frac{(2m - 3n)^2 - 1}{24}.$$

In particular,

$$\Delta_{1,1} = \Delta_{2,1} = 0.$$

and hence the Verma module $\mathcal{V}_0$ has two null vectors $|\chi_{1,1}\rangle$ and $|\chi_{2,1}\rangle$ at levels 1 and 2 respectively. The corresponding character seem to have the form

$$\chi(q) = \left(\prod_{k=1}^{\infty} \frac{1}{1-q^k}\right)(1-q-q^2) = 1 - q^5 + \ldots, \tag{194}$$

which is certainly not equal to 1. However, the modules $\mathcal{V}_1$ and $\mathcal{V}_2$ intersect. Indeed, one can check that

$$\Delta_{4,1} = \Delta_{2,3} = 1 \quad \text{and} \quad \Delta_{5,1} = \Delta_{1,3} = 2,$$

and hence the module $\mathcal{V}_1$ has two null vectors at levels 4 (total level 5) and 6 (total level 7), while $\mathcal{V}_2$ has two null vectors at levels 3 (total level 5) and 5 (total level 7). However, the corresponding descendants coincide, as one can see from the relations

$$D_{4,1}D_{1,1} = D_{1,3}D_{2,1} \quad \text{and} \quad D_{2,3}D_{1,1} = D_{5,1}D_{2,1},$$

where $D_{m,n}$ are defined in (241) and hence we have

$$L_{-\boldsymbol{\lambda}} \underbrace{\left(L_{-1}^4 - \frac{20}{3}L_{-2}L_{-1}^2 + 4L_{-2}^2 + 4L_{-3}L_{-1} - 4L_{-4}\right)}_{D_{4,1}} \underbrace{L_{-1}}_{D_{1,1}} |0\rangle =$$

$$= L_{-\boldsymbol{\lambda}} \underbrace{\left(L_{-1}^3 - 6L_{-2}L_{-1} + 6L_{-3}\right)}_{D_{1,3}} \underbrace{\left(L_{-1}^2 - \frac{2}{3}L_{-2}\right)}_{D_{2,1}} |0\rangle$$

and

$$L_{-\boldsymbol{\lambda}} \underbrace{\left(L_{-1}^6 - 14L_{-2}L_{-1}^4 + \frac{112}{3}L_{-2}^2L_{-1}^2 - \frac{512}{27}L_{-2}^3 + 14L_{-3}L_{-1}^3 - \frac{40}{3}L_{-3}L_{-2}L_{-1} - \right.}_{D_{2,3}}$$

$$\underbrace{\left. -\frac{208}{9}L_{-3}^2 - 48L_{-4}L_{-1}^2 + \frac{688}{9}L_{-4}L_{-2} + \frac{88}{9}L_{-5}L_{-1} + \frac{80}{3}L_{-6}\right)}_{D_{2,3}} \underbrace{L_{-1}}_{D_{1,1}} |0\rangle = L_{-\boldsymbol{\lambda}} \underbrace{\left(L_{-1}^5 - \frac{40}{3}L_{-2}L_{-1}^3 + \right.}_{D_{5,1}}$$

$$\underbrace{\left. + \frac{256}{9}L_{-2}^2L_{-1} + \frac{52}{3}L_{-3}L_{-1}^2 - \frac{256}{9}L_{-3}L_{-2} - \frac{104}{3}L_{-4}L_{-1} + \frac{208}{9}L_{-5}\right)}_{D_{5,1}} \underbrace{\left(L_{-1}^2 - \frac{2}{3}L_{-2}\right)}_{D_{2,1}} |0\rangle$$

It means that we have substracted states from submodules $\mathcal{V}_5$ and $\mathcal{V}_7$ in (198) twice. We add them to compensate the mismatch. Thus we have the character

$$\chi(q) = \left(\prod_{k=1}^{\infty} \frac{1}{1-q^k}\right)(1-q-q^2+q^5+q^7) = 1 + q^{12} + \ldots.$$

Thus we fixed the character up to level 12. Going deeper in the module one finds that $\mathcal{V}_5$ and $\mathcal{V}_7$ again both have a pair of null vector which coincide pairwise at the total levels 12 and 15, which implies the correction to the character (we have added them twice, so we have to substract)

$$\chi(q) = \left(\prod_{k=1}^{\infty} \frac{1}{1-q^k}\right)(1-q-q^2+q^5+q^7-q^{12}-q^{15}) = 1-q^{22}+\dots.$$

Proceeding in this way is shown by the diagram of embeddings

$$\begin{tikzcd}
 & 0 \arrow[dl, "(1,1)"'] \arrow[dr, "(2,1)"] & \\
1 \arrow[dd, "(4,1)"'] \arrow[ddrr, "(2,3)" near start] & & 2 \arrow[dd, "(5,1)"] \arrow[ddll, "(1,3)"' near start] \\
 & & \\
5 \arrow[dd, "(7,1)"'] \arrow[ddrr, "(2,5)" near start] & & 7 \arrow[dd, "(8,1)"] \arrow[ddll, "(1,5)"' near start] \\
 & & \\
12 & & 15
\end{tikzcd}$$

The fact that we will get 1 after all "addings" and "substractions" is guaranteed by the pentagonal numbers identity

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}} = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+\dots \tag{195}$$

which implies

$$\left(\prod_{k=1}^{\infty} \frac{1}{1-q^k}\right)(1-q-q^2+q^5+q^7-q^{12}-q^{15}+\dots) = 1.$$

The structure of embeddings for Verma module $\mathcal{V}_{m,n}$ for generic minimal model $\mathcal{M}_{p,q}$ can be read from (193) in a similar way. We use the identity

$$\Delta_{m+kq,n+kp} = \Delta_{-m+kq,-n+kp} \quad \text{for any} \quad k \in \mathbb{Z}.$$

We start with the module $\mathcal{V}_{m,n}$. It has two lowest lying null vectors at the levels mn and $(q-m)(p-n)$. Thus we have the embedding of Verma modules

$$\underbrace{\mathcal{V}_{-m,n} \subset \mathcal{V}_{m,n}}_{D_{m,n}} \quad \text{and} \quad \underbrace{\mathcal{V}_{-m+2q,n} \subset \mathcal{V}_{m,n}}_{D_{q-m,p-n}},$$

where we have explicitly shown the corresponding null vector creating operators. The each of Verma modules $\mathcal{V}_{-m,n}$ and $\mathcal{V}_{-m+2q,n}$ have pair of submodules

$$\underbrace{\mathcal{V}_{m+2q,n} \subset \mathcal{V}_{-m,n}}_{D_{q+m,p-n}} \quad \text{and} \quad \underbrace{\mathcal{V}_{m-2q,n} \subset \mathcal{V}_{-m,n}}_{D_{q-m,p+n}}, \tag{196}$$

and at the same time

$$\underbrace{\mathcal{V}_{m+2q,n} \subset \mathcal{V}_{-m+2q,n}}_{D_{m,2p-n}} \quad \text{and} \quad \underbrace{\mathcal{V}_{m-2q,n} \subset \mathcal{V}_{-m+2q,n}}_{D_{2q-m,n}}, \tag{197}$$

Moreover, we have the relations

$$D_{q+m,p-n}D_{m,n} = D_{m,2p-n}D_{q-m,p-n} \quad \text{and} \quad D_{q-m,p+n}D_{m,n} = D_{2q-m,n}D_{q-m,p-n},$$

which guarantee that $\mathcal{V}_{m+2q,n}$'s are actually the same in (200) and in (201) (the same is true for $\mathcal{V}_{m-2q,n}$).

Now each of $\mathcal{V}_{m+2q,n}$ and $\mathcal{V}_{m-2q,n}$ contain two submodules

$$\mathcal{V}_{-m-2q,n} \quad \text{and} \quad \mathcal{V}_{-m+4q,n},$$

etc. It is interesting to note that $\mathcal{V}_{-m-2q,n}$ and $\mathcal{V}_{m-2q,n}$ are also the submodules in $\mathcal{V}_{m,n}$

$$\underbrace{\mathcal{V}_{-m-2q,n} \subset \mathcal{V}_{m,n}}_{D_{q+m,p+n}} \quad \text{and} \quad \underbrace{\mathcal{V}_{-m+4q,n} \subset \mathcal{V}_{m,n}}_{D_{2q-m,2p-n}},$$

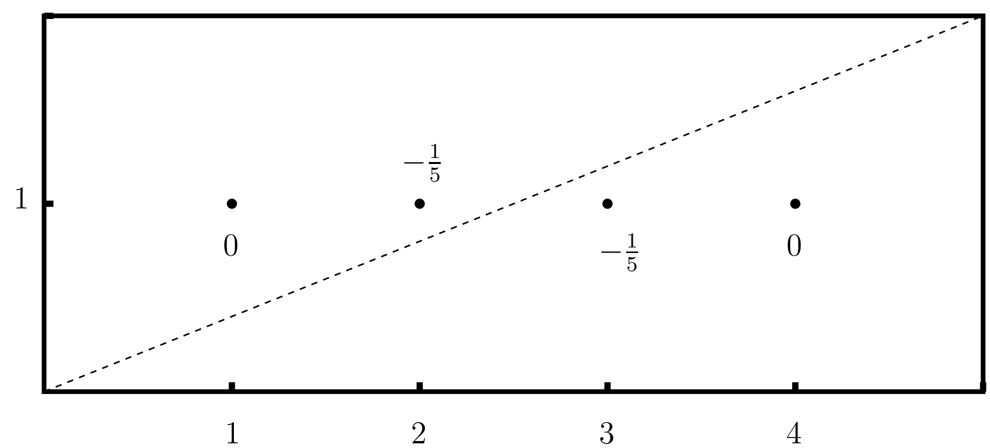
All these subtle embeddings can be drawn as follows

$$\begin{tikzcd}
 & \Delta_{m,n} \arrow[dl, "(m{,}n)"'] \arrow[dr, "(q-m{,}p-n)"] \arrow[dddl, bend right, "(m+q{,}n+p)"'] \arrow[dddr, bend left, "(2q-m{,}2p-n)"] & \\
\Delta_{-m,n} \arrow[d, "(q+m{,}p-n)"'] \arrow[drr, "(q-m{,}p+n)"] & & \Delta_{-m+2q,n} \arrow[d, "(2q-m{,}n)"] \arrow[dll, "(m{,}2p-n)"'] \\
\Delta_{m+2q,n} \arrow[d, "(m+2q{,}n)"'] \arrow[drr, "(q-m{,}3p-n)"] & & \Delta_{m-2q,n} \arrow[d, "(3q-m{,}p-n)"] \arrow[dll, "(m{,}2p+n)"'] \\
\Delta_{-m-2q,n} & & \Delta_{-m+4q,n}
\end{tikzcd}$$

The character of the corresponding irreducible module is obtained by summing all Verma modules from "odd floors" and substracting the contributions from the "even floors". The resulting character has the form (we replaced $q \to x$ to avoid misleading notations)

$$\chi_{m,n}^{(p,q)}(x) = \chi(x) \sum_{k \in \mathbb{Z}} \left(x^{\Delta_{m+2kq,n}} - x^{\Delta_{-m+2kq,n}} \right) \quad \text{where} \quad \chi(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)}. \tag{198}$$

Let us consider the simplest non-trivial minimal model $\mathcal{M}_{2,5}$ called Lee-Yang model. It has the following Kac table



There are only two primary fields $\Phi_{1,1} = \Phi_{4,1} = I$ and $\Phi_{2,1} = \Phi_{3,1} = \varphi$. Using (139) one can show that

$$\begin{aligned}
\chi_{1,1}^{(2,5)}(q) &= \prod_{k=1}^{\infty} \frac{1}{(1-q^{5k-2})(1-q^{5k-3})} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 + \dots, \\
\chi_{2,1}^{(2,5)}(q) &= q^{-\frac{1}{5}} \prod_{k=1}^{\infty} \frac{1}{(1-q^{5k-1})(1-q^{5k-4})} = q^{-\frac{1}{5}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + \dots \right).
\end{aligned} \tag{199}$$

The structure of OPE in $\mathcal{M}_{2,5}$ is very simple

$$\begin{aligned}
&I \otimes I = [I], \\
&I \otimes \varphi = [\varphi], \\
&\varphi \otimes \varphi = [I] \oplus [\varphi]
\end{aligned}$$

If we employ standard normalization of the fields, that is

$$\langle \varphi(z,\bar{z}) \varphi(w,\bar{w}) \rangle = |z-w|^{\frac{4}{5}},$$

then the only nontrivial constant in the theory is

$$\varkappa \stackrel{\text{def}}{=} C_{\varphi\varphi}^{\varphi} = C(\varphi,\varphi,\varphi).$$

In order to find $\varkappa$ we consider four-point function $\langle \varphi(z,\bar{z}) \varphi(0) \varphi(\infty) \varphi(1) \rangle$. According to (181) we have

$$\langle \varphi(z,\bar{z}) \varphi(0) \varphi(\infty) \varphi(1) \rangle = |F_1(z)|^2 + \varkappa^2 |F_2(z)|^2,$$

where (see (179))

$$F_1(z) = z^{\frac{2}{5}}(1-z)^{\frac{1}{5}} F\left(\begin{smallmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{6}{5} & \end{smallmatrix}\middle| z\right), \quad F_2(z) = z^{\frac{1}{5}}(1-z)^{\frac{1}{5}} F\left(\begin{smallmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{4}{5} & \end{smallmatrix}\middle| z\right).$$

Then the relation (183) gives

$$\varkappa^2 = \frac{\gamma\left(\frac{1}{5}\right)\gamma\left(\frac{2}{5}\right)^2\gamma\left(\frac{3}{5}\right)}{\gamma\left(\frac{4}{5}\right)\gamma\left(-\frac{1}{5}\right)} = \left\{\gamma(x)\gamma(1-x) = 1,\, \gamma(x)\gamma(-x) = -\frac{1}{x^2}\right\} = -\frac{1}{25}\gamma^3\left(\frac{1}{5}\right)\gamma\left(\frac{2}{5}\right),$$

so that

$$\varkappa = \frac{i}{5}\gamma^{\frac{3}{2}}\left(\frac{1}{5}\right)\gamma^{\frac{1}{2}}\left(\frac{2}{5}\right) \approx 1.91131i.$$

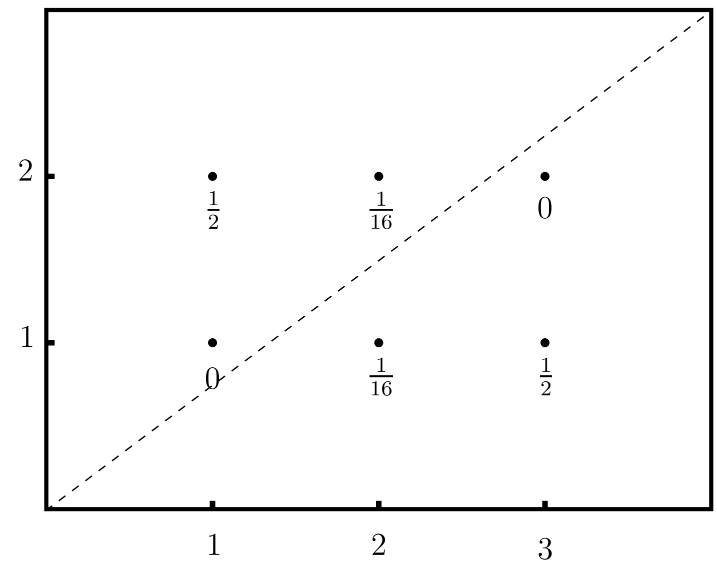
The fact that the structure constant is purely imaginary is consistent with the fact the Lee-Yang model is non-unitary.

Probs:

1. Show that the set (196) always contains negative dimensions for $q > p+1$.
2. Using (202) and Jacobi Triple Identity derive (203).

Lecture 13: Minimal Models II: Ising model, tricritical Ising model, $N=1$ SUSY CFT

In this lecture we will study minimal model $\mathcal{M}_{3,4}$ in details. It corresponds to the following Kac table



(200)

According to (194) the central charge of this theory is $c = \frac{1}{2}$. The conformal dimensions are given by

$$\Delta_{m,n} = \frac{(3m-4n)^2 - 1}{48}.$$

The fields inside the Kac table are identified by the reflection

$$\Phi_{m,n} \sim \Phi_{4-m,3-n}.$$

Thus we have three different primary families

$$\begin{aligned}
I &= \Phi_{1,1} = \Phi_{3,2} \quad \text{with} \quad \Delta = \bar{\Delta} = 0, \\
\epsilon &= \Phi_{3,1} = \Phi_{1,2} \quad \text{with} \quad \Delta = \bar{\Delta} = \frac{1}{2}, \\
\sigma &= \Phi_{2,1} = \Phi_{2,2} \quad \text{with} \quad \Delta = \bar{\Delta} = \frac{1}{16},
\end{aligned}$$

the identity operator I, the "energy" operator ϵ and the "spin" operator σ.

These fields describe critical Ising model. In order to see this, we note that the field ϵ has very special OPE

$$\epsilon\epsilon = \Phi_{1,2}\Phi_{3,1} = [\Phi_{3,2}] = [I], \qquad \epsilon\sigma = \Phi_{1,2}\Phi_{2,1} = [\Phi_{2,2}] = [\sigma] \tag{201}$$

It implies that for any correlation function of ϵ one has

$$\langle \epsilon(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\ldots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = \Big|F(z|z_1,\ldots,z_n)\Big|^2 G(z_1,\bar{z}_1,\ldots,z_n,\bar{z}_n),$$

where $\mathcal{O}_k$ stand for either ϵ, I or σ. It suggests that the field ϵ admits the holomorhic factorization

$$\epsilon = i\,\psi\bar{\psi}. \tag{202}$$

The holomorphic current $\psi(z)$ has dimension $\frac{1}{2}$ and admits the OPE

$$\psi(z)\psi(w) = \frac{1}{z-w} + \dots$$

We note that this type of OPE is only compatible with the fermionic statistic for ψ. Similar statement applies to $\bar{\psi}$. We also demand that ψ anticommutes with $\bar{\psi}$.

The pair $(\psi, \bar{\psi})$ can be treated as Majorana (real) fermion, the real part of the complex fermion studied before

$$\psi(z) = \frac{1}{\sqrt{2}}\left(\psi(z) + \psi^*(z)\right), \qquad \bar{\psi}(z) = \frac{1}{\sqrt{2}}\left(\bar{\psi}(z) + \bar{\psi}^*(z)\right)$$

The dynamics of $(\psi, \bar{\psi})$ is described by the massless Ising action

$$S = \frac{1}{2\pi}\int \left(\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi\right) d^2z. \tag{203}$$

The correlation functions of $\psi(z)$ and $\bar{\psi}(\bar{z})$ are computed by the Wick theorem

$$\langle \psi(z_1)\dots\psi(z_{2n})\rangle = \frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\dots\frac{1}{z_{2n-1}-z_{2n}} + \dots$$

and similarly for $\bar{\psi}(\bar{z})$. We note that the factor i in (206) provides canonical normalization of the energy operator

$$\langle \sigma(z,\bar{z})\sigma(w,\bar{w})\rangle = \frac{1}{|z-w|}.$$

The holomorphic stress-energy tensor for the theory (207) has the form

$$T(z) = -\frac{1}{2} : \psi\partial\psi : .$$

and it defines CFT with the central charge $c = \frac{1}{2}$. The representation of Maiorana fermion is given by the fermionic Fock module

$$\mathcal{F}^{\mathrm{NS}} = \mathrm{Span}\left(\psi_{-\boldsymbol{s}}|\varnothing\rangle = \psi_{-s_1}\psi_{-s_2}\dots|\varnothing\rangle | s_1 > s_2 > \dots\right) \tag{204}$$

where $s \in \mathbb{Z} + \frac{1}{2}$, which corresponds to NS sector. The generators ψ_s form an algebra

$$\{\psi_r, \psi_s\} = \delta_{r,-s}. \tag{205}$$

The character of (208) is given by

$$\chi^{\mathrm{NS}}(q) = \prod_{k=1}^{\infty}(1 + q^{k-\frac{1}{2}})$$

From the point of view of Minimal Model the Fock module $\mathcal{F}_{\mathrm{F}}$ corresponds to direct sum of irreducible Verma modules

$$\mathcal{F}^{\mathrm{NS}} = \mathcal{V}_{1,1} \oplus \mathcal{V}_{3,1}$$

In particular, it implies the character identity (see (202))

$$\chi^{(3,4)}_{1,1}(q) + \chi^{(3,4)}_{3,1}(q) = \chi^{\mathrm{NS}}(q)$$

which can be thought as an additional confirmation of the coincidence of two theories. Indeed, from (202) one has

$$\chi_{1,1}^{(3,4)}(q)+\chi_{3,1}^{(3,4)}(q)=\prod_{l=1}^{\infty}\frac{1}{(1-q^l)}\sum_{k\in\mathbb{Z}}\left(q^{\Delta_{1+8k,1}}-q^{\Delta_{-1+8k,1}}+q^{\Delta_{3+8k,1}}-q^{\Delta_{-3+8k,1}}\right).$$

We note that

$$\begin{aligned}
\Delta_{1+8k,1}&=\frac{3(4k)^2-4k}{4}, & \Delta_{-1+8k,1}&=\frac{3(4k-1)^2-(4k-1)}{4},\\
\Delta_{3+8k,1}&=\frac{3(4k+1)^2-(4k+1)}{4}, & \Delta_{-3+8k,1}&=\frac{3(4k-2)^2-(4k-2)}{4},
\end{aligned}$$

that is

$$\chi_{1,1}^{(3,4)}(q)+\chi_{3,1}^{(3,4)}(q)=\prod_{l=1}^{\infty}\frac{1}{(1-q^l)}\sum_{k\in\mathbb{Z}}\left(q^{\frac{3(4k)^2-4k}{4}}-q^{\frac{3(4k+3)^2-(4k+3)}{4}}+q^{\frac{3(4k+1)^2-(4k+1)}{4}}-q^{\frac{3(4k+2)^2-(4k+2)}{4}}\right). \tag{206}$$

We note that in (210) the sum goes over $4k+s$ where $s=0,1,2,3$, that is over all integers. Then one can apply the pentagonal identity (199) with $q\to -q^{\frac{1}{2}}$ which gives

$$\chi_{1,1}^{(3,4)}(q)+\chi_{3,1}^{(3,4)}(q)=\prod_{l=1}^{\infty}\frac{1}{(1-q^l)}\prod_{k=1}^{\infty}\left(1-(-q^{\frac{1}{2}})^k\right)=\prod_{k=1}^{\infty}\left(1+q^{k-\frac{1}{2}}\right)$$

Similarly, one obtains the dual identity

$$\chi_{1,1}^{(3,4)}(q)-\chi_{3,1}^{(3,4)}(q)=\prod_{k=1}^{\infty}\left(1-q^{k-\frac{1}{2}}\right).$$

Now we come to the $\sigma(z,\bar{z})$ field. According to the OPE rules (205), one has

$$\epsilon(z,\bar{z})\sigma(w,\bar{w})=\frac{1}{2|z-w|}\Big(\sigma(w,\bar{w})+\dots\Big), \tag{207}$$

where the choice of the factor

$$C_{\epsilon\sigma}^{\sigma}=\frac{1}{2}$$

will be justified below. One might ask "can we conclude from (211) that"

$$\psi(z)\sigma(w,\bar{w})\sim\frac{1}{(z-w)^{\frac{1}{2}}}\Big(\sigma(w,\bar{w})+\dots\Big)\,? \tag{208}$$

In fact no. The reason for that is the following. The OPE of the the form (212) correspond to Ramond field which is semi-local with $\psi(z)$. It means that the indexes of ψ_s are integer. Then it follows from (209) that the zero modes ψ_0 and $\bar{\psi}_0$ form the algebra

$$\psi_0^2=\bar{\psi}_0^2=\frac{1}{2},\quad \{\psi_0,\bar{\psi}_0\}=0. \tag{209}$$

This algebra does not have a one-dimensional representation. In other words the fields $\psi_0\sigma(z,\bar{z})$ and $\bar{\psi}_0\sigma(z,\bar{z})$ can not be proportional to the field $\sigma(z,\bar{z})$. The best we can do is the two-dimensional representation of the algebra (213)

$$\begin{aligned} \psi_0\sigma(z,\bar{z}) &= \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\,\mu(z,\bar{z}), \qquad & \psi_0\mu(z,\bar{z}) &= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\,\sigma(z,\bar{z}), \\ \bar{\psi}_0\sigma(z,\bar{z}) &= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\,\mu(z,\bar{z}), \qquad & \bar{\psi}_0\mu(z,\bar{z}) &= \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\,\sigma(z,\bar{z}). \end{aligned} \tag{210}$$

Equation (214) can be taken as a definition of the spin field in Ising CFT. It means that the spin field is rather a doublet and the OPE (211) is also supplemented by the dual OPE

$$\epsilon(z,\bar{z})\mu(w,\bar{w}) = -\frac{1}{2|z-w|}\Big(\mu(w,\bar{w}) + \dots\Big), \tag{211}$$

From representation point of view one has a Ramond representation of the fermionic algebra

$$\mathcal{F}^{\mathrm{R}} = \mathrm{Span}\Big(\psi_{-\boldsymbol{s}}|\pm\rangle, \quad \boldsymbol{s} = \{s_1 > s_2 > \dots > 0\},\ |-\rangle = \psi_0|+\rangle, \quad s_i \in \mathbb{Z}\Big).$$

Clearly the character of this module is given by $2\chi_{\mathrm{F}}^{\mathrm{R}}(x)$ where

$$\chi^{\mathrm{R}}(q) = q^{\frac{1}{16}}\prod_{k=1}^{\infty}(1+q^k).$$

Again, it can be checked that

$$\chi_{2,1}(q) = \chi^{\mathrm{R}}(q), \tag{212}$$

which confirms the coincidence. Indeed, one has

$$\Delta_{2+8k,1} = \frac{1}{16} + \big(3(2k)^2 + 2k\big), \quad \Delta_{-2+8k,1} = \frac{1}{16} + \big(3(2k-1)^2 + 2k - 1\big),$$

which implies via (199) the desired identity (216).

The Ising CFT can be described either by the set of fields (I,σ,ϵ) or equivalently by (I,σ,μ). In order to specify the theory completely, one has to derive the structure constants. In Ising model there is only one (see (211))

$$C_{\sigma\sigma}^{\epsilon} = C_{\sigma\epsilon}^{\sigma} = \langle\sigma(0)\sigma(1)\epsilon(\infty)\rangle = \frac{1}{2} \tag{213}$$

or (see (215))

$$C_{\mu\mu}^{\epsilon} = C_{\mu\epsilon}^{\mu} = \langle\mu(0)\mu(1)\epsilon(\infty)\rangle = -\frac{1}{2}. \tag{214}$$

It is instructive to derive (217) and (218) from BPZ equation. Consider for example the four point function

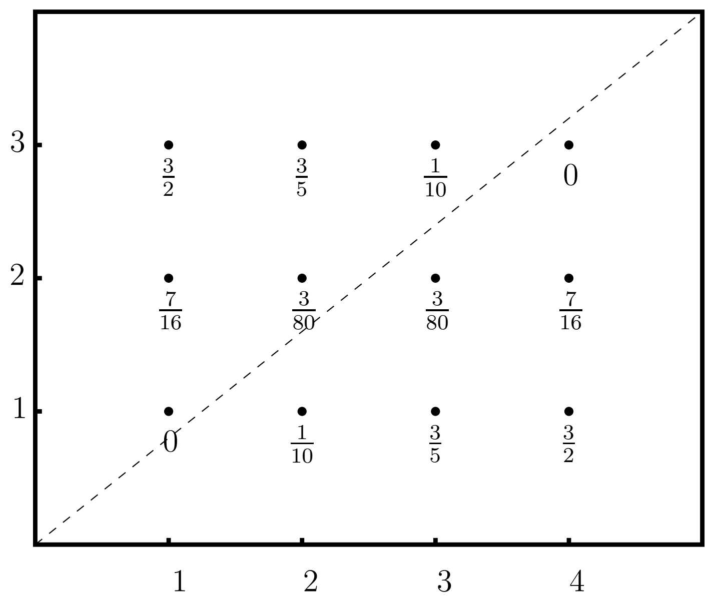
$$\langle\sigma(z,\bar{z})\sigma(0)\sigma(1)\sigma(\infty)\rangle \overset{b=i\sqrt{\frac{3}{4}}}{=} \langle\Phi_{-\frac{b}{2}}(z,\bar{z})\Phi_{-\frac{b}{2}}(0)\Phi_{-\frac{b}{2}}(1)\Phi_{-\frac{b}{2}}(\infty)\rangle.$$

The according to (178) and (183) one finds that

$$(C_{\sigma\sigma}^{\epsilon})^2 = \frac{1}{4} \implies C_{\sigma\sigma}^{\epsilon} = \frac{1}{2}.$$

Lecture 14: Minimal Models III: Tricritical Ising model, $N = 1$ SUSY CFT

The conformal symmetry can be extended in many ways. In this lecture we consider supersymmetric extension. As a motivation we consider unitary minimal model $\mathcal{M}_{4,5}$ which has been identified with the scaling limit of tricritical Ising model in [6]. It corresponds to the following Kac table



(215)

The central charge of this theory is $c = \frac{7}{10}$. The conformal dimensions are given according to (197) by

$$\Delta_{m,n} = \frac{(4m - 5n)^2 - 1}{80}.$$

The fields inside the Kac table are identified by the reflection

$$\Phi_{m,n} \sim \Phi_{5-m,4-n}.$$

We note that the field $\Phi_{4,1} = \Phi_{1,3}$ ia special: the operator product expansion of $\Phi_{4,1}$ contains only contribution of $[I] = [\Phi_{1,1}]$, as follows from the identity

$$[\Phi_{1,3}][\Phi_{4,1}] = [\Phi_{4,3}] + \cancel{[\Phi_{4,1}]} = \cancel{[\Phi_{2,3}]} + [\Phi_{4,3}] = [\Phi_{4,3}] = [\Phi_{1,1}].$$

It implies that we can construct local fields G and $\bar{G}$ of dimension $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ respectively, subject to the constraints

$$\bar{\partial} G = \partial \bar{G} = 0,$$

such that $\Phi_{1,3} = G\bar{G}$. Similarly we have

$$\left.\begin{aligned} [\Phi_{4,1}][\Phi_{2,1}] &= [\Phi_{3,1}] \\ [\Phi_{4,1}][\Phi_{3,1}] &= [\Phi_{2,1}] \end{aligned}\right\} \qquad \text{Neveu-Schwarz sector,}$$

$$\left.\begin{aligned} [\Phi_{4,1}][\Phi_{3,2}] &= [\Phi_{3,2}] \\ [\Phi_{4,1}][\Phi_{4,2}] &= [\Phi_{4,2}] \end{aligned}\right\} \qquad \text{Ramond sector.}$$

The fields $G(z)$ and $T(z) = L_{-2}I(z)$ can be regarded as generators of extended chiral symmetry[14]

$$\begin{aligned}
T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots = \frac{3\hat{c}}{4(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots, \\
T(z)G(w) &= \frac{3G(w)}{2(z-w)^2} + \frac{G'(w)}{z-w} + \dots, \\
G(z)G(w) &= \frac{2c}{3(z-w)^3} + \frac{2T(w)}{z-w} + \dots = \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w} + \dots
\end{aligned} \tag{216}$$

The algebra (220) is known as Neveu-Schwarz-Ramond algebra (NSR algebra) and appeared first in superstring theory[15]. We note that the last OPE only make sense if $G(z)$ is a Grassmann variable. An OPE of $G(z)$ with generic field must have the form

$$G(z)\mathcal{O}(w) = \sum_r \frac{G_r\mathcal{O}(w)}{(z-w)^{r+\frac{3}{2}}},$$

where $G_r\mathcal{O}(w)$ is just the notation for the new field. Then the "generators" G_r together with L_n's form the algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{8}(m^3 - m)\delta_{m,-n}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}.
\end{aligned} \tag{217}$$

Similar relations hold for antiholomorphic generators $\bar{G}_r$ and $\bar{L}_n$.

The space of fields decomposes onto the space of NS fields $\mathcal{O}_{\text{NS}}$ local with respect to $S(z)$, and the space of the Ramond fields $\mathcal{O}_{\text{R}}$ such that the correlation function

$$\langle G(z)\mathcal{O}_{\text{R}}(w,\bar{w})\dots\rangle$$

changes sign when z goes around w. In particular the fields $\Phi_{2,1}$ and $\Phi_{3,1}$ in minimal model $\mathcal{M}_{4,5}$ are NS fields and $\Phi_{3,2}$ and $\Phi_{4,2}$ are Ramond ones. We see that the indexes r, s are half-integer in NS sector and integer in R sector. One defines NS primary field by

$$T(z)\Phi(w) = \frac{\Delta\Phi(w)}{(z-w)^2} + \frac{\Phi'(w)}{z-w} + \dots \qquad G(z)\Phi(w) = \frac{\Psi(w)}{z-w} + \dots \tag{218}$$

where $\Psi = G_{-\frac{1}{2}}\Phi(w)$ is a new field. We note also that

$$G(z)\Psi(w) = \frac{2\Delta\Phi(w)}{(z-w)^2} + \frac{\Phi'(w)}{z-w} + \dots$$

[14] Last equation follows from general formula

$$\Phi_\Delta(z)\Phi_\Delta(w) \sim (z-w)^{-2\Delta}\left(1 + \frac{2\Delta(z-w)^2}{c}T(w) + \dots\right)$$

[15] We note that due to historical reasons it is customary to use the parameter $\hat{c} = \frac{2}{3}c$.

Correspondingly the primary field in Ramond sector form a doublet

$$R = \begin{pmatrix} R^{(+)} \\ R^{(-)} \end{pmatrix} \to |\Delta\rangle = \begin{pmatrix} |\Delta\rangle^{(+)} \\ |\Delta\rangle^{(-)} \end{pmatrix}$$

with the operators G_0 and $\bar{G}_0$ acting as[16]

$$G_0 \begin{pmatrix} |\Delta\rangle^{(+)} \\ |\Delta\rangle^{(-)} \end{pmatrix} = \lambda_\Delta \begin{pmatrix} 0 & e^{\frac{i\pi}{4}} \\ e^{-\frac{i\pi}{4}} & 0 \end{pmatrix} \begin{pmatrix} |\Delta\rangle^{(+)} \\ |\Delta\rangle^{(-)} \end{pmatrix}, \quad \bar{G}_0 \begin{pmatrix} |\Delta\rangle^{(+)} \\ |\Delta\rangle^{(-)} \end{pmatrix} = \lambda_\Delta \begin{pmatrix} 0 & e^{\frac{-i\pi}{4}} \\ e^{\frac{i\pi}{4}} & 0 \end{pmatrix} \begin{pmatrix} |\Delta\rangle^{(+)} \\ |\Delta\rangle^{(-)} \end{pmatrix}, \tag{219}$$

where

$$\lambda_\Delta = \sqrt{\Delta - \frac{\hat{c}}{16}} \tag{220}$$

which can be written in terms of OPE

$$G(z)R^{(\epsilon)}(w) = \frac{G_0 R^{(\epsilon)}}{(z-w)^{\frac{3}{2}}} + \frac{G_{-1}R^{(\epsilon)}(w)}{(z-w)^{\frac{1}{2}}} + \dots = \lambda_\Delta e^{\frac{i\pi\epsilon}{4}} \frac{R^{(-\epsilon)}}{(z-w)^{\frac{3}{2}}} + \frac{G_{-1}R^{(\epsilon)}(w)}{(z-w)^{\frac{1}{2}}} + \dots \tag{221}$$

One can use OPE to constraint correlation functions. Consider two-point Ward identities in NS sector

$$\begin{aligned} \langle G(\xi)\Phi_1(z_1)\Phi_2(z_2)\rangle &= \frac{\langle \Psi_1(z_1)\Phi_2(z_2)\rangle}{\xi - z_1} + \frac{\langle \Phi_1(z_1)\Psi_2(z_2)\rangle}{\xi - z_2}, \\ \langle G(\xi)\Phi_1(z_1)\Psi_2(z_2)\rangle &= \frac{\langle \Psi_1(z_1)\Psi_2(z_2)\rangle}{\xi - z_1} + \frac{2\Delta_2}{(\xi - z_2)^2}\langle \Phi_1(z_1)\Phi_2(z_2)\rangle + \frac{1}{\xi - z_2}\langle \Phi_1(z_1)\Phi_2'(z_2)\rangle. \end{aligned}$$

Using the fact that $G(\xi) \sim \frac{1}{\xi^3}$ at $\xi \to \infty$ one finds the constraint

$$\langle \Psi_1(z_1)\Phi_2(z_2)\rangle = 0, \quad \langle \Psi_1(z_1)\Psi_2(z_2)\rangle = -\langle \Phi_1(z_1)\Phi_2'(z_2)\rangle$$

One can generalize this for $n-$point correlation functions. Consider the Ward identity

$$\langle G(\xi)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle = \sum_{k=1}^{n} \frac{1}{\xi - z_k}\langle \Phi_1(z_1)\dots\Psi(z_k)\dots\Phi_n(z_n)\rangle,$$

and similar ones with $\Phi_k \to \Psi_k$. In particular, there 2 independent $3-$point correlation out of 8

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle \quad \text{and} \quad \langle \Phi_1(z_1)\Phi_2(z_2)\Psi_3(z_3)\rangle$$

Representation theory of NSR algebra (221) is very similar to the one of Virasoro algebra. It is convenient to introduce the following parametrization of the central charge and conformal dimensions of NS and R primary fields

$$\hat{c} = 1 + 2Q^2, \qquad \Delta_{\text{NS}}(\alpha) = \frac{\alpha(Q-\alpha)}{2}, \quad \Delta_{\text{R}}(\alpha) = \Delta_{\text{NS}}(\alpha) + \frac{1}{16}, \qquad Q = b + \frac{1}{b}.$$

[16] It follows from (221) that G_0 and $\bar{G}_0$ satisfy the relations

$$G_0^2 = \Delta - \frac{\hat{c}}{16}, \quad \bar{G}_0^2 = \Delta - \frac{\hat{c}}{16}, \quad \{G_0, \bar{G}_0\} = 0.$$

We denote the corresponding fields as

$$\Phi_\alpha, \quad \Psi_\alpha \quad \text{and} \quad R_\alpha^{(\epsilon)}.$$

We note that λ_Δ from (224) takes the form

$$\lambda_{\Delta(\alpha)} = \frac{i\left(\alpha - \frac{Q}{2}\right)}{\sqrt{2}}$$

The Verma module $\mathcal{V}_\Delta$ is a linear span of vectors

$$L_{-\boldsymbol{\lambda}} G_{-\boldsymbol{r}} |\Delta\rangle,$$

for ordered set $\boldsymbol{\lambda} = \lambda_1 \geq \lambda_2 \geq \ldots$ and *strictly* ordered set $\boldsymbol{r} = r_1 > r_2 > \ldots$ A singular vector is by definition a state $|\chi\rangle$ in $\mathcal{V}_\Delta$ which is killed by positive part of NSR algebra

$$L_n|\chi\rangle = G_r|\chi\rangle = 0 \quad \text{for} \quad n, r > 0.$$

A supersymmetric version of Kac theorem states that for

$$\alpha_{m,n} = -\frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2}, \quad m, n \in \mathbb{Z}_+$$

there is a singular vector at level $\frac{mn}{2}$ which appears in

$$\begin{aligned} &\text{in NS sector for} \qquad m - n \in 2\mathbb{Z}, \\ &\text{in R sector for} \qquad m - n \in 2\mathbb{Z} + 1. \end{aligned}$$

Consider first examples:

- Level $\frac{1}{2}$. The state
$$G_{-\frac{1}{2}}|\Delta\rangle$$
is a singular vector provided that $\Delta = 0 = \Delta_{\text{NS}}(0)$.
- Level 1. There are two null-vectors in Ramond sector
$$\begin{aligned} &\left(L_{-1} - \frac{2b^2}{1+2b^2} G_{-1} G_0\right)|\Delta\rangle \quad \text{for} \qquad \alpha = \alpha_{2,1} = -\frac{b}{2}, \\ &\left(L_{-1} - \frac{2b^{-2}}{1+2b^{-2}} G_{-1} G_0\right)|\Delta\rangle \quad \text{for} \qquad \alpha = \alpha_{1,2} = -\frac{1}{2b}, \end{aligned}$$
- Level $\frac{3}{2}$. Two null-vectors in NS sector
$$\begin{aligned} &\left(L_{-1} G_{-\frac{1}{2}} + b^2 G_{-\frac{3}{2}}\right)|\Delta\rangle \quad \text{for} \quad \alpha = \alpha_{3,1} = -b, \\ &\left(L_{-1} G_{-\frac{1}{2}} + b^{-2} G_{-\frac{3}{2}}\right)|\Delta\rangle \quad \text{for} \quad \alpha = \alpha_{1,3} = -b^{-1}. \end{aligned}$$

- Level 2. Two null vectors in R sector
$$\left(L_{-1}^2+\frac{2b^2}{2}L_{-2}-\frac{4b^2}{1+4b^2}L_{-1}G_{-1}G_0+\frac{b^2(1-6b^2)}{1+4b^2}G_{-2}G_0\right)|\Delta\rangle\quad\text{for}\quad \alpha=\alpha_{4,1}=-\frac{3b}{2},$$
$$\left(L_{-1}^2+\frac{2b^{-2}}{2}L_{-2}-\frac{4b^{-2}}{1+4b^{-2}}L_{-1}G_{-1}G_0+\frac{b^{-2}(1-6b^{-2})}{1+4b^{-2}}G_{-2}G_0\right)|\Delta\rangle\quad\text{for}\quad \alpha=\alpha_{1,4}=-\frac{3b^{-1}}{2},$$
  and one null-vector in NS sector
$$\left(L_{-1}^2+\frac{Q^2}{2}L_{-2}-G_{-\frac{3}{2}}G_{-\frac{1}{2}}\right)|\Delta\rangle\quad\text{for}\quad \alpha=\alpha_{2,2}=-\frac{Q}{2},$$

- Level $\frac{5}{2}$. Two null vectors in NS sector
$$\left(L_{-1}^2G_{-\frac{1}{2}}+2b^2L_{-2}G_{-\frac{1}{2}}+3b^2L_{-1}G_{-\frac{3}{2}}-b^2(1-6b^2)G_{-\frac{5}{2}}\right)|\Delta\rangle\quad\text{for}\quad \alpha=\alpha_{5,1}=-2b,$$
$$\left(L_{-1}^2G_{-\frac{1}{2}}+2b^{-2}L_{-2}G_{-\frac{1}{2}}+3b^{-2}L_{-1}G_{-\frac{3}{2}}-b^{-2}(1-6b^{-2})G_{-\frac{5}{2}}\right)|\Delta\rangle\quad\text{for}\quad \alpha=\alpha_{1,5}=-2b^{-1}.$$

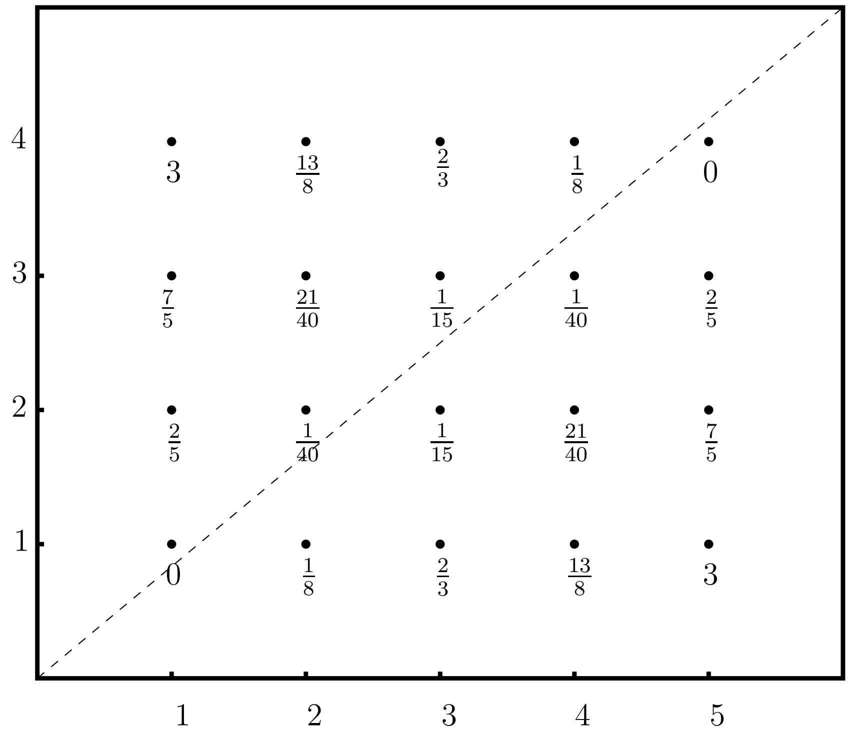
etc

It is important to derive an analog of BPZ differential equation (176) for supersymmetric case. The simplest equation arises for Ramond field degenerate at level 1
$$\frac{1+2b^2}{2b^2}\partial R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)=G_{-1}G_0R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z). \tag{222}$$
Consider the following correlation functions (here we follow [7])
$$G^{(\epsilon)}(z,\bar z)\overset{\text{def}}{=}\langle R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle \tag{223}$$
and
$$H^{(\epsilon)}(z,\bar z)\overset{\text{def}}{=}\langle G_0R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Psi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle\overset{(223)}{=}-i\frac{1+2b^2}{2\sqrt{2}b}\langle R^{(-\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Psi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle \tag{224}$$
In order to rewrite the r.h.s. of (226) we consider (see (222) and (225))
$$\frac{\overbrace{\Delta(-\frac{b}{2})-\frac{\hat c}{16}}^{-\frac{(1+2b^2)^2}{8b^2}}}{2z}\langle R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle+\langle G_{-1}G_0R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle=$$
$$=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}\sqrt{\frac{\xi}{z(\xi-z)}}\langle G(\xi)G_0R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle d\xi=$$
$$=-\frac{1}{2\pi i}\left(\oint_{\mathcal{C}_0}+\oint_{\mathcal{C}_1}\right)\sqrt{\frac{\xi}{z(\xi-z)}}\langle G(\xi)G_0R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle d\xi=$$
$$=\frac{i}{z}\langle G_0R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)G_0R^{(\epsilon)}_{\alpha_1}(0)\Phi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle-\frac{1}{\sqrt{z(1-z)}}\langle G_0R^{(\epsilon)}_{-\frac{b}{2}}(z,\bar z)R^{(\epsilon)}_{\alpha_1}(0)\Psi_{\alpha_2}(1)\Phi_{\alpha_3}(\infty)\rangle$$
In view of (227), (228) and (223) this relation can be rewritten as
$$\frac{1+2b^2}{2b^2}\left(\partial-\frac{1+2b^2}{8z}\right)G^{(\epsilon)}(z,\bar z)$$

Coming back to $\mathcal{M}_{4,5}$ model

Lecture 15: Minimal models IV: Potts model, $W-$algebras, parafermionic CFT

We consider next unitary minimal model $\mathcal{M}_{5,6}$ which is known to be related to $\mathbb{Z}_3$ Potts model [8]. It has the Kac table



(225)

The central charge of this theory is $c = \frac{4}{5}$ and Kac dimensions are

$$\Delta_{m,n} = \frac{(5m-6n)^2 - 1}{120}.$$

Similarly to the previous lecture the field $\Phi_{5,1} = \Phi_{1,4}$ can be decomposed as

$$\Phi_{5,1}(z,\bar{z}) = W(z)\bar{W}(\bar{z})$$

The current $W(z)$ of spin 3 extends the Virasoro algebra[17]

$$\begin{aligned} T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \ldots, \\ T(Z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{W'(w)}{z-w} + \ldots, \end{aligned}$$

First OPE is just for Virasoro algebra, second states that $W(z)$ is a primary field of dimension 3, while the third one

$$W(z)W(w) = \frac{c}{3(z-w)^6} + \frac{\lambda_1 T(w)}{(z-w)^4} + \frac{\lambda_2 T'(w)}{(z-w)^3} + \frac{\lambda_3 T''(w) + \lambda_4 \Lambda(w)}{(z-w)^2} + \frac{\lambda_5 T'''(w) + \lambda_6 \Lambda'(w)}{z-w} + \ldots \tag{226}$$

[17] We have $c = \frac{4}{5}$, but we can keep c arbitrary in discussions below

is an OPE expansion of the field with $\Delta = 3$ into identity operator. Here $\Lambda(z)$ is quasi-primary field which appears in OPE

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \left(\Lambda(w) + \frac{3}{10}T''(w)\right) + \ldots$$

i.e.

$$\Lambda(w) = \left(L_{-2} - \frac{3}{10}L_{-1}^2\right)T(w) = \left(L_{-2} - \frac{3}{10}L_{-1}^2\right)L_{-2}I(w). \tag{227}$$

Let us compute the coefficients λ_k in (230). We can do it exactly as before when we studied conformal properties of OPE. Namely, we act on both sides of (230) by

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_z+\mathcal{C}_z}(\xi - w)^{n+1}T(\xi)d\xi,$$

which can be interpreted as

$$\mathcal{L}_n = (z-w)^{n+1}\partial_z + 3(n+1)(z-w)^n \quad \text{or} \quad L_n.$$

Taking $n = 1$ one obtains (we use that $L_1\Lambda = 0$)

$$\frac{2\lambda_1 T(w)}{(z-w)^3} + \frac{3\lambda_2 T'(w)}{(z-w)^2} + \frac{4\lambda_3 T''(w) + 4\lambda_4\Lambda(w)}{z-w} + \cdots = \frac{4\lambda_2 T(w)}{(z-w)^3} + \frac{10\lambda_3 T'(w)}{(z-w)^2} + \frac{18\lambda_5 T''(w) + 8\lambda_6\Lambda(w)}{z-w} + \ldots,$$

which implies

$$\lambda_2 = \frac{\lambda_1}{2}, \quad \lambda_3 = \frac{3\lambda_2}{10} = \frac{3\lambda_1}{20}, \quad \lambda_5 = \frac{2\lambda_3}{18} = \frac{\lambda_1}{30}, \quad \lambda_6 = \frac{\lambda_4}{2}.$$

While taking $n = 2$ we find

$$\frac{c}{(z-w)^4} + \frac{5\lambda_1 T(w)}{(z-w)^2} + \cdots = \frac{\lambda_1 L_2 T(w)}{(z-w)^4} + \frac{\lambda_3 L_2 T''(w) + \lambda_4 L_2\Lambda(w)}{(z-w)^2} + \ldots$$

which fixes

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_4 = \frac{32}{5c+22}.$$

Altogether we obtain [9]

$$W(z)W(w) = \frac{c}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{T'(w)}{(z-w)^3} + \frac{\frac{3}{10}T''(w) + \frac{32}{5c+22}\Lambda(w)}{(z-w)^2} + \frac{\frac{1}{15}T'''(w) + \frac{16}{5c+22}\Lambda'(w)}{z-w} + \ldots$$

In terms of the modes we obtain

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \\
&[L_m, W_n] = (2m-n)W_{m+n}, \\
&[W_m, W_n] = \frac{c}{3\cdot 5!}m(m^2-1)(m^2-4)\delta_{m,-n} + \frac{16}{5c+22}(m-n)\Lambda_{m+n} + \\
&\qquad + (m-n)\left(\frac{(m+n+2)(m+n+3)}{15} - \frac{(m+2)(n+2)}{6}\right)L_{m+n}.
\end{aligned} \tag{228}$$

We note that Λ_m is not new and expressed in terms of generator L_n as follows from the definition (231)

$$\Lambda(z)+\frac{3}{10}T''(z)=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}\frac{T(\xi)T(x)}{\xi-z}d\xi.$$

After simple calculation one obtains

$$\Lambda_m=\sum_k :L_kL_{m-k}:+\frac{1}{5}x_mL_m,\quad \text{where}\quad x_{2l}=(1+l)(1-l),\quad x_{2l+1}=(2+l)(1-l).$$

Now consider W primary field

$$\begin{aligned}T(\xi)\Phi(z)&=\frac{\Delta\Phi(z)}{(\xi-z)^2}+\frac{L_{-1}\Phi}{\xi-z}+\dots,\\ W(\xi)\Phi(z)&=\frac{w\Phi(z)}{(\xi-z)^3}+\frac{W_{-1}\Phi}{(\xi-z)^2}++\frac{W_{-2}\Phi}{\xi-z}+\dots.\end{aligned}$$

We stress that while we have $L_1\Phi=\Phi'$ for Virasoro descendant, W-descendants $W_{-1}\Phi$ and $W_{-2}\Phi$ are new fields which do not have immediate relation to Φ. Consider Ward identity for $n-$point correlation function of primary fields

$$\begin{aligned}\langle W(\xi)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle=\sum_{k=1}^n\Bigg(&\frac{w_k}{(\xi-z_k)^3}\langle\Phi_1(z_1)\dots\Phi_n(z_n)\rangle+\\ &+\frac{1}{(\xi-z_k)^2}\langle\Phi_1(z_1)\dots W_{-1}\Phi_k(z_k)\dots\Phi_n(z_n)\rangle+\frac{1}{\xi-z_k}\langle\Phi_1(z_1)\dots W_{-2}\Phi_k(z_k)\dots\Phi_n(z_n)\rangle\Bigg).\end{aligned}$$

In the right hand side it involves $2n+1$ different correlation functions restricted by 5 projective Ward identities

$$\begin{aligned}W(\xi)\sim\frac{1}{\xi^6}\implies\sum_{k=1}^n\Big(&w_k\frac{l(l-1)}{2}z_k^{l-2}\langle\Phi_1(z_1)\dots\Phi_n(z_n)\rangle+lz_k^{l-1}\langle\Phi_1(z_1)\dots W_{-1}\Phi_k(z_k)\dots\Phi_n(z_n)\rangle+\\ &+z_k^l\langle\Phi_1(z_1)\dots W_{-2}\Phi_k(z_k)\dots\Phi_n(z_n)\rangle\Big)=0\quad\text{for}\quad l=0,1,2,3,4.\end{aligned}$$

- In the case of one-point function it immediately implies that

$$\langle W_{-1}\Phi\rangle=0,\quad\langle W_{-2}\Phi\rangle=0\quad\text{and}\quad w=0.$$

- For two-point function one has a system of equations

$$\begin{pmatrix}0&0&0&1&1\\0&1&1&z_1&z_2\\w_1+w_2&2z_1&2z_2&z_1^2&z_2^2\\3(w_1z_1+w_2z_2)&3z_1^2&3z_2^2&z_1^3&z_2^3\\6(w_1z_1^2+w_2z_2^2)&4z_1^3&4z_2^3&z_1^4&z_2^4\end{pmatrix}\begin{pmatrix}\langle\Phi_1\Phi_2\rangle\\\langle W_{-1}\Phi_1\Phi_2\rangle\\\langle\Phi_1W_{-1}\Phi_2\rangle\\\langle W_{-2}\Phi_1\Phi_2\rangle\\\langle\Phi_1W_{-2}\Phi_2\rangle\end{pmatrix}=0,$$

  which has a solution provided that

$$\det\begin{pmatrix}0&0&0&1&1\\0&1&1&z_1&z_2\\w_1+w_2&2z_1&2z_2&z_1^2&z_2^2\\3(w_1z_1+w_2z_2)&3z_1^2&3z_2^2&z_1^3&z_2^3\\6(w_1z_1^2+w_2z_2^2)&4z_1^3&4z_2^3&z_1^4&z_2^4\end{pmatrix}=0\implies(z_1-z_2)^6(w_1+w_2)=0,$$

and hence the two-point function takes the form

$$\langle\Phi_1(z_1)\Phi_2(z_2)\rangle \sim \frac{\delta_{\Delta_1,\Delta_2}\delta_{w_1,-w_2}}{(z_1-z_2)^{2\Delta_1}}$$

- For three-point function one has 7 functions minus 5 constraints which means that everything can be expressed as a linear combination of

$$\langle\Phi_1\Phi_2\Phi_3\rangle \quad \text{and} \quad \langle\Phi_1\Phi_2 W_{-1}\Phi_3\rangle. \tag{229}$$

Here one comes to an important difference compared to Virasoro case. In Virasoro case we had the statement that correlation functions of descendant field can always be expressed from correlation functions of primary fields by application of certain differential operators. In W case this is no longer true. For example one has two three-point functions (233) which are not related by kinematics. One can show more generic statement that any three-point function of W descendant fields can be expressed through the basic ones[18]

$$\langle\Phi_1\Phi_2 W_{-1}^k\Phi_3\rangle$$

Some simplifications appear for degenerate fields. The structure of representation theory of W-algebra (232) is very similar to the one of Virasoro algebra. The Verma module $\mathcal{V}_{\Delta,w}$ is spanned by the vectors

$$W_{-\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\Delta,w\rangle: \quad L_n|\Delta,w\rangle = W_n|\Delta,w\rangle = 0 \text{ for } n>0, \quad L_0|\Delta,w\rangle = \Delta|\Delta,w\rangle,\ W_0|\Delta,w\rangle = w|\Delta,w\rangle,$$

for two independent partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. A singular vector $|\chi\rangle \in \mathcal{V}_{\Delta,w}$ is by definition as state killed by positive part of W-algebra

$$L_n|\chi\rangle = W_n|\chi\rangle = 0 \text{ for } n>0.$$

Consider the simplest example of a singular vector at level 1

$$|\chi\rangle = (W_{-1}+\xi L_{-1})\,|\Delta,w\rangle.$$

We should impose

$$\begin{aligned} &L_1|\chi\rangle = 0 \implies (3w+2\xi\Delta)|\Delta,w\rangle = 0,\\ &W_1|\chi\rangle = 0 \implies \left(\frac{32}{5c+22}\Big(\Delta^2+\frac{1}{5}\Delta\Big)-\frac{1}{5}\Delta+3\xi w\right)|\Delta,w\rangle = 0 \end{aligned}$$

which implies the constraint between quantum numbers Δ and w

$$9w^2 = 2\Delta^2\left(\frac{32}{5c+22}\Big(\Delta+\frac{1}{5}\Big)-\frac{1}{5}\right) \tag{230}$$

If such a field present in three-point correlation function then one can express correlation function of any descendant fields from the one of primary fields.

It is convenient to introduce "Toda" like notations. Let e_1 and e_2 be the simple roots of $\mathfrak{sl}(3)$, that is their Gram matrix is

$$(e_i\cdot e_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

[18] Try to argue this from Ward identities.

the Weyl vector ρ

$$\rho = e_1 + e_2,$$

fundamental weights ω_1 and ω_2

$$(\omega_i, e_j) = \delta_{ij} \implies \omega_1 = \frac{2}{3}e_1 + \frac{1}{3}e_2 \quad \omega_2 = \frac{2}{3}e_2 + \frac{1}{3}e_1,$$

and weights of fundamental representation

$$h_1 = \omega_1, \quad h_2 = \omega_1 - e_1, \quad h_3 = \omega_1 - e_1 - e_2 \implies \sum_{k=1}^{3} h_k = 0, \quad (h_i \cdot h_j) = \delta_{ij} - \frac{1}{3}.$$

Then we use parametrization of central charge and quantum numbers Δ and w

$$c = 2+12(\mathcal{Q},\mathcal{Q}) = 2+24Q^2, \quad \Delta = \frac{(\alpha, 2\mathcal{Q}-\alpha)}{2}, \quad w(\alpha) = i\sqrt{\frac{48}{5c+22}}(\alpha-\mathcal{Q},h_1)(\alpha-\mathcal{Q},h_2)(\alpha-\mathcal{Q},h_3),$$

where

$$\mathcal{Q} = Q\rho, \qquad Q = b + \frac{1}{b}.$$

This wild parametrization provides a solution to (234) if

$$\alpha = \varkappa\omega_1 \quad \text{or} \quad \alpha = \varkappa\omega_2.$$

We should call such field *semi-degenerate*. It is clear that $n-$point correlation function with $(n-2)$ semi-degenerate fields, for example

$$\langle \Phi_{\alpha_1}(z_1)\Phi_{\varkappa_1\omega_1}(z_2)\dots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle, \tag{231}$$

can be computed using OPE as

$$\begin{aligned}
&\langle \Phi_{\alpha_1}(z_1)\Phi_{\varkappa_1\omega_1}(z_2)\dots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle = \\
&\quad = \sum_{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}} C^{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}}_{\alpha_1,\varkappa_1\omega_1}(z_1-z_2)\langle W_{-\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}\Phi_\alpha(z_2)\dots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle = \\
&= \sum_{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}} C^{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}}_{\alpha_1,\varkappa_1\omega_1}(z_1-z_2)\sum_{\beta,\boldsymbol{\nu},\boldsymbol{\sigma}} C^{\beta,\boldsymbol{\nu},\boldsymbol{\sigma}}_{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu},\varkappa_2\omega_2}(z_2-z_3)\langle W_{-\boldsymbol{\sigma}}L_{-\boldsymbol{\nu}}\Phi_\beta(z_2)\dots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle = \dots,
\end{aligned}$$

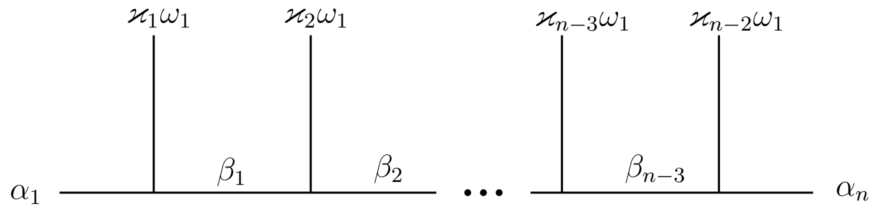
The structure constant at each step is related by lowering the index to the three-point correlation function

$$\langle \big(W_{-\boldsymbol{\sigma}}L_{-\boldsymbol{\nu}}\Phi_{\alpha_1}(z_1)\big)\big(W_{-\boldsymbol{\lambda}}L_{-\boldsymbol{\mu}}\Phi_{\alpha_2}(z_2)\Phi_{\varkappa\omega_1}(z_3)\big)\rangle.$$

Since it contains the semi-degenerate field it can be reduced to differential operator acting on correlation function of primary fields

$$\langle \Phi_{\alpha_1}(z_1,\bar{z}_1)\Phi_{\alpha_2}(z_2,\bar{z}_2)\Phi_{\varkappa\omega_1}(z_3,\bar{z}_3)\rangle = \frac{C(\alpha_1,\alpha_2,\varkappa\omega_1)}{\prod_{i<j}|z_i-z_j|^{2\gamma_{ij}}}.$$

The correlation function (235) belongs to the class of computable correlation functions, just as in Virasoro case. In particular, the conformal block



is completely determined by kinematics.

Now, we consider another aspect of the theory $\mathcal{M}_{5,6}$. Namely, consider the field $\Phi_{3,1} = \Phi_{3,4}$ with conformal dimension $\Delta = \frac{2}{3}$. Its OPE has the form

$$\Phi_{3,1}\Phi_{3,1} = [\Phi_{1,1}] + [\Phi_{3,1}] + [\Phi_{5,1}]$$

It has been noticed by Cardy [10, 11] that one can build self-consistent theory assuming that the field $\Phi_{3,1}$ enters the theory with multiplicity 2. One can choose a basis of these two fields which admits holomorphic factorization

$$\mathbf{\Psi}(z)\bar{\mathbf{\Psi}}(\bar{z}) \quad \text{and} \quad \mathbf{\Psi}^+(z)\bar{\mathbf{\Psi}}^+(\bar{z}),$$

where $\mathbf{\Psi}(z)$ and $\mathbf{\Psi}^+(z)$ are the so called $\mathbb{Z}_3$ *parafermionic* fields [12]

$$\begin{gathered}\mathbf{\Psi}(z)\mathbf{\Psi}(w) = \frac{C}{(z-w)^{\frac{2}{3}}}\left(\mathbf{\Psi}^+(w) + \dots\right), \qquad \mathbf{\Psi}^+(z)\mathbf{\Psi}^+(w) = \frac{C}{(z-w)^{\frac{2}{3}}}\left(\mathbf{\Psi}(w) + \dots\right), \\ \mathbf{\Psi}(z)\mathbf{\Psi}^+(w) = \frac{1}{(z-w)^{\frac{4}{3}}}\left(1 + \frac{5}{3}T(w)(z-w)^2 + \dots\right)\end{gathered} \tag{232}$$

Here the factor $\frac{5}{3} = \frac{2\Delta}{c}$ is universal and the structure constant has to be fixed from the associativity condition of the operator algebra. We note that the form of parafermionic algebra (236) is only consistent with the fractional statistic for the field $\mathbf{\Psi}(z)$

$$\mathbf{\Psi}(z)\mathbf{\Psi}(w) = e^{\frac{2i\pi}{3}}\mathbf{\Psi}(w)\mathbf{\Psi}(z).$$

In order to find C we consider $N-$point correlation functions of $\mathbf{\Psi}(z)$. We note that the algebra (236) implies that the correlation function is only non-zero if n is divisible by 3

$$\langle\mathbf{\Psi}(z_1)\dots\mathbf{\Psi}(z_{3n})\rangle = \frac{P_n^{(3)}(z_1,\dots,z_{3n})}{\prod_{i<j}(z_i-z_j)^{\frac{2}{3}}}, \tag{233}$$

where $P_n^{(3)}(z_1,\dots,z_{3n})$ is the symmetric homogeneous polynomial which satisfies the following three properties

- $P_n^{(3)}(\lambda z_1,\dots,\lambda z_{3n}) = \lambda^{3n(n-1)}P_n(z_1,\dots,z_{3n})$
- $P_n^{(3)}(z_1,\dots,z_{2n}) = z_1^{2(n-1)} + \dots$ at $z_1 \to \infty$
- $P_n^{(3)}(z_1,\dots,z_{3(n-1)},x,x,x) = C^2\prod_{k=1}^{3(n-1)}(z_k-x)^2P_{n-1}^{(3)}(z_1,\dots,z_{3(n-1)})$

It can be proven that the polynomial with such properties exist and unique

$$P_n^{(3)}(z_1,\dots,z_{3n}) = C^{2(n-1)}\mathrm{Sym}_{\boldsymbol{z}}\left[\prod_{i<j\in I}(z_i-z_j)^2\prod_{i<j\in II}(z_i-z_j)^2\prod_{i<j\in III}(z_i-z_j)^2\right],$$

where I, II and III are three groups of n points.

We note that for Maiorana fermion $\psi(z)$ we have the formula similar to (237)

$$\langle\psi(z_1)\dots\psi(z_{2n})\rangle = \frac{2^{1-n}P_n^{(2)}(z_1,\dots,z_{2n})}{\prod_{i<j}(z_i-z_j)},$$

where

$$P_n^{(2)}(z_1,\dots,z_{2n}) = \mathrm{Sym}_{\boldsymbol{z}}\left[\prod_{i<j\in I}(z_i-z_j)^2\prod_{i<j\in II}(z_i-z_j)^2\right],$$

This formula is a consequence of the bosonization map

$$\psi(z) = \frac{1}{\sqrt{2}}(\psi(z)+\psi^*(z)) = \frac{1}{\sqrt{2}}\left(e^{i\varphi(z)}+e^{-i\varphi(z)}\right) \tag{234}$$

Generalization of (238) is straightforward

$$\boldsymbol{\Psi}(z) = \frac{1}{\sqrt{3}}\left(e^{i(\boldsymbol{h}_1\cdot\boldsymbol{\varphi}(z))}+e^{i(\boldsymbol{h}_2\cdot\boldsymbol{\varphi}(z))}+e^{i(\boldsymbol{h}_3\cdot\boldsymbol{\varphi}(z))}\right),$$

where $\boldsymbol{\varphi}(z) = (\varphi_1(z),\varphi_2(z))$ is the two-component bosonic field and $\boldsymbol{h}_k$ are proportional to the weights of fundamental representation of $\mathfrak{sl}(3)$, that is 3 linearly dependent vectors in $\mathbb{R}^2$

$$(\boldsymbol{h}_i\cdot\boldsymbol{h}_j) = 2\left(\delta_{ij}-\frac{1}{3}\right) \quad\Longrightarrow\quad \sum_{k=1}^{3}\boldsymbol{h}_k = 0.$$

Probs:

1.

Lecture 16: Friedan Qiu and Shenker theorem

The theorem [5] states that the Verma module $\mathcal{V}_\Delta$ does not have vectors of negative norm only in two cases:

- For $\Delta\geq 0$ and $c\geq 1$
- For unitary minimal model

$$c=1-\frac{6}{p(p+1)},\quad \Delta_{m,n}=\frac{(mp-n(p+1))^2-1}{4p(p+1)}\quad\text{for}\quad 0<n<m<p+1.$$

The proof substantially uses the Kac determinant formula

$$\det\Gamma^{(N)}\sim\prod_{m,n}(\Delta-\Delta_{m,n})^{p(N-mn)} \tag{235}$$

We remind the meaning of (239). Consider generic state in the Verma module $\mathcal{V}_\Delta$ at level N

$$|\rho\rangle=\sum_{|\boldsymbol{\lambda}|=\rho}C_{\boldsymbol{\lambda}}L_{-\boldsymbol{\lambda}}|\Delta\rangle.$$

Then its norm is

$$\langle\rho|\rho\rangle=\sum_{\boldsymbol{\lambda},\boldsymbol{\mu}}\langle\Delta|L_{\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\Delta\rangle C_{\boldsymbol{\lambda}}C_{\boldsymbol{\mu}}=\Gamma^{(N)}_{\boldsymbol{\lambda},\boldsymbol{\mu}}C_{\boldsymbol{\lambda}}C_{\boldsymbol{\mu}}. \tag{236}$$

We know that at $\Delta=\Delta_{m,n}$ there is a singular vector at level mn

$$|\chi_{m,n}\rangle=D_{m,n}|\Delta_{m,n}\rangle\quad\text{where}\quad D_{m,n}=L_{-1}^{mn}+c_1(b)L_{-2}L_{-1}^{mn-2}+c_2(b)L_{-3}L_{-1}^{mn-2}+\dots, \tag{237}$$

with

$$c_1=\frac{mn}{6}\left((m^2-1)b^2+(n^2-1)b^{-2}\right)\quad\text{etc.}$$

Moreover, for any two partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ the following holds $|\boldsymbol{\lambda}|=|\boldsymbol{\nu}|+mn$

$$\langle\Delta|L_{\boldsymbol{\lambda}}L_{-\boldsymbol{\nu}}D_{m,n}|\Delta\rangle\sim(\Delta-\Delta_{m,n}),$$

which implies that

$$\det\Gamma^{(mn+|\boldsymbol{\mu}|)}\sim(\Delta-\Delta_{m,n})^{|\boldsymbol{\nu}|}.$$

Thus the product in (239) exhausts all the required zeroes with the correct multiplicities. It remains to show that it gives the correct asymptotic at $\Delta\to\infty$. It is clear that the degree of $\langle\Delta|L_{\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\Delta\rangle$ in Δ is not greater than $l(\boldsymbol{\lambda})$ and $l(\boldsymbol{\mu})$ and that $\langle\Delta|L_{\boldsymbol{\lambda}}L_{-\boldsymbol{\lambda}}|\Delta\rangle\sim\Delta^{l(\boldsymbol{\lambda})}$. It implies that

$$\det\Gamma^{(N)}\sim\Delta^{\sum_{|\boldsymbol{\lambda}|=N}l(\boldsymbol{\lambda})}$$

and hence we expect the combinatorial fact

$$\sum_{|\boldsymbol{\lambda}|=N}l(\boldsymbol{\lambda})=\sum_{m,n}p(N-mn),$$

which can be proven by elementary methods. Namely representing

$$\boldsymbol{\lambda} = \{\underbrace{1, \dots, 1}_{n_1}, \underbrace{\{2, \dots, 2}_{n_2}\}, \dots\}$$

we have

$$\sum_{|\boldsymbol{\lambda}|=N} l(|\boldsymbol{\lambda}|) = \sum_{\sum_k kn_k=N} \sum_k n_k$$

Now we come to the first part of the theorem. It follows from three simple facts:

Fact 1:

$$\langle\Delta|L_1L_{-1}|\Delta\rangle = 2\Delta\langle\Delta|\Delta\rangle \implies \Delta \geq 0,$$
$$\langle\Delta|L_nL_{-n}|\Delta\rangle = \Big(2n\Delta + \frac{c}{12}(n^3 - n)\Big)\langle\Delta|\Delta\rangle \implies c \geq 0.$$

Fact 2: The Kac determinant $\det G_N$ is positive for $\Delta > 0$ and $c \geq 1$. Indeed, for $c > 25$ all Kac values are negative, while for $1 < c < 25$ we have $\Delta_{m,n} = \Delta^*_{n,m}$ for $m \neq n$ and $\Delta_{m,m} < 0$.

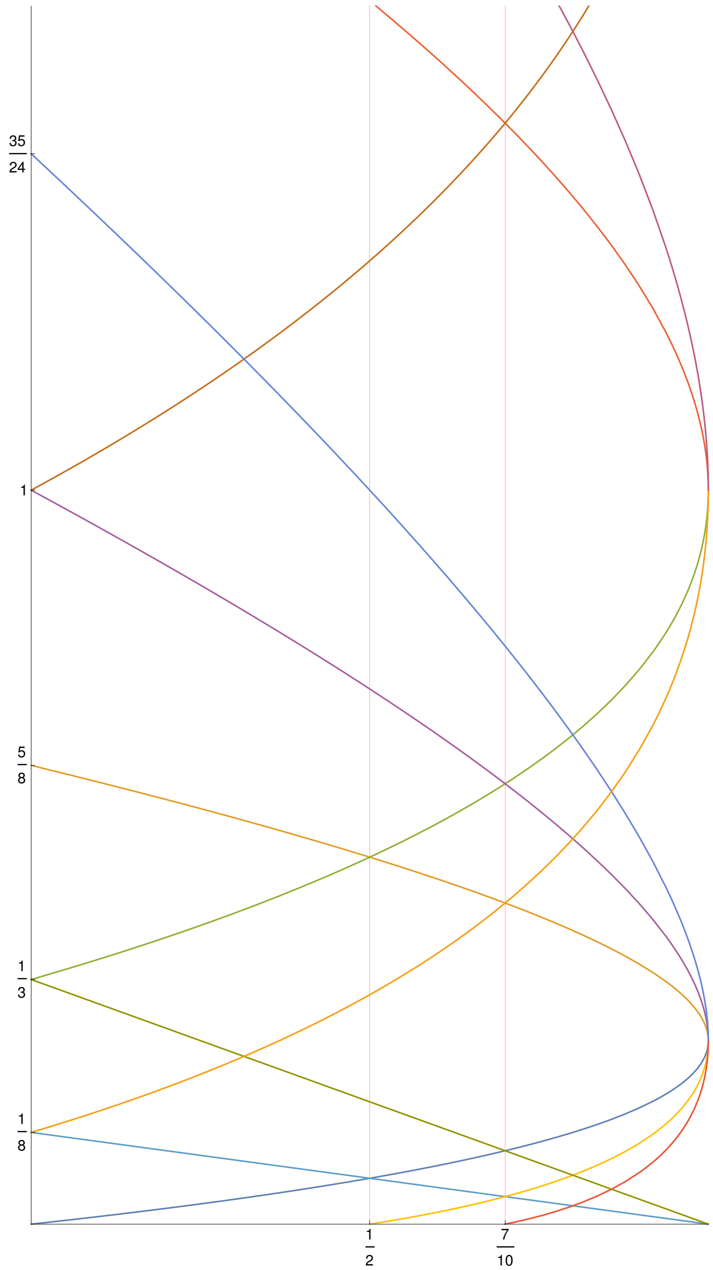
Fact 3: The Shapovalov form is positive in the limit $\Delta \to \infty$. Indeed, consider the generic state $|\rho\rangle$. Its norm is given by (240). In the limit $\Delta \to \infty$ only the states with maximal $l(\boldsymbol{\lambda})$ will contribute, but for these states we have

$$\Gamma_{\boldsymbol{\mu},\boldsymbol{\lambda}} = \Delta^{l(\boldsymbol{\lambda})}\left(\xi\delta_{\boldsymbol{\mu},\boldsymbol{\lambda}} + O\Big(\frac{1}{\Delta}\Big)\right),$$

for some positive ξ.

From these three facts we understand that the Gramm matrix Γ is positive definite for large $\Delta > 0$ and for $c > 1$ its determinant is strictly positive. It means that it can not become negative since in that case it should cross 0, but this does not happen.

Now we come to the second part of the theorem. The idea is similar, using Kac determinant formula (239) we will eliminate domains in the semi-strip $(\Delta \geq 0, 0 \leq c \leq 1)$ level by level.



35/24
1
5/8
1/3
1/8
1/2
7/10

Lecture 16: Modular bootstrap I: minimal models, ADE classification, Verlinde formula

So far, we have discussed CFT's on the sphere, but for various reasons, especially for the purposes of string theory, it is worth to consider CFT on arbitrary Riemann surface even with boundary. In this lecture we consider the simplest case of the torus.

The easiest way to obtain the torus is to cut a piece of a cylinder and identify its ends. One can obtain the cylinder as a map from the plane (without two points). We use cylinder coordinate frame as in (79) with new coordinates t and σ related to the complex coordinate z by exponential map

$$z = Re^{-\frac{iu}{R}}, \quad u = \sigma + it \implies ds^2 = e^{\frac{2t}{R}}\left(dt^2 + d\sigma^2\right).$$

Now in fact there are two options to use the Hamitonian formalism. The one, which we already used, is known as the radial quantization. We take $t \in [-\infty, \infty]$ as a time coordinate and $\sigma \in [0, 2\pi R]$ a space one. It has to singular points $z = 0$ and $z = \infty$. Then the correlation function of local fields is related to the Green function as follows

$$\langle \mathcal{O}_1(\sigma_1, t_1) \ldots \mathcal{O}_N(\sigma_N, t_N)\rangle = \frac{\langle 0|\mathcal{T}_t\left[\mathcal{O}_1(\sigma_1, t_1) \ldots \mathcal{O}_N(\sigma_N, t_N)\right]|0\rangle}{\langle 0|0\rangle}, \tag{238}$$

where the Hamiltonian H has the form

$$H = \frac{1}{2\pi R}\int_0^{2\pi R} T_{tt} d\sigma = \frac{1}{R}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right). \tag{239}$$

We note that the momentum operator

$$P = \frac{1}{R}(L_0 - \bar{L}_0)$$

should have quantized eigenvalues n/R.

But one can also consider the same system in the framework of angular quantization. Namely, we interpret t as the spatial coordinate which spans the whole real line and σ as the time. The angular nature of σ manifests itself in different compared to (242) representation for the correlation functions

$$\langle \mathcal{O}_1(\sigma_1, t_1) \ldots \mathcal{O}_N(\sigma_N, t_N)\rangle = \frac{\mathrm{Tr}\Big[\mathcal{T}_\sigma\left[\mathcal{O}_1(\sigma_1, t_1) \ldots \mathcal{O}_N(\sigma_N, t_N)\right] e^{-2\pi R H'}\Big]_{\mathcal{H}'}}{\mathrm{Tr}\Big[e^{-2\pi R H'}\Big]_{\mathcal{H}'}},$$

where H' is the angular Hamiltonian

$$H' = \frac{1}{2\pi}\int_{-\infty}^{\infty} T_{\sigma\sigma} dt,$$

and the trace goes over the Hilbert space $\mathcal{H}'$ of angular Hamiltonian.

In order to obtain the theory on the torus, one has to compactify $t \sim t + 2\pi R'$ with some radius R'. It can be interpreted as either the system of size R in radial quantization (242) at finite temperature $1/R'$

$$\langle \mathcal{O}_1(\sigma_1, t_1) \ldots \mathcal{O}_N(\sigma_N, t_N)\rangle_{\text{torus}} = \frac{\mathrm{Tr}\Big[\mathcal{T}_\tau\left[\mathcal{O}_1(\sigma_1, t_1) \ldots \mathcal{O}_N(\sigma_N, t_N)\right] e^{-2\pi R' H}\Big]_{\mathcal{H}}}{\mathrm{Tr}\Big[e^{-2\pi R' H}\Big]_{\mathcal{H}}},$$

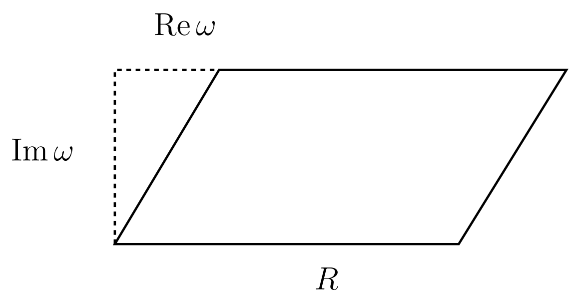
or as the system at finite temperature $1/R$ in angular quantization in finite volume R

$$\langle \mathcal{O}_1(\sigma_1,t_1)\dots\mathcal{O}_N(\sigma_N,t_N)\rangle_{\text{torus}} = \frac{\operatorname{Tr}\Big[\mathcal{T}_\sigma\left[\mathcal{O}_1(\sigma_1,t_1)\dots\mathcal{O}_N(\sigma_N,t_N)\right]e^{-2\pi RH'}\Big]_{\mathcal{H}'}}{\operatorname{Tr}\Big[e^{-2\pi RH'}\Big]_{\mathcal{H}'}}.$$

The result should be independent on a way we arrive to it, but puts non-trivial constraints on spectra and fusion coefficients. This is known under the name of modular bootstrap. We will study it for the partition function

$$Z(R,R') \overset{\text{def}}{=} \operatorname{Tr}\Big[e^{-2\pi RH'}\Big]_{\mathcal{H}'} = \operatorname{Tr}\Big[e^{-2\pi R'H}\Big]_{\mathcal{H}} \implies Z(R,R') = Z(R',R).$$

In fact, it will be more convenient to consider general torus with complex moduli as shown on the picture



Noticing that the operators H and P commute and that $e^{2i\pi aP}$ translates through the distance a, one can write in this case

$$\langle\dots\rangle_{\text{torus}} = \frac{\operatorname{Tr}\Big[\dots e^{-2\pi(\operatorname{Im}\omega\, H - i\operatorname{Re}\omega\, P)}\Big]_{\mathcal{H}}}{\operatorname{Tr}\Big[e^{-2\pi(\operatorname{Im}\omega\, H - i\operatorname{Re}\omega\, P)}\Big]_{\mathcal{H}}} = \frac{\operatorname{Tr}\Big[\dots q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\Big]_{\mathcal{H}}}{\operatorname{Tr}\Big[q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\Big]_{\mathcal{H}}}, \tag{240}$$

where we have used (243)-(243) and defined

$$q = e^{2i\pi\tau} \quad \text{with} \quad \tau = \frac{\omega}{R}.$$

Consider general properties of the average (244). First, since the eigenvalues of P are n/R with $n\in\mathbb{Z}$ one can shift ω by an integer amount of R. That is $\langle\dots\rangle_{\text{torus}}$ is invariant under $\mathcal{T}$ transformation

$$\mathcal{T}:\ \tau\to\tau+1.$$

At the same time the replacement $(\omega_1,\omega_2)=(R,\omega)$ by (ω_2,ω_1) describes the same torus and hence $\langle\dots\rangle_{\text{torus}}$ should be invariant under $\mathcal{S}$ transformation

$$\mathcal{S}:\ \tau\to-\frac{1}{\tau}.$$

We note that the same transformations should be applied to $\bar{\tau}$ since they are complex conjugate of each other.

These two transformations $\mathcal{T}$ and $\mathcal{S}$ satisfy the relations

$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = 1$$

and are known to generate the $PSL(2,\mathbb{Z})$ group or the modular group

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad a,b,c,d\in\mathbb{Z}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim -\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{241}$$

We note that equivalently the torus can be regarded as the quotient of the complex plane by the lattice generated by two elements

$$z \sim z + m\omega_1 + n\omega_2, \quad m,n\in\mathbb{Z}.$$

From this point of view it is clear that the torus defined by (ω_1,ω_2) and by

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \implies \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

are equivalent. Then the corresponding modular parameters $\tau=\omega_1/\omega_2$ and $\tau=\omega_1'/\omega_2'$ are related by (245).

Let us start to consider the partition function

$$Z(\tau) \overset{\text{def}}{=} \text{Tr}\Big[q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\Big],$$

where the trace goes over some Hilbert space. Consider some simplest examples.

Free boson. In this case we take the Hilbert space which consists of all Fock spaces $\mathcal{F}_P\otimes\mathcal{F}_P$. Then one has

$$Z(\tau) = \left|\frac{q^{-\frac{1}{24}}}{\prod_k(1-q^k)}\right|^2 \int' |q|^{P^2}dP = \frac{1}{|\eta(\tau)|^2}\int' |q|^{P^2}dP, \tag{242}$$

where we introduced the so called Dedekind η-function

$$\eta(\tau) \overset{\text{def}}{=} q^{\frac{1}{24}}\prod_{k=1}^{\infty}(1-q^k).$$

From the definition of $\eta(\tau)$ function we immediately see that

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \tag{243}$$

and hence the partition function (246), which involves only absolute values squared is ultimately invariant under $\mathcal{T}$ modular transformation. Under modular transformation $\mathcal{S}$ it behaves as follows

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{244}$$

In order to see it, it is convenient to define more general objects known as theta constants

$$\vartheta_2(\tau) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \overset{(139)}{=} 2q^{\frac{1}{8}}\prod_{k=0}^{\infty}(1+q^{k+1})^2(1-q^{k+1}),$$

$$\vartheta_3(\tau) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} \overset{(139)}{=} \prod_{k=0}^{\infty}(1+q^{k+\frac{1}{2}})^2(1-q^{k+1}),$$

$$\vartheta_4(\tau) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} \overset{(139)}{=} \prod_{k=0}^{\infty}(1-q^{k+\frac{1}{2}})^2(1-q^{k+1}),$$

where we have used the Jacobi triple identity to rewrite the sum in terms of infinite product. Using the Poisson resumation formula[19]

$$\sum_{n\in\mathbb{Z}} e^{-\pi\alpha n^2+\beta n} = \frac{1}{\sqrt{\alpha}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{\alpha}\left(n+\frac{\beta}{2i\pi}\right)^2} \tag{245}$$

one can derive modular properties of theta constants and Dedekind $\eta-$function

$$\begin{aligned}
\vartheta_2(\tau+1) &= e^{\frac{i\pi}{4}}\vartheta_2(\tau) \qquad & \vartheta_2\left(-\frac{1}{\tau}\right) &= \sqrt{-i\tau}\vartheta_4(\tau), \\
\vartheta_3(\tau+1) &= \vartheta_4(\tau) \qquad & \vartheta_3\left(-\frac{1}{\tau}\right) &= \sqrt{-i\tau}\vartheta_3(\tau), \\
\vartheta_4(\tau+1) &= \vartheta_3(\tau) \qquad & \vartheta_4\left(-\frac{1}{\tau}\right) &= \sqrt{-i\tau}\vartheta_2(\tau), \\
\eta(\tau+1) &= e^{\frac{i\pi}{12}}\eta(\tau) \qquad & \eta\left(-\frac{1}{\tau}\right) &= \sqrt{-i\tau}\eta(\tau)
\end{aligned} \tag{246}$$

We note that

$$\frac{\vartheta_2\vartheta_3\vartheta_4}{2\eta^3} = 1.$$

and hence we obtain (247) and (248).

We see that if we treat the integral in (246) literally

$$\int_{-\infty}^{\infty} |q|^{P^2} dP = \int_{-\infty}^{\infty} e^{-4\pi \mathrm{Im}\tau\, P^2} dP = \frac{1}{2\sqrt{\mathrm{Im}\tau}},$$

then the partition function

$$Z(\tau) = \frac{1}{2\sqrt{\mathrm{Im}\tau}} \frac{1}{|\eta(\tau)|^2}$$

is invariant under $\mathcal{S}$ transformation as well.

Ising model. There are three characters in this case $\chi_{1,1}^{(3,4)}(q)$, $\chi_{3,1}^{(3,4)}(q)$ and $\chi_{2,1}^{(3,4)}(q)$ (including the factor $q^{-\frac{c}{24}} = q^{-\frac{1}{48}}$ which we dropped before)

$$\begin{aligned}
\chi_{1,1}^{(3,4)}(q) + \chi_{3,1}^{(3,4)}(q) &= q^{-\frac{1}{48}} \prod_{k=1}^{\infty}(1+q^{k+\frac{1}{2}}) = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}, \\
\chi_{1,1}^{(3,4)}(q) - \chi_{3,1}^{(3,4)}(q) &= q^{-\frac{1}{48}} \prod_{k=1}^{\infty}(1-q^{k+\frac{1}{2}}) = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}, \\
\chi_{2,1}^{(3,4)}(q) &= q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1+q^{k}) = \sqrt{\frac{\theta_2(\tau)}{2\eta(\tau)}}.
\end{aligned}$$

[19] It follows from application of the identity

$$\sum_{k\in\mathbb{Z}} \delta(x-k) = \sum_{k\in\mathbb{Z}} e^{2i\pi kx}$$

to the function $e^{-\pi\alpha x^2+\beta x}$.

Using the modular properties of the theta constants (250) one can show that the combination

$$Z(\tau)=\frac{1}{2}\left(\left|\frac{\theta_3}{\eta}\right|+\left|\frac{\theta_4}{\eta}\right|+\left|\frac{\theta_2}{\eta}\right|\right)=|\chi^{(3,4)}_{1,1}(q)|^2+|\chi^{(3,4)}_{3,1}(q)|^2+|\chi^{(3,4)}_{2,1}(q)|^2$$

form modular invariant partition function.

Generic unitary minimal model. We consider $\mathcal{M}_{p,p+1}$ minimal model with

$$c=1-\frac{6}{p(p+1)},\qquad \Delta_{m,n}=\frac{(mp-n(p+1))^2-1}{4p(p+1)}.$$

There are $p(p-1)/2$ primary fields in Kac table $0<n<m<p+1$ and the character of each representation is given by Rocha-Caridi-Feigin-Fuks formula (compare to (202))

$$\chi^{(p,p+1)}_{m,n}(\tau)=\frac{1}{\eta(\tau)}\sum_{k\in\mathbb{Z}}\left(q^{\frac{((m+2k(p+1))p-n(p+1))^2}{4p(p+1)}}-q^{\frac{((-m+2k(p+1))p-n(p+1))^2}{4p(p+1)}}\right)=$$
$$=\frac{1}{\eta(\tau)}\left(\Theta_{pm-n(p+1),p(p+1)}(\tau)-\Theta_{pm+n(p+1),p(p+1)}(\tau)\right),\quad (247)$$

where

$$\Theta_{r,s}(\tau)\overset{\text{def}}{=}\sum_{k\in\mathbb{Z}}q^{s\left(k+\frac{r}{2s}\right)^2},$$

and we used the property $\Theta_{-r,s}(\tau)=\Theta_{r,s}(\tau)$.

Using Poisson resumation formula (249) one can find modular transformation properties of $\Theta_{r,s}(\tau)$

$$\begin{aligned}&\Theta_{r,s}(\tau+1)=e^{\frac{i\pi r^2}{4s}}\Theta(\tau),\\&\Theta_{r,s}(-\frac{1}{\tau})=\sqrt{-i\tau}\sum_{r'=-s+1}^{s}\frac{1}{\sqrt{2s}}e^{-\frac{i\pi rr'}{s}}\Theta_{r',s}(\tau).\end{aligned}\qquad (248)$$

First equation is obvious. For the second one we use

$$\Theta_{r,s}\left(-\frac{1}{\tau}\right)=\sum_{k\in\mathbb{Z}}e^{-\frac{2i\pi s}{\tau}\left(k+\frac{r}{2s}\right)^2}\overset{(249)}{=}\frac{\sqrt{-i\tau}}{\sqrt{2s}}\sum_{k'\in\mathbb{Z}}e^{2i\pi\tau s\left(\frac{k'}{2s}\right)^2+\frac{i\pi rk'}{s}}.$$

It is convenient to represent

$$k'=-2sk-r'\quad\text{with}\quad k\in\mathbb{Z}\quad\text{and}\quad r'=-s+1,\dots,s.\qquad (249)$$

Then the last sum in (253) can be rewritten as

$$\sum_{k'\in\mathbb{Z}}e^{2i\pi\tau s\left(\frac{k'}{2s}\right)^2+\frac{i\pi rk'}{s}}=\sum_{r'=-s+1}^{s}\sum_{k\in\mathbb{Z}}e^{2i\pi\tau s\left(k+\frac{r'}{2s}\right)^2-\frac{i\pi rr'}{s}}=\sum_{r'=-s+1}^{s}e^{-\frac{i\pi rr'}{s}}\Theta_{r',s}(\tau).$$

We note that using the symmetry $\Theta_{r,s}(\tau)=\Theta_{-r,s}(\tau)$ one can rewrite

$$\Theta_{r,s}(-\frac{1}{\tau})=\frac{\sqrt{-i\tau}}{\sqrt{2s}}\left(\Theta_{0,s}(\tau)+\sum_{r'=1}^{s-1}\left(e^{\frac{i\pi rr'}{s}}+e^{-\frac{i\pi rr'}{s}}\right)\Theta_{r',s}(\tau)+(-1)^r\Theta_{s,s}(\tau)\right)$$

Plugging (252) into the character formula (251), one finds

$$\chi_{m,n}^{(p,p+1)}(\tau+1) = e^{2i\pi\left(\Delta_{m,n}-\frac{c}{24}\right)}\chi_{m,n}^{(p,p+1)}(q)(\tau)$$

and

$$\chi_{m,n}^{(p,p+1)}\left(-\frac{1}{\tau}\right) = \frac{1}{\eta(\tau)}\frac{1}{\sqrt{2p(p+1)}}\sum_{r=1}^{p(p+1)-1} 4\sin\frac{\pi mr}{p+1}\sin\frac{\pi nr}{p}\,\Theta_{r,p(p+1)}(\tau).$$

Now, we use the following lemma which belongs to Cardy [10]. Namely, one can notice that the set

$$pm' \pm (p+1)n' \quad \text{with} \quad 0 < n' < m' < p+1,$$

spans all integers r' from 1 to $p(p+1)$ not divisible by p and $p+1$ modulo $2p(p+1)$ and $r' \to -r'$. Thus we obtain[20]

$$\chi_{m,n}^{(p,p+1)}\left(-\frac{1}{\tau}\right) = \mathcal{S}_{m,n}^{m',n'}\chi_{m',n'}^{(p,p+1)}(\tau), \qquad \mathcal{S}_{m,n}^{m',n'} = \frac{4}{\sqrt{2p(p+1)}}(-1)^{mn'+m'n+1}\sin\frac{\pi pmm'}{p+1}\sin\frac{\pi(p+1)nn'}{p}$$

We see that the matrix $\mathcal{S}$ is symmetric and real. Moreover from its definition it follows that

$$\mathcal{S}\mathcal{S} = I \implies \mathcal{S}^{-1} = \mathcal{S}.$$

The expression for the partition function is

$$Z(\tau) = \sum_{m,n,m',n'} \mathcal{N}_{m,n,m',n'}\chi_{m,n}(\tau)\chi_{m',n'}(\bar{\tau}), \tag{250}$$

where $\mathcal{N}_{m,n,m',n'}$ is an integer number called the multiplicity, that is the number of times the representation with the highest weights $(\Delta,\bar{\Delta}) = (\Delta_{m,n},\Delta_{m',n'})$ is present. The modular of the partition function $Z(\tau)$ is equivalent to the set of conditions

$$\mathcal{T}\mathcal{N}\mathcal{T}^{-1} = \mathcal{S}\mathcal{N}\mathcal{S}^{-1} = 1,$$

where $\mathcal{T}$ and $\mathcal{S}$ are the matrices of elementary modular transformations. In addition we require

$$\mathcal{N}_{1,1,1,1} = 1,$$

that is we assume that the identity operator I is unique.

One solution to (254) which may always be found is

$$\mathcal{N}_{m,n,m',n'} = \delta_{m,m'}\delta_{n,n'},$$

which corresponds to the situation where all primary fields are scalar, that is $\Delta = \bar{\Delta}$, and all of them are taken just ones (with multiplicity 1). However, as was noticed by Cardy, there are other solutions.

[20]For generic $\mathcal{M}_p, q$ model one has

$$\mathcal{S}_{m,n}^{m',n'} = \frac{4}{\sqrt{2pq}}(-1)^{mn'+m'n+1}\sin\frac{\pi pmm'}{q}\sin\frac{\pi qnn'}{p}$$

Necessary condition comes from $\mathcal{T}$ symmetry, which demands that only the operators with the integer spin may occur

$$\Delta_{m,n} - \Delta_{m',n'} = s \in \mathbb{Z}.$$

Inspecting the Kac tables for $\mathcal{M}_{3,4}$ and $\mathcal{M}_{4,5}$ theories ((204) and (219) respectively), one finds that this does not happen. However for the model $\mathcal{M}_{5,6}$ (see (229)) we see that

$$\Delta_{5,1} - \Delta_{1,1} = 3, \qquad \Delta_{5,2} - \Delta_{1,2} = 1,$$

and hence $\mathcal{N}_{5,1,1,1}$ and $\mathcal{N}_{5,2,1,2}$ might be non-zero. It can be shown that

$$Z(\tau) = |\chi_{1,1}^{(5,6)}(\tau) + \chi_{5,1}^{(5,6)}(\tau)|^2 + |\chi_{1,2}^{(5,6)}(\tau) + \chi_{5,2}^{(5,6)}(\tau)|^2 + 2|\chi_{3,1}^{(5,6)}(\tau)|^2 + 2|\chi_{3,2}^{(5,6)}(\tau)|^2$$

is invariant under $\mathcal{S}$ transformation. We note that in this non-diagonal solution the fields $\Phi_{4,r}$ are not present, while the fields $\Phi_{3,r}$ enter with multiplicity 2. This model corresponds to Z_3 parafermionic CFT [12]. Two copies of the field $\Phi_{3,1}$ correspond to parafermionic currents (236), while two fields $\Phi_{3,2}$ with $\Delta = \frac{1}{15}$ to the "energy operators" σ_1 and $\sigma_2 = \sigma_1^+$ from [12].

General classification of modular invariant partition functions has been done in [13].

References

[1] J. Polchinski, *Scale and conformal invariance in Quantum Field Theory*, *Nucl. Phys.* **B303** (1988) 226.

[2] D. Friedan, E. J. Martinec and S. H. Shenker, *Conformal invariance, supersymmetry and String theory*, *Nucl. Phys.* **B271** (1986) 93.

[3] Al. B. Zamolodchikov, *Conformal symmetry in two-dimensions: an explicit reccurence formula for the conformal partial wave amplitude*, *Comm. Math. Phys.* **96** (1984) 419.

[4] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B241** (1984) 333.

[5] D. Friedan, Z.-a. Qiu and S. H. Shenker, *Conformal invariance, unitarity and two-dimensional critical exponents*, *Phys.Rev.Lett.* **52** (1984) 1575.

[6] D. Friedan, Z.-a. Qiu and S. H. Shenker, *Superconformal invariance in two-dimensions and the tricritical Ising model*, *Phys.Lett.* **B151** (1985) 37.

[7] R. H. Poghossian, *Structure constants in the N=1 superLiouville field theory*, *Nucl. Phys. B* **496** (1997) 451 [hep-th/9607120].

[8] V. Dotsenko, *Critical behavior and associated conformal algebra of the $\mathbb{Z}_3$ Potts model*, *Nucl. Phys. B* **235** (1984) 54.

[9] A. B. Zamolodchikov, *Infinite additional symmetries in two-dimensional conformal quantum field theory*, *Theor. Math. Phys.* **65** (1985) 1205.

[10] J. L. Cardy, *Operator content of two-dimensional conformally invariant theories*, *Nucl.Phys.* **B270** (1986) 186.

[11] J. L. Cardy, *Effect of boundary conditions on the operator content of two-dimensional conformally invariant theories*, *Nucl. Phys. B* **275** (1986) 200.

[12] V. A. Fateev and A. B. Zamolodchikov, *Parafermionic currents in the two-dimensional conformal quantum field theory and selfdual critical points in $\mathbb{Z}_n$ invariant statistical systems*, *Sov. Phys. JETP* **62** (1985) 215.

[13] A. Cappelli, C. Itzykson and J. B. Zuber, *Modular Invariant Partition Functions in Two-Dimensions*, *Nucl. Phys. B* **280** (1987) 445.