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Lecture 1: Elements of classical field theory, Noether theorem, stress-energy tensor

Consider classical field theory in $\mathbb{R}^D$ given by the action

$$S = \int_{\mathbb{R}^D} \mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x})) d^D \boldsymbol{x}, \tag{1}$$

where $\Phi(\boldsymbol{x})$ is not necessarily just one field, it can be a collection of fields, not necessarily a scalar field, it can carry representation indices etc. The function $\mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}))$ is called the Lagrangian density. It can be taken more or less arbitrary. There are two conditions, which we require. First, is the locality, $\mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}))$ should contain interactions only in the same point, that is we forbid terms in the Lagrangian of the form $\Phi(\boldsymbol{x})\Phi(\boldsymbol{x}+\boldsymbol{a})$ etc. Also, we assume that the Langrangian does not include higher derivatives. This is what in principle can be violated, but we will not do it.

Having defined an action, one obtains equations of motion from the least action principle $\delta S = 0$. They are (for simplicity Φ here is just one scalar field)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0. \tag{2}$$

The main task in classical theory is to solve equations of motion subject to certain boundary conditions, initial data etc/

Special role is played by the Integrals of Motion, or conservation laws. Their existence is related to the Noether theorem, a relation between symmetries and conservation laws, which we briefly review now. Suppose our theory has a family of continuous transformations

$$\Phi(\boldsymbol{x}) \to \tilde{\Phi}(\boldsymbol{x}) = \mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})),$$

such that the action does not change $S[\Phi(\boldsymbol{x})] = S[\tilde{\Phi}(\boldsymbol{x})]$. For example, consider $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\boldsymbol{x}+\boldsymbol{a})$. This is a continuous transformation corresponding to the translation $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$. It is continuous because the vector $\boldsymbol{a}$ varies. It can be taken arbitrary small, in this case the transformation $\mathcal{F}$ will be in the vicinity of the identity transformation. Since our action is an integral over entire space and the Lagrangian $\mathcal{L}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}))$ does not depend explicitly on coordinates, which we assume, the action is invariant.

Another example: $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\Lambda \boldsymbol{x})$, where $\Lambda \in SO(D)$, corresponds to rotations (Lorentz transformation). The action is invariant if the derivatives enter the action in Lorentz invariant way. The most general invariant Lagrangian of one bosonic field with at most two derivatives is

$$\mathcal{L} = U(\Phi)(\partial_\mu \Phi)^2 + V(\Phi), \tag{3}$$

where U and V are arbitrary functions.

Another important example of the symmetry involves only transformation of the fields. For example, consider complex scalar field

$$X = \Phi_1 + i\Phi_2,$$

with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu X \partial_\mu X^* + V(|X|^2).$$

This action is manifestly invariant under arbitrary $U(1)$ rotation

$$X \to e^{i\alpha} X,$$

again a continuous symmetry.

Given a symmetry, one derives Noether theorem. Suppose that our theory admits a family of transformations indexed by a continuous parameter ϵ

$$\Phi(\boldsymbol{x}) \to \tilde{\Phi}(\boldsymbol{x}) = \mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\boldsymbol{x}) + \epsilon f\big(\boldsymbol{x}, \Phi(\boldsymbol{x})\big) + O(\epsilon^2),$$

such that the action does not change $S[\Phi(\boldsymbol{x})] = S[\tilde{\Phi}(\boldsymbol{x})]$. Consider the following infinitesimal transformation

$$\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x}) f\big(\boldsymbol{x}, \Phi(\boldsymbol{x})\big) + O(\epsilon^2)$$

where, what is important, $\epsilon(\boldsymbol{x})$ depends on a position $\boldsymbol{x}$. Then the corresponding variation of the action S takes the form (here we assume that the action contains at most first derivatives)

$$\delta S = \int \Big(\epsilon(\boldsymbol{x}) J(\boldsymbol{x}) + \partial_\mu \epsilon(\boldsymbol{x})\, K_\mu(\boldsymbol{x})\Big) d^D\boldsymbol{x},$$

with some $J(\boldsymbol{x})$ and $K_\mu(\boldsymbol{x})$. By assumption, $\delta S = 0$ for constant ϵ. This is possible if (we assume no boundary issues here)

$$J(\boldsymbol{x}) = \partial_\mu J_\mu(\boldsymbol{x}),$$

and hence

$$\delta S = \int \Big(K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})\Big)\partial_\mu \epsilon(\boldsymbol{x}) d^D\boldsymbol{x} = -\int \partial_\mu \Big(K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})\Big)\epsilon(\boldsymbol{x}) d^D\boldsymbol{x}.$$

What we just made is an arbitrary variation of the action. It should vanish on-shell (2). But the function $\epsilon(\boldsymbol{x})$ is arbitrary. This implies that $j_\mu(\boldsymbol{x}) = K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})$ satisfies the continuity equation

$$\partial_\mu j_\mu(\boldsymbol{x}) = \partial_\mu \left(K_\mu(\boldsymbol{x}) - J_\mu(\boldsymbol{x})\right) \overset{\text{on-shell}}{=} 0. \tag{4}$$

The current j_μ is usually referred as Noether current.

The continuity equation (4) then implies the conservation law. Namely, by Stokes theorem we have

$$\oint_{\partial\mathcal{M}} j_\mu(\boldsymbol{x}) d\sigma_\mu = \int_{\mathcal{M}} \partial_\mu j_\mu(\boldsymbol{x}) d^D\boldsymbol{x} = 0,$$

for any closed "surface" $\partial\mathcal{M}$. In particular, if we take $\mathcal{M}$ to be very large cylinder between two time slices $x_1 = t_1$ and $x_1 = t_2$, we get[1]

$$Q_{t_1} = Q_{t_2}, \quad \text{where} \quad Q_t = \int j_1(\boldsymbol{x}) d^{D-1}\boldsymbol{x}\Big|_t.$$

[1]In Euclidean FT the choice of the time slice is not canonically defined.

Among other Noether currents, the one, called the stress-energy tensor, will be primarily important for us. It is conserved due to invariance of the action under translations. Consider variation of the action (1) under arbitrary coordinate transformations $\boldsymbol{x} \to \boldsymbol{x} + \epsilon(\boldsymbol{x})$. In other words, we compute the response of the action (1) with respect to the substitution $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x} + \boldsymbol{\epsilon}(\boldsymbol{x})) = \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x}) + \dots$. We have

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} \left[\epsilon_\nu \Big(\frac{\partial \mathcal{L}}{\partial \Phi} \partial_\nu \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\mu \partial_\nu \Phi \Big) + \partial_\mu \epsilon_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi \right] d^D \boldsymbol{x}. \tag{5}$$

First term in (5) equals to $\epsilon_\nu \partial_\mu (\delta_{\mu\nu} \mathcal{L})$ as a reflection of the fact that the action does not depend on $\boldsymbol{x}$ explicitly and hence for constant ϵ_μ the variation should vanish. After integrating by parts we get

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} \partial_\mu \epsilon_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta_{\mu\nu} \mathcal{L} \right) d^D \boldsymbol{x} \stackrel{\text{def}}{=} \int_{\mathbb{R}^D} \partial_\mu \epsilon_\nu T_{\mu\nu} d^D \boldsymbol{x} = - \int_{\mathbb{R}^D} \epsilon_\nu \partial_\mu T_{\mu\nu} d^D \boldsymbol{x}. \tag{6}$$

On shell the variation (6) should vanish. Since the function $\epsilon_\mu(\boldsymbol{x})$ is arbitrary it implies the continuity condition for $T_{\mu\nu}$

$$\partial_\mu T_{\mu\nu} = 0,$$

and hence conservation of energy and momentum

$$E \stackrel{\text{def}}{=} \int T_{11} d^{D-1} \boldsymbol{x}, \qquad P_i \stackrel{\text{def}}{=} \int T_{1i} d^{D-1} \boldsymbol{x}, \quad i \neq 1.$$

Derivation given above leads to the definition of the stress-energy tensor as a response to the infinitesimal coordinate change

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D} \partial_\mu \epsilon_\nu T_{\mu\nu} d^D \boldsymbol{x} \tag{7}$$

In Lorentz invariant FT involving only scalar fields the canonical stress-energy tensor always comes out to be symmetric $T_{\mu\nu} = T_{\nu\mu}$. In general this is not the case, but in rotationally invariant theories $T_{\mu\nu}$ can always be made symmetric. It can be seen as follows. Consider $\epsilon_\nu = \omega_{\nu\lambda} x_\lambda$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, then from (7) we have

$$\delta_{\boldsymbol{\epsilon}} S = \frac{1}{2} \int_{\mathbb{R}^D} \left[\partial_\mu \omega_{\nu\lambda} \big(x_\lambda T_{\mu\nu} - x_\nu T_{\mu\lambda} \big) - \omega_{\mu\nu} \big(T_{\mu\nu} - T_{\nu\mu} \big) \right] d^D \boldsymbol{x}. \tag{8}$$

For constant $\omega_{\mu\nu}$ this variation should vanish for rotationally invariant theories, which implies

$$T_{\mu\nu} - T_{\nu\mu} = \partial_\lambda f_{\lambda\mu\nu}, \qquad f_{\lambda\mu\nu} = -f_{\lambda\nu\mu}.$$

Now we define modified tensor

$$\tilde{T}_{\mu\nu} \stackrel{\text{def}}{=} T_{\mu\nu} - \partial_\lambda B_{\lambda\mu\nu} \quad \text{where} \quad B_{\lambda\mu\nu} = \frac{1}{2} \big(f_{\lambda\mu\nu} - f_{\mu\lambda\nu} - f_{\nu\lambda\mu} \big). \tag{9}$$

We note that the tensor $B_{\lambda\mu\nu}$ is antisymmetric in first two indexes $B_{\lambda\mu\nu} = -B_{\mu\lambda\nu}$, which implies

$$\partial_\mu \partial_\lambda B_{\lambda\mu\nu} \equiv 0.$$

At the same time, we have

$$B_{\lambda\mu\nu} - B_{\lambda\nu\mu} = f_{\lambda\mu\nu},$$

and hence the modified stress-energy tensor (9) is symmetric

$$\tilde{T}_{\mu\nu} = \tilde{T}_{\nu\mu}.$$

This tensor is known as Belifante tensor. Now, integrating by parts (8) one finds conservation of the angular momentum current

$$\partial_\mu \big(x_\nu T_{\mu\lambda} - x_\lambda T_{\mu\nu} + f_{\lambda\mu\nu} \big) = \partial_\mu \big(x_\nu \tilde{T}_{\mu\lambda} - x_\lambda \tilde{T}_{\mu\nu} \big) = 0.$$

We saw that $T_{\mu\nu}$ is not canonically defined. It is related to the intrinsic ambiguity in the definition (7). Indeed, in our example one can change the transformation rules $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu \Phi(\boldsymbol{x}) + \dots$ to the more general ones

$$\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu \Phi(\boldsymbol{x}) + \partial_\mu \epsilon_\nu(\boldsymbol{x}) \Sigma_{\mu\nu}[\Phi(\boldsymbol{x})] + \dots$$

where $\Sigma_{\mu\nu}[\Phi(\boldsymbol{x})]$ are some functions of $\Phi(\boldsymbol{x})$ and $\dots$ may contain higher derivatives of $\boldsymbol{\epsilon}(\boldsymbol{x})$. The stress-energy tensor changes as

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta_{\mu\nu}\mathcal{L} + \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Phi)} \right) \right) \Sigma_{\mu\nu} + \dots \tag{10}$$

We note that the additional term in (10) is proportional to equations of motion and hence both tensors coincide on-shell.

One can take an alternative point of view that the change of fields $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x} + \boldsymbol{\epsilon}(\boldsymbol{x}))$ can be supplemented by the change of coordinates $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\epsilon}(\boldsymbol{x})$ or $\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{\epsilon}(\boldsymbol{y}) + \dots$, such that the fields do not change, but we have to replace (infinitesimally)

$$\frac{\partial}{\partial x^\mu} = (\delta_{\mu\nu} + \partial_\mu \epsilon_\nu) \frac{\partial}{\partial y^\mu}, \qquad d^D \boldsymbol{x} = (1 - \partial_\nu \epsilon_\nu) d^D \boldsymbol{y}.$$

Of course this variation leads to the same conclusion (6) with points redefinition $\boldsymbol{x} \to \boldsymbol{y}$. We note that transformation $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{\epsilon}$ induces variation of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} = -(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu).$$

So, we come to an idea to define the stress-energy tensor as a response to the infinitesimal variation of the background metric. Namely, we assume that the action (1) admits a covariant extension

$$S[\Phi] \to S[\Phi, g], \quad \text{such that} \quad \delta S = \int \sqrt{g}\, T_{\mu\nu} \delta g^{\mu\nu} d^D \boldsymbol{x}. \tag{11}$$

From this definition it is clear that $T_{\mu\nu} = T_{\nu\mu}$. Note, that in flat space the definition (11) is still ambiguous, as one can add terms to the action which vanish at $g_{\mu\nu} \to \delta_{\mu\nu}$. For example, one can add the so called *dilaton* term

$$\int W(\Phi) R \sqrt{g}\, d^D \boldsymbol{x}, \tag{12}$$

where R is the scalar curvature and $W(\Phi)$ is arbitrary. This term disappears in the flat space, however it affects the form of $T_{\mu\nu}$. The most general term, which can be added to the action, and which gives non-vanishing contribution to $T_{\mu\nu}$ in the limit $g_{\mu\nu} \to \delta_{\mu\nu}$, but vanishes in this limit, is

$$\int R^{\mu\sigma\nu\rho} Y_{\mu\sigma\nu\rho}(\Phi) \sqrt{g}\, d^D \boldsymbol{x}, \tag{13}$$

where $R^{\mu\sigma\nu\rho}$ is the Riemann tensor for the background metric and $Y_{\mu\sigma\nu\rho}(\Phi)$ is some local tensor field which is antisymmetric in $(\mu\sigma)$ and in $(\nu\rho)$, but symmetric with respect to exchange of these pairs. This term gives the following contribution to the stress-energy tensor

$$T_{\mu\nu} \to T_{\mu\nu} + \partial_\sigma \partial_\rho Y_{\mu\sigma\nu\rho}.$$

This intrinsic ambiguity can not be resolved unless we require smth else. The theory in which the terms like (13) and similar are absent are minimal covariant extension.

From now we assume that the theory has a symmetric stress-energy tensor defined by (7). Important class of theories obey the property of scale invariance. Let us probe if the Poincaré invariant action (3) is scale invariant as well. Namely, let $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \lambda^{-\Delta}\Phi(\lambda \cdot \boldsymbol{x})$, where Δ is the so called scaling dimension, and assume, for simplicity, that $U = 1$. Then we immediately see that the first term in the action (3) is invariant if

$$\Delta = \frac{D-2}{2}.$$

Then it is clear that V has to be a power Φ^n, where

$$n = \frac{2D}{D-2}.$$

We see, that n is rarely an integer. The only exceptions are: $n = 6$ for $D = 3$, $n = 4$ for $D = 4$ and $n = 3$ for $D = 6$.

The case $D = 2$ is exceptional because scalar field is dimensionless in this case, but we can not built scale invariant theory with power like potential. As a compensation, we have Liouville theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 + e^\Phi, \tag{14}$$

which is scale invariant with $\mathcal{F}(\boldsymbol{x}, \Phi(\boldsymbol{x})) = \Phi(\lambda \cdot \boldsymbol{x}) + 2\log\lambda$.

Another scale invariant theory in two dimensions is known as non-linear sigma model

$$\mathcal{L} = G_{ab}(\mathbf{\Phi})\partial_\mu \Phi^a \partial_\mu \Phi^b.$$

Here $\mathbf{\Phi} = (\Phi^1, \ldots, \Phi^N)$ is the $N-$component bosonic field and $G_{ab}(\mathbf{\Phi})$ is some function (since Φ is dimensionless). Usually, one interprets $\mathbf{\Phi}$ as coordinates on some "target" Riemanian manifold $\mathcal{M}$ and $G_{ab}(\mathbf{\Phi})$ as a metric on it.

The last, but not the least example is the Yang-Mills theory (here fundamental fields A_μ take values in Lie group $SU(N)$)

$$S = \int \mathrm{Tr}(F_{\mu\nu}^2)\, d^D x \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

To insure scale invariance both terms in $F_{\mu\nu}$ should have the same scale dimension. From this condition we find $\Delta = 1$. Then the action transforms

$$S \to \lambda^{4-D} S,$$

and hence this theory is scale invariant only in 4 dimensions.

Going back to the definition (7) we see that if the theory is scale invariant then we should have

$$\int \Theta \, d^D\boldsymbol{x} = 0 \quad \text{where} \quad \Theta \overset{\text{def}}{=} T_{\mu\mu},$$

which requires $\Theta = \partial_\mu \theta_\mu$, so that

$$D_\mu = \theta_\mu - x_\nu T_{\mu\nu}$$

is a conserved current called the scale current.

Interestingly, the scale invariance might imply the extended symmetry. For example, one can notice that if θ_μ in turn is a gradient $\theta_\mu = \partial_\mu L$ then one can redefine $T_{\mu\nu}$

$$T_{\mu\nu} \to \tilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{D-1}\left(\partial_\mu \partial_\nu - \delta_{\mu\nu}\partial^2\right) L \tag{15}$$

to make it traceless[2]. This redefinition of $T_{\mu\nu}$ corresponds to the dilaton term (12) in the curved space.

Let us consider $\lambda\Phi^4$ theory in four dimensions

$$S = \int \left(\frac{1}{2}(\partial_\mu \Phi)^2 + \lambda \Phi^4\right) d^4\boldsymbol{x}.$$

While computing the stress-energy tensor, we are free to choose the transformation rule for the field Φ. The only condition is that it is reduced to shifts $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x} + \boldsymbol{\epsilon})$ for constant $\boldsymbol{\epsilon}$. As we saw above Φ has dimension 1, i.e. we should take the replacement

$$\Phi(\boldsymbol{x}) \to \left(1 + \frac{\partial_\mu \epsilon_\mu}{4}\right) \Phi(\boldsymbol{x} + \boldsymbol{\epsilon}).$$

Then the formula (10) implies that

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \delta_{\mu\nu}\left(\frac{1}{2}(\partial\Phi)^2 + \lambda\Phi^4\right) + \frac{1}{4}(4\lambda\Phi^3 - \partial^2\Phi)\Phi\delta_{\mu\nu},$$

where the last term corresponds to $\Sigma_{\mu\nu}$ term in (10). We see that this stress-energy tensor satisfies conditions specified above, that is $\Theta = -\frac{1}{2}\partial^2\Phi^2$, and hence the improved stress-energy tensor (15) is traceless.

The vanishing of Θ signals for larger symmetry called the conformal symmetry, whose infinitesimal form is

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \sim \delta_{\mu\nu}. \tag{16}$$

It will be studied in the next lecture.

Probs:

1. Consider Liouville theory (14)
   - Compute stress-energy tensor and show that it can be made traceless
   - Consider embedding of (14) into background metric. Adjust dilaton term (12) in such a way that $\Theta = 0$.

[2] In fact, for $D > 2$ it is enough to have $\Theta = \partial_\mu \partial_\nu L_{\mu\nu}$ then the improved tensor

$$T_{\mu\nu} + \frac{1}{D-2}\left(\partial_\mu \partial_\lambda L_{\lambda\nu} + \partial_\nu \partial_\lambda L_{\lambda\mu} - \partial^2 L_{\mu\nu} - \delta_{\mu\nu}\partial_\lambda \partial_\rho L_{\lambda\rho}\right) + \frac{1}{(D-2)(D-1)}\left(\delta_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu\right) L_{\lambda\lambda}$$

is traceless (see [1] and next lecture for more details)

Lecture 2: Conformal group

The transformation (16) is the infinitesimal form of the conformal transformation, that is an invertible map $\boldsymbol{x} \to \boldsymbol{x}'$ which leaves the metric $\delta_{\mu\nu}$ invariant up to a scale

$$\delta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(\boldsymbol{x})\delta_{\mu\nu}. \tag{17}$$

We note that $\Lambda(x) = 1$ corresponds to the Poincaré group consisting of rotations and translations. These transformations preserve the distances, while the general conformal transformations only preserve the angles.

In infinitesimal form (16) we have the condition

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(\boldsymbol{x})\eta_{\mu\nu}.$$

Contracting this equation with $\delta^{\mu\nu}$ we find that $f(\boldsymbol{x}) = \frac{2}{D}(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon})$ and hence

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{D}(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon})\delta_{\mu\nu}. \tag{18}$$

Applying ∂^ν we get

$$\left(1 - \frac{2}{D}\right)\partial_\mu(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon}) + \partial^2\epsilon_\mu = 0.$$

Furthermore we take ∂_ν and symmetrize $\mu \leftrightarrow \nu$ to find

$$\left(1 - \frac{2}{D}\right)\partial_\mu\partial_\nu(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon}) + \frac{1}{2}\partial^2\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right) = 0.$$

Finally, using (18), we find[3]

$$\left((D-2)\partial_\mu\partial_\nu + \delta_{\mu\nu}\partial^2\right)(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon}) = 0 \Longrightarrow (D-1)\partial^2(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon}) = 0. \tag{19}$$

Another useful identity is obtained from (18) by taking $\partial_\mu(18)_{\nu\rho} + \partial_\mu(18)_{\mu\rho} - \partial_\rho(18)_{\mu\nu}$

$$2\partial_\mu\partial_\nu\epsilon_\rho = \frac{2}{D}\left(\delta_{\rho\mu}\partial_\nu + \delta_{\rho\nu}\partial_\mu - \delta_{\mu\nu}\partial_\rho\right)(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon}) \tag{20}$$

We see from (19) that the cases $D = 2$ and $D > 2$ are different. Let us consider the case $D > 2$ first. Equation (19) implies that the function $(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon})$ is linear and then equation (20) implies that $\partial_\mu\partial_\nu\epsilon_p$ is a constant. Henceforth $\epsilon_\mu(\boldsymbol{x})$ are quadratic functions

$$\epsilon_\mu(\boldsymbol{x}) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\lambda}x^\nu x^\lambda, \tag{21}$$

[3] From (19) we see that for conformal invariance it is enough to have $\Theta = \partial_\mu\partial_\nu L_{\mu\nu}$ for $D > 2$ and $\Theta = \partial^2 L$ for $D = 2$. It follows from the fact that for conformal transformations the variation (7) takes the form

$$\delta_{\boldsymbol{\epsilon}} S = \int_{\mathbb{R}^D}(\boldsymbol{\partial}\cdot\boldsymbol{\epsilon})\Theta\, d^D\boldsymbol{x},$$

and hence in the virtue of (19) it can be integrated to zero if either $\Theta = \partial_\mu\partial_\nu L_{\mu\nu}$ or $\Theta = \partial^2 L$ holds.

subject to the condition (18). The constant term a_μ is not constrained at all. It represents the infinitesimal translations. The linear term $b_{\mu\nu}$ obeys

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D}\delta_{\mu\nu}(\delta^{\lambda\sigma}b_{\lambda\sigma}).$$

General solution is

$$b_{\mu\nu} = \alpha\delta_{\mu\nu} + \omega_{\mu\nu} \quad \text{where} \quad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

The antisymmetric part represents infinitesimal rotations, while the pure trace part corresponds to the scale transformation $x'^\mu = (1+\alpha)x^\mu$.

So, we are left over with the quadratic term $c_{\mu\nu\lambda}$. Inserting (21) into (20), one finds

$$c_{\mu\nu\rho} = \delta_{\nu\rho}\zeta_\mu - \delta_{\mu\rho}\zeta_\nu - \delta_{\mu\nu}\zeta_\rho \quad \text{where} \quad \zeta_\mu = -\frac{1}{D}c^\rho_{\rho\mu},$$

which corresponds to the infinitesimal transformation

$$x'^\mu = x^\mu - 2(\boldsymbol{x}\cdot\boldsymbol{\zeta})x^\mu + \zeta^\mu\boldsymbol{x}^2 + \ldots,$$

called the *Special Conformal Transformation.*

Finite conformal transformation can be obtained by exponentiation. They and the corresponding generators acting on functions are summarized in the following table

	Transformation	Generators
Translation	$x'^\mu = x^\mu + a^\mu$	$\mathcal{P}_\mu = -i\partial_\mu,$
Rotation	$x'^\mu = \Omega^\mu_\nu x^\nu$	$\mathcal{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu),$
Dilatation	$x'^\mu = \lambda x^\mu$	$\mathcal{D} = -i(\boldsymbol{x}\cdot\boldsymbol{\partial}),$
SCT	$x'^\mu = \dfrac{x^\mu + \boldsymbol{x}^2\zeta^\mu}{1+2(\boldsymbol{\zeta}\cdot\boldsymbol{x}) + \boldsymbol{\zeta}^2\boldsymbol{x}^2}$	$\mathcal{K}_\mu = -i(\boldsymbol{x}^2\partial_\mu - 2x_\mu(\boldsymbol{x}\cdot\boldsymbol{\partial}))$

One can check that SCT is indeed a conformal transformation with the scale factor $\Lambda = (1 + 2(\boldsymbol{\zeta}\cdot\boldsymbol{x}) + \boldsymbol{\zeta}^2\boldsymbol{x}^2)^2$. More intuitive way of thinking about the SCT comes from the formula

$$\frac{x'^\mu}{\boldsymbol{x}'^2} = \frac{x^\mu}{\boldsymbol{x}^2} + \zeta^\mu.$$

It means that we can define the special conformal transformations by combining an inversion with a translation and then another inversion

$$x^\mu \longrightarrow \frac{x^\mu}{\boldsymbol{x}^2} \longrightarrow \frac{x^\mu}{\boldsymbol{x}^2} + \zeta^\mu \longrightarrow \frac{\frac{x^\mu}{\boldsymbol{x}^2} + \zeta^\mu}{(\frac{\boldsymbol{x}}{\boldsymbol{x}^2} + \boldsymbol{\zeta})^2} = \frac{x^\mu + \boldsymbol{x}^2\zeta^\mu}{1 + 2(\boldsymbol{\zeta}\cdot\boldsymbol{x}) + \boldsymbol{\zeta}^2\boldsymbol{x}^2}. \tag{22}$$

One can easily check that the inversion obeys (17), so does the SCT.

We note that the SCT is not globally well defined in $\mathbb{R}^D$. In particular the point

$$y^\mu = -\frac{\zeta^\mu}{\boldsymbol{\zeta}^2}$$

is mapped to infinity. Therefore in order to define SCT globally, one usually considers conformal compactification of $\mathbb{R}^D$ which is S^D, or the D-sphere.

It is interesting to identify the conformal algebra for $D>2$, that is the Lie algebra of $(\mathcal{P}_\mu, \mathcal{L}_{\mu\nu}, \mathcal{D}, \mathcal{K}_\mu)$. First, we compute the number of generators. Keeping in mind that $\mathcal{L}_{\mu\nu}$ is antisymmetric, we have

$$D + \frac{D(D-1)}{2} + 1 + D = \frac{(D+2)(D+1)}{2}.$$

Interestingly, it is equal to the size of rotation algebra in $D+2$ dimension. This is actually the case up to a signature issues. The generators $\mathcal{D}$, $\mathcal{K}_\mu$ and $\mathcal{P}_\mu$, $\mathcal{L}_{\mu\nu}$ admit the following commutation relations

$$\begin{aligned} &[\mathcal{D},\mathcal{P}_\mu] = i\mathcal{P}_\mu, && [\mathcal{D},\mathcal{L}_{\mu\nu}] = 0, && [\mathcal{D},\mathcal{K}_\mu] = -i\mathcal{K}_\mu, \\ &[\mathcal{K}_\mu,\mathcal{K}_\nu] = 0, && [\mathcal{K}_\mu,\mathcal{P}_\nu] = -2i\left(\delta_{\mu\nu}\mathcal{D} + \mathcal{L}_{\mu\nu}\right), && [\mathcal{K}_\lambda,\mathcal{L}_{\mu\nu}] = i(\delta_{\lambda\mu}\mathcal{K}_\nu - \delta_{\lambda\nu}\mathcal{K}_\mu), \end{aligned} \tag{23}$$

plus those of the Poincaré algebra

$$[\mathcal{P}_\mu,\mathcal{P}_\nu] = 0, \quad [\mathcal{P}_\lambda,\mathcal{L}_{\mu\nu}] = i(\delta_{\lambda\mu}\mathcal{P}_\nu - \delta_{\lambda\nu}\mathcal{P}_\mu), \quad [\mathcal{L}_{\mu\nu},\mathcal{L}_{\rho\sigma}] = i(\delta_{\nu\rho}\mathcal{L}_{\mu\sigma} + \delta_{\mu\sigma}\mathcal{L}_{\nu\rho} - \delta_{\mu\rho}\mathcal{L}_{\nu\sigma} - \delta_{\nu\sigma}\mathcal{L}_{\mu\rho}). \tag{24}$$

These commutation relations can be brought to the convenient form by defining

$$\mathcal{J}_{\mu\nu} = \mathcal{L}_{\mu\nu}, \quad \mathcal{J}_{D+2\,D+1} = \mathcal{D}, \quad \mathcal{J}_{D+1\,\mu} = \frac{1}{2}(\mathcal{P}_\mu + \mathcal{K}_\mu), \quad \mathcal{J}_{D+2\,\mu} = \frac{1}{2}(\mathcal{P}_\mu - \mathcal{K}_\mu),$$

where $\mathcal{J}_{MN} = -\mathcal{J}_{NM}$. Then the new generators satisfy the relations of $SO(D+1,1)$ Lie algebra

$$[\mathcal{J}_{MN},\mathcal{J}_{RS}] = i\left(\eta_{NR}\mathcal{J}_{MS} + \eta_{MS}\mathcal{J}_{NR} - \eta_{MR}\mathcal{J}_{NS} - \eta_{NS}\mathcal{J}_{MR}\right),$$

where η_{MN} is the diagonal matrix with Minkowski signature $(1,1,\ldots,1,-1)$. It can be seen by explicit calculations and we leave it as an exercise, but it is better to derive it from the following arguments. Consider the vector

$$\boldsymbol{X} = (x^1,\ldots,x^D,\frac{1-\boldsymbol{x}^2}{2},\frac{1+\boldsymbol{x}^2}{2}) \in \mathbb{R}^{D+1,1}$$

It is not arbitrary, but subject to two additional constraints. First it is a "light-like" vector $\boldsymbol{X}^2 = \boldsymbol{x}^2 + (\frac{1-\boldsymbol{x}^2}{2})^2 - (\frac{1+\boldsymbol{x}^2}{2})^2 = 0$. Second constraint is the condition $X^{D+1} + X^{D+2} = 1$, which defines the section of the light-cone. This section is parameterized by $\boldsymbol{x}$, which are our original coordinates in $\mathbb{R}^D$. One can easily check that the induced metric on $\mathbb{R}^D$ coincides with the flat metric. The group $SO(D+1,1)$ acts on $\mathbb{R}^{D+1,1}$ by linear transformations

$$X^M \to \Lambda^M_N X^N. \tag{25}$$

We want somehow to project this action to our section: $\boldsymbol{X}^2 = 0$, $X^{D+1} + X^{D+2} = 1$. Since the first constraint is preserved by the action, it is not problematic. The second constraint transforms

$$1 = X^D + X^{D+1} \to \lambda(\boldsymbol{X}),$$

where $\lambda(\boldsymbol{X})$ is some linear function. So, we just replace (25) by the transformation

$$X^M \to \lambda^{-1}(\boldsymbol{X})\Lambda^M_N X^N, \tag{26}$$

which certainly preserves both constraints. It remains to show that the transformation $X^M \to \lambda(\boldsymbol{X})X^M$ is a conformal transformation of the light-cone (and hence of the section as well). Indeed

$$\Big(d(\lambda\boldsymbol{X})\cdot d(\lambda\boldsymbol{X})\Big) = \Big(\big(\lambda d\boldsymbol{X} + (\boldsymbol{\nabla}\lambda\cdot d\boldsymbol{X})\boldsymbol{X}\big)\cdot\big(\lambda d\boldsymbol{X} + (\boldsymbol{\nabla}\lambda\cdot d\boldsymbol{X})\boldsymbol{X}\big)\Big) = \lambda^2\Big(d\boldsymbol{X}\cdot d\boldsymbol{X}\Big),$$

where we used $\boldsymbol{X}^2=0$, $(\boldsymbol{X}\cdot d\boldsymbol{X})=0$. That is, that linear transformations $\Lambda \in SO(D+1,1)$ corresponds via (26) to the conformal transformations of $\mathbb{R}^D$, which we summarized in (23)-(24). For example

$$\begin{pmatrix} 1 & 0 & 0 & \ldots\ldots & & 0 & 0 \\ 0 & 1 & 0 & \ldots\ldots & & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & 0 & 1 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & 0 & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & 0 & 0 & \frac{\lambda+\lambda^{-1}}{2} & \frac{\lambda-\lambda^{-1}}{2} \\ \ldots & \ldots & \ldots & 0 & 0 & \frac{\lambda-\lambda^{-1}}{2} & \frac{\lambda+\lambda^{-1}}{2} \end{pmatrix} \tag{27}$$

corresponds to dilations $\boldsymbol{x} \to \lambda \boldsymbol{x}$ etc.

It is interesting to construct conformal invariants, that is functions of $F(x_1,\ldots,x_N)$ which are invariant with respect to all conformal transformations. Poincare invariance implies that $F(x_1,\ldots,x_N)$ may depend only on relative distances $|\boldsymbol{x}_i-\boldsymbol{x}_j|$, then the scale invariance implies that it can only depend on the ratios

$$\frac{|\boldsymbol{x}_i-\boldsymbol{x}_j|}{|\boldsymbol{x}_k-\boldsymbol{x}_l|}.$$

Finally, to insure the invariance under the SCT it is enough due to (22) to ensure the invariance with respect to inversions. Using

$$(\boldsymbol{x}_i-\boldsymbol{x}_j)^2 \xrightarrow{x^\mu \to \frac{x^\mu}{\boldsymbol{x}^2}} \frac{(\boldsymbol{x}_i-\boldsymbol{x}_j)^2}{\boldsymbol{x}_i^2\boldsymbol{x}_j^2},$$

we see that we can built an invariant only through 4 points

$$u=\frac{|\boldsymbol{x}_1-\boldsymbol{x}_2||\boldsymbol{x}_3-\boldsymbol{x}_4|}{|\boldsymbol{x}_1-\boldsymbol{x}_3||\boldsymbol{x}_2-\boldsymbol{x}_4|}, \quad v=\frac{|\boldsymbol{x}_1-\boldsymbol{x}_2||\boldsymbol{x}_3-\boldsymbol{x}_4|}{|\boldsymbol{x}_2-\boldsymbol{x}_3||\boldsymbol{x}_1-\boldsymbol{x}_4|}.$$

It means that the conformaly invariant function of four points is actually a function of two invariants $F(u,v)$. In general, there are $N(N-3)/2$ invariants for N points. Indeed, the number of $|\boldsymbol{x}_i-\boldsymbol{x}_j|$'s is $N(N-1)/2$. Then write a monomial

$$\prod_{i<j}|\boldsymbol{x}_i-\boldsymbol{x}_j|^{m_{ij}}.$$

Conformal invariance demands that each individual degree in $\boldsymbol{x}_k$ is 0. That is

$$\sum_{j=1}^{k-1} m_{jk}+\sum_{j=k+1}^{N} m_{kj}=0.$$

So we have N equations for $N(N-1)/2$ unknowns: $N(N-1)/2-N=N(N-3)/2$.

We note that these monomials are not algebraically independent for $N \geq D+2$. Indeed, after all we have N points in D-dimensional space constrained by the conformal group. Hence the number of algebraically independent cross-rations has to be

$$ND-\frac{(D+2)(D+1)}{2}.$$

In particular, one has only $2(N-3)$ independent cross-ratios for $D=2$ and $N-3$ for $D=1$. For example for $D=1$ and for $x_1 > x_2 > x_3 > x_4$

$$u = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_2-x_4)}, \quad v = \frac{(x_1-x_2)(x_3-x_4)}{(x_2-x_3)(x_1-x_4)} \implies v = \frac{u}{1-u}.$$

The counting above does not work for $N < D+2$, since there is a residual subgroup of the conformal group which leaves the N points invariant. So that we have, $N(N-3)/2$ invariants for $N < D+2$ and $ND - \frac{(D+2)(D+1)}{2}$ for $N \geq D+2$.

Now, we consider the conformal group in two dimensions. We already saw that $D=2$ is special (see eqs (18), (19), (20)). Namely, the condition (18) reads

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1, \tag{28}$$

which are nothing else as the Cauchy-Riemann equations in complex analysis: the complex function whose real and imaginary parts satisfy (28) is holomorphic. Namely, we introduce the notations

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \quad \partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2), \quad \epsilon = \epsilon_1 + i\epsilon_2, \quad \epsilon = \epsilon_1 - i\epsilon_2.$$

Then (28) is equivalent to the statement

$$\partial\bar{\epsilon} = \bar{\partial}\epsilon = 0. \tag{29}$$

What we just obtained is merely the simple fact that in two dimensions any holomorphic function $f(z)$ give rise to the conformal transformation

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dzd\bar{z} \xrightarrow{z=f(w),\, \bar{z}=\bar{f}(\bar{w})} \left|\frac{df}{dw}\right|^2 dwd\bar{w},$$

or in infinitesimal form $f(z) = z + \epsilon(z)$. We note that doing holomorphic maps we regard the variables z and $\bar{z}$ as complex conjugated, so that the metric remains real.

We see that in $D=2$ the conformal group is infinite dimensional and consists of all holomorphic maps with the group multiplication being the composition of maps. This is to be compared to the finite-dimensional conformal group in $D>2$ dimensions, which is isomorphic to $SO(D+1,1)$. It is precisely this infiniteness, which makes the $D=2$ case so special. Imposing this infinite symmetry, we got infinitely many constraints on correlation functions and in some cases can compute them exactly. There is, however, some subtlety which is related to the global definition of holomorphic maps. In fact, the Cauchy-Riemann conditions (28)-(29) are defined only locally. They just kinematically guarantee that the function $\epsilon(z)$ depends only on one variable z, but do not demand the corresponding map to be defined everywhere and be invertible. By definition, the conformal group consists of all invertible and globally defined maps (keeping in mind that SCT requires to add "infinity" point to the manifold). We will therefore distinguish between global and local conformal transformations in two dimensions. Let us construct holomorphic invertible globally defined mappings $f(z)$. Clearly, $f(z)$ could not have any essential singularities of branch points. Hence the only admissible singularities are the poles and then $f(z)$ is a rational function

$$f(z) = \frac{P(z)}{Q(z)}.$$

The polynomial $P(z)$ could not have distinct zeroes because in this case the inverse image of 0 is not well defined. The multiple zeroes are also not allowed, because the inverse function will be multiple valued. So, the only possibility is the linear functions. The same arguments apply to the denominator of $f(z)$ when looking at the behavior near ∞. We conclude that

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ad - bc = 1.$$

The last condition has been applied in order to fix the freedom $(a, b, c, d) \to (\lambda a, \lambda b, \lambda c, \lambda d)$ which does not change $f(z)$. For each global map $f(z)$ one associates a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}),$$

where $\mathrm{SL}(2, \mathbb{C})$ is the group of complex 2×2 matrices with unit determinant. One can easily verify, that the composition of maps $f_2(f_1(z))$ corresponds to the matrix multiplication $M_2 M_1$

$$f_2(f_1(z)) = \frac{a_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + b_2}{c_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + d_2} = \frac{(a_1 a_2 + b_2 c_1)z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2)z + (c_2 b_1 + d_2 d_1)}.$$

Furthermore, we note that even after imposing the condition $ad - bc = 1$ there is still a redundant symmetry $(a, b, c, d) \to (-a, -b, -c, -d)$ and we have to eliminate it. We conclude that the group of global conformal transformations in two dimensions coincides with the Möbius group $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$. The Möbius group is a continuous 6-parametric group, which is known to coincide with the Lorentz group in four dimensions group $SO(3, 1)$ (more precisely its identity component) according to spinor map. Namely, we note that there is a natural map from $\mathbb{R}^{1,3}$ to the space of 2×2 Hermitian matrices

$$(t, x, y, z) \to H = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}$$

such that the quadratic form becomes the determinant $t^2 - x^2 - y^2 - z^2 = \det H$. Now we can let $M \in \mathrm{SL}(2, \mathbb{C})$ act on Hermitian matrices by conjugation (the spin homomorphism)

$$H \to M^+ H M.$$

The kernel of the spin homomorphism consists of two matrices $M = \pm \mathbb{I}$ and hence we come to the conclusion that the Möbius group $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ is isomorphic to the identity component of $SO(1, 3) = SO(3, 1)$, which is consistent with our previous findings.

Probs:

1. Find Noether current corresponding to the special conformal transformation (22).

2. Show, that (27) corresponds to dilations. Identify other elements of the conformal group as elements of $SO(D + 1, 1)$.

Lecture 3: Stress-energy tensor in QFT, conformal Ward identities

We consider Euclidean, $SO(D)$ invariant field theory with *symmetric* stress-energy tensor $T_{\mu\nu} = T_{\nu\mu}$ defined by (7). Quantization of the theory amounts to consider functional integrals of the form

$$\langle X \rangle \overset{\text{def}}{=} \frac{1}{Z} \int X e^{-S[\Phi]} [\mathcal{D}\Phi], \tag{30}$$

where X is a composite field. Usually, we take it in the form

$$X = \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_N(\boldsymbol{x}_N),$$

where $\mathcal{O}_r(\boldsymbol{x}_r)$'s are some local fields

$$\mathcal{O}(\boldsymbol{x}) = \mathcal{F}(\Phi(\boldsymbol{x}), \partial_\mu \Phi(\boldsymbol{x}), \dots)$$

In principle, function $\mathcal{F}(\dots)$ can be arbitrary. The only property in which we insist, is that the fields finitely separated in the space are not allowed. The collection of all local fields is usually thought of as a vector space. One can imagine it as

$$\mathcal{A} = \text{span}\{\Phi^N(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu\partial_\nu\Phi(\boldsymbol{x}), \Phi^N(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x})\partial_\nu\Phi(\boldsymbol{x}), \dots\} \tag{31}$$

As we learn in Quantum Field Theory course the composite fields like $\Phi^N(\boldsymbol{x})$ require renormalization. So, the fields in (31) can be regarded as symbols for the true quantum fields. We will usually denote them as $\mathcal{O}_j(\boldsymbol{x})$, $j = 1, \dots, \infty$, meaning that they form a basis in infinitedimensional vector space $\mathcal{A}$.

At the moment we do not specify $\mathcal{O}_j(\boldsymbol{x})$'s in (30) and try to work in general. The symbolic integration in the right hand side in (30) is known to lack mathematically rigorous definition. Nevertheless, we assume that the functional integral (30) exists and shares some properties of ordinary integral. In particular, since in (30) we integrate over all functions $\Phi(\boldsymbol{x})$ we assume that the measure of integration is invariant with respect to translations

$$\mathcal{D}(\Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x})) = \mathcal{D}(\Phi(\boldsymbol{x})),$$

where $\epsilon(\boldsymbol{x})$ is an arbitrary function. The value of the functional integral (30) should not change. It leads to the following identity

$$\sum_{k=1}^{N} \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \delta_\epsilon \mathcal{O}_k(\boldsymbol{x}_k) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle = \int_{\mathbb{R}^D} \epsilon(\boldsymbol{x}) \langle \text{EOM}(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle \, d^D\boldsymbol{x}, \tag{32}$$

where

$$\text{EOM}(\boldsymbol{x}) = \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right)$$

is the composite field which vanishes on-shell in classical field theory. Now, we note that the function $\epsilon(\boldsymbol{x})$ is arbitrary. In particular, it can be taken to have no support at the point $\boldsymbol{x}_k$. Then the left hand side of (32) should vanish by assumption of locality. Thus we have

$$\langle \text{EOM}(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle = 0 \quad \text{for} \quad \boldsymbol{x} \neq \boldsymbol{x}_k. \tag{33}$$

A field with this property, that is any correlation function involving this field vanishes unless its position $\boldsymbol{x}$ coincides with one of the other insertion points, is called the redundant field. Equation of the form (33) is usually referred as vanishing of correlation function up to contact terms.

There are, in principle, infinitely many redundant fields in QFT. Formally, their existence is related to the more general transformations of integration variable in (30)

$$\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x}) + \epsilon(\boldsymbol{x}) F[\Phi(\boldsymbol{x})]. \tag{34}$$

Generally, we do not known if the measure transforms covariantly under this change. If we would known a Jacobian of this transformation we would find a new redundant field similar to EOM($\boldsymbol{x}$). An important class of transformations (34) comes from the symmetries of the theory. Natural assumption would be that if the action has some symmetry, then the measure should share the same symmetry as well. For example, we expect that

$$\mathcal{D}(\Phi(\boldsymbol{x}+\boldsymbol{\epsilon})) = \mathcal{D}(\Phi(\boldsymbol{x})), \tag{35}$$

as a manifestation of the fact that the change $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{\epsilon}$ just relabels the coordinates in the functional integral. It is easy to justify the invariance (35) for a constant $\boldsymbol{\epsilon}$. But what if $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{x})$ is a function, as in $2D$ CFT? In general, this is the source of anomaly. We will discuss it later in our course.

Exactly, for the transformation $\Phi(\boldsymbol{x}) \to \Phi(\boldsymbol{x} + \boldsymbol{\epsilon}(\boldsymbol{x})) = \Phi(\boldsymbol{x}) + \epsilon_\mu(\boldsymbol{x})\partial_\mu\Phi(\boldsymbol{x}) + \dots$ we do not expect measure issues. Therefore we have an identity

$$\sum_{k=1}^{N} \langle \mathcal{O}_1(\boldsymbol{x}_1)\dots\delta_{\boldsymbol{\epsilon}}\mathcal{O}_k(\boldsymbol{x}_k)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle = \int_{\mathbb{R}^D} \partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x}. \tag{36}$$

In quantum field theory we take (36) as a definition of the stress-energy tensor.

From very general grounds one can assume that variation of local field can depend on $\boldsymbol{\epsilon}(\boldsymbol{x})$ and *finitely* many its derivatives

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(\boldsymbol{x}) = \epsilon_\mu(\boldsymbol{x})\partial_\mu\mathcal{O}(\boldsymbol{x}) + \partial_\mu\epsilon_\nu\mathcal{O}^{\mu\nu}(\boldsymbol{x}) + \dots, \tag{37}$$

where $\mathcal{O}^{\mu\nu}(\boldsymbol{x})$ etc are some local fields. The fact that there are only derivatives of $\boldsymbol{\epsilon}(\boldsymbol{x})$ in (37) reflects general assumption of locality. The fact that there are finitely many terms in (37) is an assumption that the spectra of dimensions of local fields is bounded from below.

Now, let $\mathbb{B}_k$ be the small ball surrounding the point $\boldsymbol{x}_k$, such that $\mathbb{B}_i \cap \mathbb{B}_j = \varnothing$. Then we split the integral in the r.h.s. in (36) as

$$\int_{\mathbb{R}^D} = \sum_{k=1}^{N}\int_{\mathbb{B}_k} + \int_{\bar{\mathbb{R}}^D},$$

where $\bar{\mathbb{R}}^D \cup \mathbb{B}_1 \cup \dots \cup \mathbb{B}_N = \mathbb{R}^D$. The last integral can be transformed by parts

$$\int_{\bar{\mathbb{R}}^D} \partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x} = -\int_{\bar{\mathbb{R}}^D} \epsilon_\nu(\boldsymbol{x})\langle \partial_\mu T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x} + \text{b.t.} \tag{38}$$

Where by b.t. we denoted the boundary terms. They are the sum of integrals over the boundaries of all balls $\mathbb{B}_k$. Now, let us take $\boldsymbol{\epsilon}(\boldsymbol{x})$ of very special form (with no support at $\boldsymbol{x} = \boldsymbol{x}_k$)

$$\boldsymbol{\epsilon}(\boldsymbol{x})\Big|_{\mathbb{B}_k} = 0 \quad \text{for all} \quad k = 1, \dots, N.$$

Then the first term in the right hand side of (38) is the only one who contributes and hence we have

$$\langle \partial_\mu T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle = 0 \quad \text{if} \quad \boldsymbol{x}\in\bar{\mathbb{R}}^D. \tag{39}$$

We note that we can take the balls $\mathbb{B}_k$ arbitrary small and hence (39) is valid for all $\boldsymbol{x}\neq\boldsymbol{x}_k$, i.e. the correlation function (39) vanishes everywhere except for some delta functions supported at the insertion points $\boldsymbol{x}_1\dots\boldsymbol{x}_N$. That is $\partial_\mu T_{\mu\nu}$ is a redundant field.

Having in mind (39), we conclude that

$$\sum_{k=1}^N\langle\mathcal{O}_1(\boldsymbol{x}_1)\dots\delta_{\boldsymbol{\epsilon}}\mathcal{O}_k(\boldsymbol{x}_k)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle = \sum_{k=1}^N\int_{\mathbb{B}_k}\partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^D\boldsymbol{x}+\text{b.t.}$$

Now, we specify everything to the case of $D=2$ and scale $T_{\mu\nu}\to\frac{1}{2\pi}T_{\mu\nu}$ for future convenience. Using the Green theorem

$$\int_{\mathcal{D}}\partial_\mu A^\mu d^2\boldsymbol{x} = \oint_{\partial\mathcal{D}}\varepsilon_{\mu\nu}A^\mu dx^\nu,$$

we find

$$\begin{aligned}\sum_{k=1}^N\langle\mathcal{O}_1(\boldsymbol{x}_1)\dots\delta_{\boldsymbol{\epsilon}}\mathcal{O}_k(\boldsymbol{x}_k)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle &= \frac{1}{2\pi}\sum_{k=1}^N\int_{\mathbb{B}_k}\partial_\mu\epsilon_\nu(\boldsymbol{x})\langle T_{\mu\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)\rangle d^2\boldsymbol{x}-\\ &-\frac{1}{2\pi}\sum_{k=1}^N\oint_{\partial\mathbb{B}_k}\epsilon_\nu(\boldsymbol{x})\varepsilon_{\mu\lambda}\langle T_{\lambda\nu}(\boldsymbol{x})\mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_N(\boldsymbol{x}_N)dx^\mu,\end{aligned} \tag{40}$$

where the contour integral goes in the counterclockwise direction. Since, $\boldsymbol{\epsilon}$ is arbitrary we can take it non-zero only in the vicinity of the point $\boldsymbol{x}_k$. In this case only one term of the sum contributes in (40). We can rewrite (40), formally erasing an average sign, as

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathcal{D}_{\boldsymbol{x}}}\partial_\mu\epsilon_\nu(\boldsymbol{y})T_{\mu\nu}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})d^2\boldsymbol{y}-\frac{1}{2\pi}\oint_{\mathcal{C}_{\boldsymbol{x}}}\epsilon_\nu(\boldsymbol{y})\varepsilon_{\mu\lambda}T_{\lambda\nu}(\boldsymbol{y})\mathcal{O}(\boldsymbol{x})dy^\mu, \tag{41}$$

where $\mathcal{D}_{\boldsymbol{x}}$ is a small disk surrounding the point $\boldsymbol{x}$ and $\mathcal{C}_{\boldsymbol{x}}$ is its boundary.

Now, suppose that our theory is conformally invariant, that is $T_{\mu\nu}$ is traceless. In this case the first term in (41) does not contribute for conformal transformations. Moreover, taking into account peculiarities of $2D$ geometry

$$\begin{aligned}\epsilon_1(\boldsymbol{x})+i\epsilon_2(\boldsymbol{x}) &= \epsilon(z), & T_{11}(\boldsymbol{x})-T_{22}(\boldsymbol{x})-2iT_{12}(\boldsymbol{x}) &= T(z),\\ \epsilon_1(\boldsymbol{x})-i\epsilon_2(\boldsymbol{x}) &= \bar{\epsilon}(\bar{z}), & T_{11}(\boldsymbol{x})-T_{22}(\boldsymbol{x})+2iT_{12}(\boldsymbol{x}) &= \bar{T}(\bar{z}),\end{aligned}$$

we find that

$$\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_N((z_1,\bar{z}_1))\rangle \quad \text{and} \quad \langle\bar{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_N((z_1,\bar{z}_1))\rangle \tag{42}$$

are holomorphic and antiholomorphic functions respectively. Moreover variation of the field $\mathcal{O}(z,\bar{z})$ under the conformal change of coordinates $\boldsymbol{\epsilon}=(\epsilon,\bar{\epsilon})$: $z\to z+\epsilon(z)$, $\bar{z}\to\bar{z}+\bar{\epsilon}(\bar{z})$ is

$$\delta_{\boldsymbol{\epsilon}}\mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z}\epsilon(\zeta)\,T(\zeta)\mathcal{O}(z,\bar{z})d\zeta+\frac{1}{2\pi i}\oint_{\mathcal{C}_{\bar{z}}}\bar{\epsilon}(\bar{\zeta})\,\bar{T}(\bar{\zeta})\mathcal{O}(z,\bar{z})d\bar{\zeta},$$

where both contours $\mathcal{C}_z$ and $\mathcal{C}_{\bar{z}}$ go in the counterclockwise direction. It is important, that correlation functions (42) not only holomorphic (antiholomorphic), but also single valued. It allows us to define the holomorphic variation of local fields (assuming that $\epsilon(\zeta)$ is single-valued as well)

$$\delta_\epsilon \mathcal{O}(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} \epsilon(\zeta)\, T(\zeta)\mathcal{O}(z,\bar{z})d\zeta.$$

Consider infinitesimal transformation of a very special form

$$\epsilon_n(\zeta) = \alpha(\zeta - z)^{n+1} \qquad \alpha \ll 1, \quad n \geq -1.$$

Variation of $\mathcal{O}(z,\bar{z})$ under this special conformal transformation we denote by $L_n\mathcal{O}(z,\bar{z})$: $\delta\mathcal{O} = \alpha L_n\mathcal{O}(z,\bar{z})$. For generic ϵ the variation $\delta_\epsilon\mathcal{O}$ can be expressed in terms of $(L_k\mathcal{O})$ as

$$\delta_\epsilon\mathcal{O} = \epsilon(L_{-1}\mathcal{O}) + \epsilon'(L_0\mathcal{O}) + \frac{\epsilon''}{2!}(L_1\mathcal{O}) + \frac{\epsilon'''}{3!}(L_2\mathcal{O}) + \ldots \tag{43}$$

At least two of these new fields $L_n\mathcal{O}(z,\bar{z})$ we can identify

$$L_{-1}\mathcal{O}(z,\bar{z}) = \partial\mathcal{O}(z,\bar{z}), \qquad L_0\mathcal{O}(z,\bar{z}) = \Delta_{\mathcal{O}}\mathcal{O}(z,\bar{z}),$$

where $\Delta_{\mathcal{O}}$ is called the conformal dimension of the field $\mathcal{O}$. Other fields $L_n\mathcal{O}(z,\bar{z})$ are some new fields which a priori are unrelated to the original one $\mathcal{O}(z,\bar{z})$. In general, we expect that (43) contains only finitely many derivative terms, that is there should exists such $N > 0$, that

$$L_N\mathcal{O}(z,\bar{z}) = 0.$$

It is clear that conformal dimensions of the fields $(L_k\mathcal{O})$ are given by

$$\Delta_{\mathcal{O}^{(k)}} = \Delta_{\mathcal{O}} - k.$$

We assume that the spectra of conformal dimensions $\{\Delta_j\}$ is bounded from below. Actually, we might require even more and forbid negative conformal dimensions at all. It guaranties for example that the two-point functions

$$\langle\mathcal{O}(z,\bar{z})\mathcal{O}(z',\bar{z}')\rangle \sim \frac{1}{|z-z'|^{4\Delta_{\mathcal{O}}}},$$

will fall at infinity. In any case, this restriction implies that for any local operator $\mathcal{O}$ there should exists an integer ν, such that $\mathcal{O}^{(\nu+1)} = 0$. That is, for any local field $\mathcal{O}$ with conformal dimension $\Delta_{\mathcal{O}}$ there are only *finitely* many fields with dimensions $\Delta_{\mathcal{O}} - k$. One can say it other way around. Namely, all conformal dimensions can be represented in the form

$$\Delta_{\mathcal{O}} = \Delta_n + k, \qquad k = 0, 1, 2, \ldots,$$

where Δ_n are some master dimensions. Now, let $\Phi_n(z)$ be the local field corresponding to such master dimension. Then it should have the most simple variation (43)

$$\delta_\epsilon\Phi_n(z) = \epsilon(z)\partial\Phi(z) + \Delta_n\epsilon'(z)\Phi_n(z).$$

Fields with such a property are called primary fields. They obviously satisfy $L_n\Phi = 0$ for all $n > 0$. Under generic, not infinitesimal, holomorphic transformation primary fields behave as generalized tensor fields

$$\Phi(z) \to \left(\frac{dw}{dz}\right)^{\Delta}\Phi(w).$$

From now on the notation $\Phi(z)$ will stick for primary field.

Consider the Ward identity

$$\sum_{k=1}^{N}\langle\mathcal{O}_1(z_1)\ldots\delta_\epsilon\mathcal{O}_k(z_k)\ldots\mathcal{O}_N(z_N)\rangle=\frac{1}{2\pi i}\sum_{k=1}^{N}\oint_{\mathcal{C}_{z_k}}\epsilon(\zeta)\langle T(\zeta)\mathcal{O}_1(z_1)\ldots\mathcal{O}_N(z_N)\rangle d\boldsymbol{\zeta}, \tag{44}$$

We assume that the correlation function

$$\langle T(\zeta)\mathcal{O}_1(z_1)\ldots\mathcal{O}_N(z_N)\rangle.$$

is a single-valued function of ζ falling sufficiently fast at infinity with only possible singularities, the poles at the insertion point z_k. Then, comparing (44) and (43), we find

$$\langle T(\zeta)\mathcal{O}_1(z_1)\ldots\mathcal{O}_N(z_N)\rangle=\sum_{j=1}^{N}\sum_{k=0}^{\nu_j}\frac{k!}{(\zeta-z_j)^{k+1}}\langle\mathcal{O}_1(z_1)\ldots\mathcal{O}_{j-1}(z_{j-1})\mathcal{O}_j^{(k-1)}(z_j)\mathcal{O}_{j+1}(z_{j+1})\mathcal{O}_N(z_N)\rangle. \tag{45}$$

The singular part of this relation is inherited from (44) and (43) unambiguously. The regular part is absent in order to insure proper behavior at infinity. As we will see, the absence of the regular part in (45) is necessary, but not a sufficient condition. The formula (45) is known under the name of conformal Ward identity. It has a particularly neat form for primary fields

$$\langle T(\zeta)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle=\sum_{j=1}^{N}\left(\frac{\Delta_k}{(\zeta-z_k)^2}+\frac{\partial_k}{\zeta-z_k}\right)\langle\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle. \tag{46}$$

One can rewrite (46) in the form of operator product expansion (OPE)

$$T(\zeta)\Phi(z)=\frac{\Delta\Phi(z)}{(\zeta-z)^2}+\frac{\partial\Phi(z)}{\zeta-z}+\ldots \tag{47}$$

where by $\ldots$ we denote terms regular at $\zeta\to z$. Similarly, from (45) we find that

$$T(\zeta)\mathcal{O}(z)=\cdots+\frac{L_2\mathcal{O}(z)}{(\zeta-z)^4}+\frac{L_1\mathcal{O}(z)}{(\zeta-z)^3}+\frac{\Delta_{\mathcal{O}}\mathcal{O}(z)}{(\zeta-z)^2}+\frac{\partial\mathcal{O}(z)}{\zeta-z}+\ldots \tag{48}$$

Now, as we saw before, the conformal dimension of the field $\mathcal{O}$ differs from the conformal dimension of some primary field Φ by an integer positive amount. It suggests that, may be, $\mathcal{O}$ can be obtained from Φ. To do so, we consider regular part of (47)

$$T(\zeta)\Phi(z)=\frac{\Delta\Phi(z)}{(\zeta-z)^2}+\frac{\partial\Phi(z)}{\zeta-z}+L_{-2}\Phi(z)+(\zeta-z)L_{-3}\Phi(z)+(\zeta-z)^2L_{-4}\Phi(z)+\ldots, \tag{49}$$

where $L_{-k}\Phi(z)$ are, by definition, some new local fields (note that $L_{-1}\Phi(z)=\partial\Phi(z)$). Their existence can be justified by functional integral arguments and from (46). For example

$$L_{-2}\Phi(z)\approx T(z)\Phi(z),$$

where the symbol $\approx$ means complicated things related to the regularization of product of operators in QFT etc. We will make it simpler and just postulate, that (49) defines the new fields $L_{-k}\Phi(z)$, which will be called descendant fields (but not only them). It can be expressed as follows

$$L_{-k}\Phi(z)=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\zeta-z)^{1-k}T(\zeta)\Phi(z)d\zeta \tag{50}$$

Using (46), one finds that

$$\langle L_{-k}\Phi(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = \frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{1-k}\langle T(\xi)\Phi(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = \\ = \hat{\mathcal{L}}_{-k}\langle\Phi(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle, \quad (51)$$

where the differential operator $\hat{\mathcal{L}}(z,z_k)$ is given by

$$\hat{\mathcal{L}}_{-k} = \sum_{j=1}^{N}\left[\frac{(k-1)\Delta_j}{(z_j-z)^k} - \frac{\partial_j}{(z_j-z)^{k-1}}\right].$$

The last line in (51) is obtained by Cauchy formula and we leave it as an exercise.

It is interesting to derive conformal properties of the descendant field $L_{-k}\Phi(z)$. In order to do that, we have to derive ones for $T(z)$ itself. Consider the product

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{\Lambda(w)}{(z-w)^3} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots \quad (52)$$

Few comments are in order. First, the conformal dimension of $T(z)$ is 2. Since it is a conserved current it does not acquire quantum corrections. This can be easily seen by comparing scaling properties of both sides of (48). Second, the most singular term in (52) is proportional to the field of dimension 0. We assume that there is only one such field, namely the identity operator, and hence c in (52) is just a number. More singular terms are forbidden because of our assumption $\Delta \geq 0$. The field $\Lambda = L_1 T$ should have dimension 1. Since the product $T(z)T(w)$ is symmetric we have also

$$T(z)T(w) = \frac{c}{2(w-z)^4} + \frac{\Lambda(z)}{(w-z)^3} + \frac{2T(z)}{(w-z)^2} + \frac{T'(z)}{w-z} + \dots \quad (53)$$

Comparing (53) with (52) we find that $\Lambda = 0$. Therefore, under our assumptions,

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots, \quad (54)$$

which is equivalent to the infinitesimal conformal transformation

$$\delta_\epsilon T(z) = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \frac{c}{12}\epsilon'''(z). \quad (55)$$

This infinitesimal transformation can be“exponentiated” to

$$T(z) \to \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w,z\}, \quad (56)$$

where $\{w,z\}$ is the Schwarzian derivative

$$\{w,z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \underset{w=z+\epsilon}{=} \epsilon''' + \dots$$

In order to validate (56) we have to check the group property. It follows from the following property of the Schwarzian derivative

$$\{w, z\} = \left(\frac{d\zeta}{dz}\right)^2 \{w, \zeta\} + \{\zeta, z\}. \tag{57}$$

Moreover, one can check that $\{f, z\}$ vanishes on functions $f(z) = \frac{az+b}{cz+d}$, which correspond to global conformal transformations. The fields with transformation laws like (55), i.e. which behave as primary fields under Möbius transformations, are called quasi-primary or conformal.

Probs:

1. Prove Schwarzian identity (57). Solve the equation

$$\{f(z), z\} = 0.$$

Lecture 4: Conformal families, Virasoro algebra

In the last lecture we have defined descendant fields (50) and showed that correlation functions with one such field and arbitrary number of primary fields can be expressed through the correlation function with primary fields only by some differential operator (51). In order to compute more general correlation functions with multiple insertions of descendant fields

$$\langle L_{-k_1}\Phi(z_1)L_{-k_2}\Phi_2(z_2)\Phi_3(z_3)\dots\Phi_N(z_N)\rangle,$$

we have to use Ward identities with multiple T insertions

$$\langle T(\zeta)T(\eta)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle = \\ = \left[\sum_{j=1}^{N}\left(\frac{\Delta_k}{(\zeta-z_k)^2}+\frac{\partial_k}{\zeta-z_k}\right)+\left(\frac{2}{(\zeta-\eta)^2}+\frac{\partial_\eta}{\zeta-\eta}\right)+\frac{c}{2(\zeta-\eta)^4}\right]\langle T(\eta)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle, \quad (58)$$

which follow from the OPE of T with itself (54). In principle, using the multipoint analog of (58), we can compute arbitrary correlation function of the form

$$\langle L_{-k_1}\Phi_1(z_1)L_{-k_2}\Phi_2(z_2)L_{-k_3}\Phi_3(z_3)\dots L_{-k_N}\Phi_N(z_N)\rangle = \mathcal{D}\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle.$$

It is given by some "hard to find", but explicit, differential operator $\mathcal{D}$, applied to the correlation function involving primary fields only.

It is useful to find conformal transformation properties of the field $L_{-k}\Phi(z)$. In a very general form it is

$$T(\zeta)(L_{-k}\Phi(z)) = \dots + \frac{(L_2L_{-k}\Phi(z))}{(\zeta-z)^4}+\frac{(L_1L_{-k}\Phi(z))}{(\zeta-z)^3}+\frac{(L_0L_{-k}\Phi(z))}{(\zeta-z)^2}+\frac{(L_{-1}L_{-k}\Phi(z))}{\zeta-z}+\dots \quad (59)$$

The fields appearing in the singular part of (59) are, as we will see shortly, not new fields. The fields from the regular part are new and will be denoted by $L_{-l}L_{-k}\Phi^{(-l,-k)}(z)$. From Ward identity we have

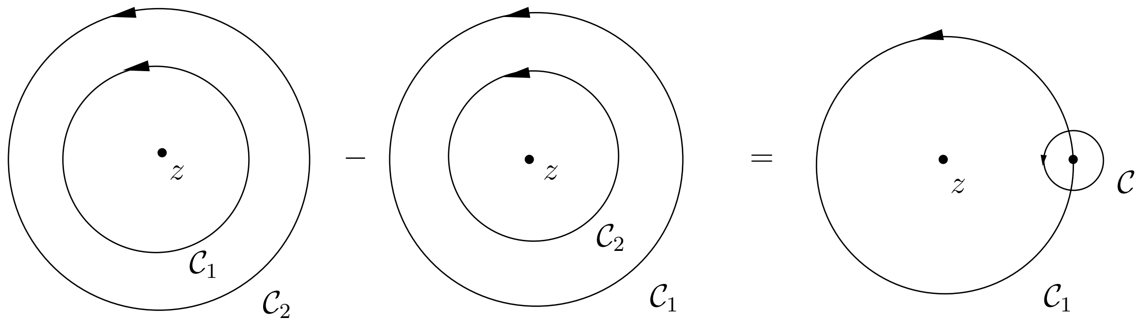
$$L_{-l}L_{-k}\Phi(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\eta-z)^{1-l}T(\eta)\Phi^{(-k)}(z)d\eta = \\ = \frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\eta-z)^{1-l}T(\eta)\left(\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\zeta-z)^{1-k}T(\zeta)\Phi(z)d\zeta\right)d\eta.$$

This procedure can be repeated, producing an infinite tower of descendant fields

$$L_{-k_1}\dots L_{-k_n}\Phi(z). \quad (60)$$

The descendant fields (60) are not all linearly independent. To see this consider the commutator

$$[L_m,L_n]\mathcal{O}(z) = L_mL_n\mathcal{O}(z)-L_nL_m\mathcal{O}(z) = \\ = \frac{1}{2\pi i}\oint_{\mathcal{C}_2}(\eta-z)^{1+m}T(\eta)\left(\frac{1}{2\pi i}\oint_{\mathcal{C}_1}(\zeta-z)^{1+n}T(\zeta)\mathcal{O}(z)d\zeta\right)d\eta- \\ -\frac{1}{2\pi i}\oint_{\mathcal{C}_1}(\zeta-z)^{1+n}T(\zeta)\left(\frac{1}{2\pi i}\oint_{\mathcal{C}_2}(\eta-z)^{1+m}T(\eta)\mathcal{O}(z)d\eta\right)d\zeta.$$



Two integrals above look the same. The only difference is the order of contours $\mathcal{C}_1$ and $\mathcal{C}_2$. In the first integral the contour $\mathcal{C}_1$ goes first around z and then the contour $\mathcal{C}_2$ encircles both the point z and the contour $\mathcal{C}_1$. In the second integral the role of $\mathcal{C}_1$ and $\mathcal{C}_2$ is exchanged. Transforming both contours as shown on the picture we find

$$[L_m, L_n]\mathcal{O}(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}_1}(\zeta - z)^{1+n}\left(\frac{1}{2\pi i}\oint_{\mathcal{C}}(\eta - z)^{1+m}T(\eta)T(\zeta)\mathcal{O}(z)d\eta\right)d\zeta =$$
$$= \frac{1}{2\pi i}\oint_{\mathcal{C}_1}(\zeta - z)^{1+n}\left(\frac{1}{2\pi i}\oint_{\mathcal{C}}(\eta - z)^{1+m}\left(\frac{c}{2(\eta-\zeta)^4} + \frac{2T(\zeta)}{(\eta-\zeta)^2} + \frac{T'(\zeta)}{\eta-\zeta} + \dots\right)\mathcal{O}(z)d\eta\right)d\zeta =$$
$$= \frac{1}{2\pi i}\oint_{\mathcal{C}_1}(\zeta - z)^{1+n}\left(\frac{c}{12}(m^3-m)(\zeta-z)^{-2+m} + 2(m+1)(\zeta-z)^m T(\zeta) + (\zeta-z)^{1+m}T'(\zeta)\right)d\eta. \quad (61)$$

In the second line we used conformal Ward identity for the field T itself. Evaluating the first integral and integrating by part the third one in (61), one arrives to the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \quad (62)$$

known as Virasoro algebra. Since the relations (62) are valid when applied to any field $\mathcal{O}$, we simply erased $\mathcal{O}$ in (62).

From (62) we see that

$$L_0 L_{-k}\Phi(z) = (\Delta + k)L_{-k}\Phi(z), \quad L_n L_{-k}\Phi(z) = (n+k)L_{n-k}\Phi(z) \quad \text{for} \quad n = 1, \dots, k-1,$$
$$L_k L_{-k}\Phi(z) = \left(2k + \frac{c}{12}(k^3 - k)\right)\Phi(z).$$

We come to an important conclusion. The conformal transformation properties of descendant field $L_{-k}\Phi(z)$ involve *only* descendant fields build out of the same primary field Φ. The same is true for generic field (60). This fact leads us to the notion of the conformal family $[\Phi]$, i.e. the set (infinite) of all descendants fields (60). It is clear that because of the relations (62) there are linear relations among (60). The conformal family $[\Phi]$ consists of ordered ones

$$[\Phi] = \text{Span}\left(L_{-k_1}L_{-k_2}\dots L_{-k_n}\Phi(z) |\, k_1 \geq k_2 \geq k_3 \geq \dots\right)$$

Since generic descendant can be obtained from the primary field by successive applications of (50), correlation functions involving descendants can be expressed from correlation function of primary fields only by means of some differential operators. Correlation functions of primaries are further constrained by the so called projective Ward identities. They follow from the fact that any correlation function involving $T(z)$ should fall at infinity as

$$\langle T(z)\dots\rangle \sim \frac{1}{z^4} \quad \text{at} \quad z\to\infty. \tag{63}$$

Writing (63), we assumed that no field has been placed at $z=\infty$. Then $z=\infty$ should be regular point, as all other points. If we introduce local coordinate $z=\frac{1}{w}$, we have

$$\langle T(z)\dots\rangle = \left(\frac{dw}{dz}\right)^2 \langle T(w)\dots\rangle = w^4\langle T(w)\dots\rangle \sim \frac{1}{z^4} \quad \text{at} \quad z\to\infty.$$

In the second equality we used transformation law for $T(z)$ derived before (56). We note that the anomalous term c does not contribute for inversion $z=1/w$. Now, let us apply (63) to the Ward identity (46). Terms of order $1/\zeta$, $1/\zeta^2$ and $1/\zeta^3$ in the right hand side in (46) should vanish

$$\begin{aligned}
&\sum_{k=1}^N \partial_k\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0,\\
&\sum_{k=1}^N \left(\Delta_k + z_k\partial_k\right)\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0,\\
&\sum_{k=1}^N \left(2z_k\Delta_k + z_k^2\partial_k\right)\langle\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0
\end{aligned} \tag{64}$$

Let us study the consequences of these equations (here we do not write $\bar{z}$ dependence of correlation functions for simplicity). The one-point function vanishes unless $\Delta=0$

$$\langle\Phi(z)\rangle \sim \delta_{\Delta,0}.$$

In that case it is a constant. Remember, that due to our assumption, there a unique primary field with $\Delta=0$, the identity operator. Now, let us study the two-point function $\langle\Phi_1(z_1)\Phi_2(z_2)\rangle$. First equation in (64) forces it to depend on the difference of variables only

$$\langle\Phi_1(z_1)\Phi_2(z_2)\rangle = F(z_1-z_2),$$

second equation implies

$$F(z_1-z_2) = \frac{\Lambda(\Delta_1,\Delta_2)}{(z_1-z_2)^{\Delta_1+\Delta_2}},$$

while the third one gives $\Lambda(\Delta_1,\Delta_2)=\mathcal{N}^2(\Delta_1)\delta_{\Delta_1,\Delta_2}$. We note that the factor $\mathcal{N}^2(\Delta_1)$ can always be set equal to one by changing normalizations of the fields. Thus, we have

$$\langle\Phi_1(z_1)\Phi_2(z_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{(z_1-z_2)^{2\Delta_1}}.$$

We call this canonical normalization of the two-point correlation function. The three-point function is given by

$$\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle = C(\Delta_1,\Delta_2,\Delta_3)\prod_{i<j}(z_i-z_j)^{-\Delta_{ij}},$$

where $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$ etc and $C(\Delta_1, \Delta_2, \Delta_3)$ is some constant. In fact, remembering the anti-holomorphic part of the correlation function, this constant is a first "dynamical" quantity we wish to compute. It contains actual information about the theory, explicit Lagrangian for example. We will return to the problem of computation of $C(\Delta_1, \Delta_2, \Delta_3)$ later in this course. Going further, we consider four-point function $\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4)\rangle$. One can show that generic solution has the form

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4)\rangle = \prod_{i<j}(z_i - z_j)^{\gamma_{ij}} F(z), \quad z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \quad \text{and} \quad \sum_j \gamma_{ij} = -2\Delta_i.$$

In general

$$\langle \Phi_1(z_1) \ldots \Phi_N(z_N)\rangle = \prod_{i<j}(z_i - z_j)^{\gamma_{ij}} F(\boldsymbol{z}), \quad \text{where} \quad \sum_j \gamma_{ij} = -2\Delta_i,$$

and $F(\boldsymbol{z})$ is some function of $N-3$ cross ratios.

We will use the projective invariance to set the positions of three points to 0, 1 and ∞. For 4−point correlation function of spinless primary fields (that is $\Delta_k = \bar{\Delta}_k$) one has

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\Phi_3(z_3, \bar{z}_3)\Phi_4(z_4, \bar{z}_4)\rangle = \prod_{i<j}|z_i - z_j|^{2\gamma_{ij}} F(z, \bar{z}),$$

where

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}, \quad \sum_j \gamma_{ij} = -2\Delta_i.$$

The choice of γ_{ij}'s is not unique, which is related to the obvious freedom

$$\prod_{i<j}|z_i - z_j|^{2\gamma_{ij}} \to \prod_{i<j}|z_i - z_j|^{2\gamma_{ij}}|z|^{2A}|1 - z|^{2B},$$

which certainly does not spoil the condition $\sum_j \gamma_{ij} = -2\Delta_i$. We fix this freedom by demanding that the prefactor does not change the behavior of correlation function at $z_1 \to z_2$ and $z_1 \to z_3$

$$\prod_{i<j}|z_i - z_j|^{2\gamma_{ij}} = |z_1 - z_4|^{-2\Delta_1}|z_2 - z_3|^{2(\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3)}|z_2 - z_4|^{2(\Delta_1 + \Delta_3 - \Delta_2 - \Delta_4)}|z_3 - z_4|^{2(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)}.$$

In this case the function $F(z, \bar{z})$ can be expressed through the limit

$$F(z, \bar{z}) = \lim_{\zeta \to \infty} \zeta^{2\Delta_4}\langle \Phi_1(z, \bar{z})\Phi_2(0)\Phi_3(1)\Phi_4(\zeta, \bar{\zeta})\rangle.$$

Combining alltogether we obtain

$$\begin{aligned}&\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\Phi_3(z_3, \bar{z}_3)\Phi_4(z_4, \bar{z}_4)\rangle = \\ &\quad = |z_1 - z_4|^{-2\Delta_1}|z_2 - z_3|^{2(\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3)}|z_2 - z_4|^{2(\Delta_1 + \Delta_3 - \Delta_2 - \Delta_4)}|z_3 - z_4|^{2(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)} \times \\ &\quad\quad \times \lim_{\zeta \to \infty} \zeta^{2\Delta_4}\langle \Phi_1(z, \bar{z})\Phi_2(0)\Phi_3(1)\Phi_4(\zeta, \bar{\zeta})\rangle \quad \text{where} \quad z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.\end{aligned} \tag{65}$$

There is an instructive way to derive projective Ward identities as follows. We remind the variation formula for correlation function of generic fields, not necessarily primary ones,

$$\delta_\epsilon\langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_N(z_N)\rangle = \frac{1}{2\pi i}\sum_{k=1}^{N}\oint_{\mathcal{C}_{z_k}} \epsilon(\zeta)\langle T(\zeta)\mathcal{O}_1(z_1) \ldots \mathcal{O}_N(z_N)\rangle d\boldsymbol{\zeta}.$$

Here $\epsilon = \epsilon(z)$ is a infinitesimal holomorphic function. We saw that the only integral globally defined holomorphic functions are

$$f(z) = \frac{az+b}{cz+d}. \tag{66}$$

Any further conformal transformations must have singularities and can not be one-to-one. So, let us assume that $f(z)$ has a singularity at $z = z_0$. While deriving (41) we implicitly assumed that this singular point is one of the positions z_k of the field insertions in (41). We saw, that singular conformal transformations produce descendant fields. Note, that we can also assume that $z_0 \neq z_k$ and treat this point as a place where the trivial operator I is inserted. Thus, after a singular conformal transformation one can produce a non-trivial descendant field from "nothing", like $T(z) = L_{-2}I(z)$ for example. It means, that the formula (41) should be understood in this generalized sense: where is some number of identity fields in the set of fields $\mathcal{O}_k$. After this remark, we note that the right hand side of (41) can be written as

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_\infty} \epsilon(\zeta)\langle T(\zeta)\mathcal{O}_1(z_1)\dots\mathcal{O}_N(z_N)\rangle d\boldsymbol{\zeta},$$

which vanishes for all functions $\epsilon(z) = \alpha + \beta z + \gamma z^2$. This function corresponds to the infinitesimal form of global conformal transformation (66) with $a = 1+\beta/2$, $b = \alpha$, $c = -\gamma$ and $d = 1-\beta/2$. We note that infinitesimal conformal transformations $\epsilon(z) = \alpha + \beta z + \gamma z^2$ correspond to $SL(2)$ subalgebra of Virasoro algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm}, \quad [L_1, L_{-1}] = 2L_0.$$

Consider quasiprimary fields

$$L_0\phi(z) = \Delta\phi(z), \qquad L_1\phi(z) = 0. \tag{67}$$

The multipoint correlation function involving only such fields should also satisfy the projective Ward identities (64). We note that the condition (67) is less restrictive than the condition for primary field, where we require $L_n\Phi(z) = 0$ for all $n > 0$. In particular $T(z)$ is a quasiprimary field since $L_1T(z) = 0$, but not a primary one $L_2T(z) = c/2 \neq 0$. The projective Ward identities (64) correspond to the following finite identity between correlation functions

$$\langle\phi_1(z_1)\dots\phi_N(z_N)\rangle = \prod_{k=1}^{N}(cz_k+d)^{-2\Delta_k}\Big\langle\phi_1\left(\frac{az_1+b}{cz_1+d}\right)\dots\phi_N\left(\frac{az_N+b}{cz_N+d}\right)\Big\rangle.$$

Probs:

1. Find explicitly differential operator $\mathcal{D}_{l,k}$ defined by

$$\langle L_{-l}L_{-k}\Phi(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = \mathcal{D}_{l,k}\langle\Phi(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle$$

2. Consider the field $T^2(w)$ which appears as a regular term in (54)

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + T^2(w) + O((z-w)).$$

Find conformal transformation law for this field $\delta_\epsilon T^2 = \dots$. Show, that one can find a combination $\Lambda = T^2 + \lambda T''$ which is quasi-primary field. Write an integral transformation law for Λ (similar to (56)).

Lecture 5: Representations of Virasoro algebra, null-vectors

In this lecture we consider general properties of representations of Virasoro algebra. It will be convenient to work in Hamiltonian formalism. For that we have to choose one of the Cartesian coordinates, say y to be Euclidean time, while the other x to be the space coordinate. There are many other choices related to the previous one by rotations. We will use cylindric coordinate frame with new coordinates τ and σ related to the complex coordinate z by exponential map

$$z = e^{-iu}, \quad u = \sigma + i\tau \implies ds^2 = e^{2\tau}\left(d\tau^2 + d\sigma^2\right). \tag{68}$$

We take $\tau \in [-\infty, \infty]$ as a time coordinate and $\sigma \in [0, 2\pi]$ a space one. We see that the map (68) is a conformal one, but not globally defined. It has to singular points $z = 0$ and $z = \infty$.

We used path integral (30) as the very definition of correlation functions. Instead one can use Hamiltonial formalism. The dictionary between the two approaches reads as

$$\langle \mathcal{O}_1(\sigma_1, \tau_1) \ldots \mathcal{O}_N(\sigma_N, \tau_N) \rangle = \langle 0 | \mathcal{T}\left[\mathcal{O}_1(\sigma_1, \tau_1) \ldots \mathcal{O}_N(\sigma_N, \tau_N)\right] | 0 \rangle,$$

where $\mathcal{T}$ stands for the chronological ordering. The Hamiltonian H has the form

$$H = \frac{1}{2\pi} \int_0^{2\pi} T_{\tau\tau} d\sigma = L_0 + \bar{L}_0 - \frac{c}{12}, \tag{69}$$

where the constant shift comes from the Schwarzian in transformation law for $T(z)$.

The vacuum state $|0\rangle$ is an eigenstate of Hamiltonian (69) which obeys

$$L_n |0\rangle = 0 \quad \text{for} \quad n \geq -1.$$

It follows from the fact that the conformal transformations corresponding to L_n with $n \geq -1$ are regular at $z = 0$. Similarly, we have

$$\langle 0 | L_n = 0 \quad \text{for} \quad n \leq 1.$$

It is quite natural to define Hermitian conjugation

$$(L_n)^+ = L_{-n},$$

which is consistent with the requirement that $T(z)$ real in Minkowsky space-time (assuming that both Δ and c are reals). The primary field with dimension Δ generates bra-ket states according to the rule

$$|\Delta\rangle \stackrel{\text{def}}{=} \Phi_\Delta |0\rangle \qquad \langle \Delta | \stackrel{\text{def}}{=} \langle 0 | \Phi(\infty) = \lim_{z \to \infty} \langle 0 | \Phi(z) z^{2L_0}.$$

From the definition of primary fields these states satisfy the conditions

$$L_n |\Delta\rangle = 0, \qquad \langle \Delta | L_{-n} = 0 \quad \text{for} \quad n > 0.$$

We define the Verma module $\mathcal{V}_\Delta$

$$L_{-\boldsymbol{\lambda}} |\Delta\rangle \stackrel{\text{def}}{=} L_{-\lambda_1} \ldots L_{-\lambda_n} |\Delta\rangle : \quad L_n |\Delta\rangle = 0 \quad \text{for} \quad n > 0, \quad L_0 \Delta\rangle = \Delta |\Delta\rangle, \qquad \lambda_1 \geq \lambda_2 \geq \ldots$$

is decomposed into the direct sum of finite dimensional subspaces (here $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \dots$)

$$\mathcal{V}_{\Delta,N} = \text{span}\{L_{-\boldsymbol{\lambda}}|\Delta\rangle : |\boldsymbol{\lambda}| = N\},$$

which are eigenspaces of the operator L_0:

$$L_0 L_{-\boldsymbol{\lambda}}|\Delta\rangle = (\Delta + |\boldsymbol{\lambda}|)L_{-\boldsymbol{\lambda}}|\Delta\rangle$$

On first few levels one has

$$\begin{aligned}
&|\Delta\rangle \quad \text{for} \quad N=0,\\
&L_{-1}|\Delta\rangle \quad \text{for} \quad N=1,\\
&L_{-2}|\Delta\rangle \quad \text{and} \quad L_{-1}^2|\Delta\rangle \quad \text{for} \quad N=2,\\
&L_{-3}|\Delta\rangle,\, L_{-2}L_{-1}|\Delta\rangle \quad \text{and} \quad L_{-1}^3|\Delta\rangle \quad \text{for} \quad N=3,\\
&L_{-4}|\Delta\rangle,\, L_{-3}L_{-1}|\Delta\rangle,\, L_{-2}^2|\Delta\rangle,\, L_{-2}L_{-1}^2|\Delta\rangle \quad \text{and} \quad L_{-1}^4|\Delta\rangle \quad \text{for} \quad N=4.
\end{aligned}$$

In general there are $p(N)$ states in $\mathcal{V}_{\Delta,N}$, where $p(N)$ is the number of partitions of N. It is convenient to define the character (holomorphic block of the partition function)

$$\chi_\Delta(q) \overset{\text{def}}{=} \text{Tr}\left(q^{L_0-\frac{c}{24}}\right)\Big|_{\mathcal{V}_\Delta}.$$

Then we have

$$\chi_\Delta(q) = q^{\Delta-\frac{c}{24}}\sum_{N=0}^{\infty} p(N)q^N = \frac{q^{\Delta-\frac{c}{24}}}{\prod_{k=1}^{\infty}(1-q^k)}$$

So far, we assumed that the values of the conformal dimension Δ and of the central charge c are generic. In this case the Verma module $\mathcal{V}_\Delta$ is irreducible. However, interesting things happen for quantized values of Δ. Remember, that we have postulated that $\Phi_{\Delta=0} = I$ is an identity operator and hence

$$\partial I = L_{-1}I = 0,$$

as it should be for coordinate independent field. But does that consistent with the conformal symmetry? Evidently, we have to check that

$$L_n L_{-1}|\Delta\rangle = 0 \quad \text{for} \quad n > 0. \tag{70}$$

Well, in our case $\Delta = 0$, but we leave it arbitrary in order to see how does that happen. Actually, the condition (70) is satisfied for all $n > 1$ identically. We only have to demand it for $n = 1$

$$0 = L_1 L_{-1}|\Delta\rangle = 2\Delta|\Delta\rangle.$$

We see that $\Delta = 0$ is necessary condition for the vector $L_{-1}|\Delta\rangle$ to vanish. But not sufficient or course. We can claim that for $\Delta = 0$ one can remove the state $L_{-1}|\Delta\rangle$, as well as all its descendants

$$L_{-\boldsymbol{k}}L_{-1}|\Delta\rangle,$$

from our Hilbert space without violating the conformal symmetry. We call such a state a null-vector. The fact that the null-vector vanishes leads us to the trivial conclusion that any correlation function involving the identity operator should satisfy

$$\partial_z\langle I(z)\Phi_1(z_1)\dots\Phi_N(z_N)\rangle = 0.$$

Now we try to generalize this. On level 2 we have two states $L_{-1}^2|\Delta\rangle$ and $L_{-2}|\Delta\rangle$. Probably, we can find their linear combination which vanishes, or, at least, can be safely removed from $\mathcal{V}_\Delta$. We have to require

$$L_n\left(L_{-1}^2+\lambda L_{-2}\right)|\Delta\rangle=0 \quad \text{for} \quad n>0. \tag{71}$$

We note that here we have to impose two conditions (71) with $n=1$ and $n=2$. For $n=1$ we have

$$(4\Delta+2+3\lambda)L_{-1}|\Delta\rangle=0 \quad \Longrightarrow \quad \lambda=-\frac{2(2\Delta+1)}{3}.$$

For $n=2$ we have

$$6\Delta-\frac{2(2\Delta+1)}{3}\left(4\Delta+\frac{c}{2}\right)=0 \quad \Longrightarrow \quad \Delta=\frac{1}{16}\left(5-c\pm\sqrt{(c-1)(c-25)}\right). \tag{72}$$

Going further, we consider a descendant on third level

$$|\chi\rangle=\left(\lambda_1L_{-1}^3+\lambda_2L_{-1}L_{-2}+\lambda_3L_{-3}\right)|\Delta\rangle.$$

If it is a null-vector it has to obey $L_1|\chi\rangle=L_2|\chi\rangle=L_3|\chi\rangle=0$, but since $L_3=[L_2,L_1]$ it is enough to impose only first two conditions. Simple algebra gives

$$\begin{aligned}
L_1|\chi\rangle &= \left(6(\Delta+1)\lambda_1+3\lambda_2\right)L_{-1}^2|\Delta\rangle+\left(2(\Delta+2)\lambda_2+4\lambda_3\right)L_{-2}|\Delta\rangle,\\
L_2|\chi\rangle &= \Big(6(3\Delta+1)\lambda_1+\big(4\Delta+\frac{c}{2}+9\big)\lambda_2+5\lambda_3\Big)L_{-1}|\Delta\rangle.
\end{aligned}$$

We have three linear equations for three unknowns $(\lambda_1,\lambda_2,\lambda_3)$. So, the determinant should vanish

$$12\left(3(\Delta+1)^2+(c-13)(\Delta+1)+12\right)=0,$$

which has two solutions

$$\Delta=\frac{1}{6}\left(7-c\pm\sqrt{(c-1)(c-25)}\right). \tag{73}$$

We see that the expressions for null-vectors (72) and (73) look very similar. One can simplify them by introducing Liouville like parametrization of the central charge and of conformal dimension

$$c=1+6Q^2,\quad Q=b+\frac{1}{b},\quad \Delta=\Delta(\alpha)=\alpha(Q-\alpha).$$

Then the singular vectors appear at the values

$$\alpha=-\frac{b}{2},\qquad \alpha=-\frac{b^{-1}}{2}$$

on level 2 and

$$\alpha=-b,\qquad \alpha=-b^{-1}$$

on level 3. Corresponding null-vectors have the form

$$\left(L_{-1}^2+b^2L_{-2}\right)|\Delta\rangle \quad \text{and} \quad \left(L_{-1}^3+4b^2L_{-1}L_{-2}+2b^2(2b^2-1)L_{-3}\right)|\Delta\rangle,$$

and similar expressions for $b \to b^{-1}$. One can compute null-vectors on higher levels in a similar manner. General result states that at level N, for any two positive integers m and n such that $N = mn$, there exist a null vector $|\chi_{m,n}\rangle$ with (this result is known as Kac-Feigin-Fuks theorem)

$$\Delta = \Delta_{m,n} = \Delta(\alpha_{m,n}), \qquad \alpha_{m,n} = -\frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2}. \tag{74}$$

For generic values of the central charge c, $|\chi_{m,n}\rangle$ is the only one singular vector in the Verma module $\mathcal{V}_{\Delta_{m,n}}$ with $\Delta = \Delta_{m,n} + mn = \Delta_{m,-n}$. We can define the factor space

$$\mathcal{V}_{\Delta_{m,n}} / \mathcal{V}_{\Delta_{m,-n}}$$

without violating the conformal symmetry. The character of the corresponding factor space is

$$\chi'_{m,n}(q) = \frac{q^{\Delta_{m,n} - \frac{c}{24}}(1 - q^{mn})}{\prod_{k=1}^{\infty}(1 - q^k)}.$$

It is convenient to think about representation theory of Virasoro algebra with the help of Shapovalov form, that is Hermitian form defined by

$$\langle\Delta|\Delta\rangle = 1, \qquad (L_n)^+ = L_{-n}.$$

We introduce Gram matrix

$$G_{\boldsymbol{\lambda},\boldsymbol{\mu}} \overset{\text{def}}{=} \langle\Delta|L_{\boldsymbol{\mu}} L_{-\boldsymbol{\lambda}}|\Delta\rangle \tag{75}$$

Clearly, it is block diagonal matrix $G = \{G_0, G_1, G_2 \ldots\}$ with block sizes $p(N) \times p(N)$. The degeneracies of this matrix are closely related to the reducibility of the corresponding Verma module. For example, one has

$$G_1 = 2\Delta,$$

and hence the determinant $\det G_1$ vanishes for degenerate dimension $\Delta = \Delta_{1,1}$. In general, it is clear that any descendant of a singular vector is orthogonal to everything else in Verma module

$$\langle\Delta_{m,n}|L_{\boldsymbol{\mu}} L_{-\boldsymbol{\lambda}}|\chi_{m,n}\rangle = 0 \quad \text{for all} \quad \boldsymbol{\lambda} \text{ and } \boldsymbol{\mu}.$$

That is we have $p(N - mn)$ singular vectors on level N of the form $L_{-\boldsymbol{\lambda}}|\chi_{m,n}\rangle$ with $|\boldsymbol{\lambda}| = N$, which implies that the determinant of the Shapovalov form on level N vanishes as

$$\det G_N \sim \prod_{m,n} (\Delta - \Delta_{m,n})^{p(N-mn)} \tag{76}$$

Let us assume that we have a field $\Phi(z)$ with $\Delta = \Delta_{2,1}$

$$(L_{-1}^2 + b^2 L_{-2})|\Delta_{2,1}\rangle,$$

and consider the following correlation function

$$\Psi(z|z_1, \ldots, z_N) \overset{\text{def}}{=} \langle\Phi(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle.$$

This function satisfies partial differential equation

$$\left[\partial_z^2 + b^2 \sum_{k=1}^{N} \left(\frac{\Delta_k}{(z - z_k)^2} + \frac{\partial_k}{z - z_k}\right)\right] \Psi(z|z_1, \ldots, z_N) = 0.$$

In the case of $N = 3$ this partial differential equation actually becomes an ordinary differential equation. Indeed, in this case the projective Ward identities allow one to express derivatives ∂_k through ∂. As a result we have a hypergeometric equation for correlation function

$$\left[\frac{d^2}{dz^2} + b^2\left(\sum_{k=1}^{3}\left(\frac{\Delta_k}{(z-z_k)^2} - \frac{1}{z-z_k}\frac{d}{dz}\right) - \sum_{i<j}\frac{\Delta_{1,2}+\Delta_{ij}}{(z-z_i)(z-z_j)}\right)\right]\Psi(z|z_1,z_2,z_3) = 0. \tag{77}$$

It is convenient to change $\Psi(z|z_1,z_2,z_3) = \prod_{k=1}^{3}(z-z_k)^{-\frac{b^2}{2}}\tilde{\Psi}(z|z_1,z_2,z_3)$ in such a way that the term with first derivative vanishes. Then (77) reduces to

$$\left(\frac{d^2}{dz^2} + \mathcal{T}(z)\right)\tilde{\Psi}(z|z_1,z_2,z_3) = 0, \qquad \mathcal{T}(z) = \sum_{k=1}^{3}\left(\frac{\delta_k}{(z-z_k)^2} + \frac{c_k}{z-z_k}\right). \tag{78}$$

The parameters δ_k are given by

$$\delta_k = b^2\left(\Delta_k - \frac{1}{2}\right) - \frac{b^4}{4},$$

and three "accessory" parameters c_k are subject to three linear equations following from condition of vanishing of singularity at infinity

$$\mathcal{T}(z) = \frac{1}{z^4} \quad \text{at} \quad z \to \infty,$$

and hence are uniquely determined. Let us look for the solution to this equation in the form

$$\tilde{\Psi}(z|z_1,z_2,z_3) = (z-z_1)^{\lambda}\left(1 + a_1(z-z_1) + \dots\right) \quad \text{at} \quad z \to z_1.$$

In the leading order we obtain two solutions for λ

$$\lambda = b\alpha_1 - \frac{b^2}{2} \quad \text{and} \quad \lambda = 1 - b\alpha_1 + \frac{b^2}{2}.$$

These two exponents correspond to the following behavior of correlation function

$$\Psi(z|z_1,z_2,z_3) = (z-z_1)^{\Delta(\alpha_1 \pm \frac{b}{2}) - \Delta(\alpha_1) - \Delta(-\frac{b}{2})}\left(1 + \dots\right) \quad \text{at} \quad z \to z_1$$

We will interpret this as a fact that the degenerate field $\Phi_{-\frac{b}{2}}$ "fusses" with general field as

$$[\Phi_{-\frac{b}{2}}][\Phi_\alpha] = [\Phi_{\alpha-\frac{b}{2}}] + [\Phi_{\alpha+\frac{b}{2}}]. \tag{79}$$

Probs:

1. Compute singular vectors on level 4.
2. Let $|\chi\rangle$ be the null-vector at level N. How many equations provide the constraints $L_1|\chi\rangle = L_2|\chi\rangle = 0$? Count the number of equations on level 5 and explain that the excess equations are algebraically dependent from non-excess ones.

Lecture 6: Free bosonic CFT I: path integral approach

Let us start with the theory of free massless bosonic field

$$S[\varphi]=\frac{1}{8\pi}\int\big(\partial_\mu\varphi(\boldsymbol{x})\big)^2d^2\boldsymbol{x}. \tag{80}$$

First of all, we notice that this is our "patient": the theory is conformally invariant (at least classically). This follows from the identity

$$\int\partial_\mu\varphi(\boldsymbol{x})\partial_\mu\varphi(\boldsymbol{x})d^2\boldsymbol{x}=-2\int\partial\varphi(z,\bar{z})\bar{\partial}\varphi(z,\bar{z})dzd\bar{z},\quad z=x_1+ix_2,\quad \bar{z}=x_1-ix_2.$$

In this complex form it is obvious, that the action is invariant under conformal transformations

$$z=f(\zeta),\qquad \bar{z}=f^*(\bar{\zeta}).$$

The stress-energy tensor

$$T_{\mu\nu}\overset{\text{def}}{=}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi-\delta_{\mu\nu}\mathcal{L}=\frac{1}{4\pi}\left(\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}\delta_{\mu\nu}\big(\partial_\mu\varphi\big)^2\right),$$

is indeed traceless $T_{\mu\mu}=0$ and hence the components

$$T=-\frac{\pi}{2}\left(T_{11}-T_{22}-2iT_{12}\right)=-\frac{1}{2}(\partial\varphi)^2,\quad \bar{T}=-\frac{\pi}{2}\left(T_{11}-T_{22}+2iT_{12}\right)=-\frac{1}{2}(\bar{\partial}\varphi)^2,$$

obey

$$\bar{\partial}T=\partial\bar{T}=0.$$

on-shell.

Now let us study the theory (80) quantum mechanically. It is easy, since the theory is Gaussian. There are however some subtleties. Consider the partition function

$$Z=\int[\mathcal{D}\varphi]e^{-S}.$$

This integral diverges since the action does not contain the zero mode φ_0 of the field φ: $Z\sim\int d\varphi_0$. We define the measure $[\mathcal{D}\varphi]'$ simply as an integral over all non-zero modes of the field φ.

Moreover, anticipating that we will have to deal with infrared divergencies, we will consider our theory in a finite volume. That is we impose the periodic conditions $\varphi(x_1,x_2+2\pi R)=\varphi(x_1,x_2)$. Let us compute the two-point function in this theory

$$G(\boldsymbol{x}-\boldsymbol{y})\overset{\text{def}}{=}\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{y})\rangle=\frac{1}{Z}\int[\mathcal{D}\varphi]\varphi(\boldsymbol{x})\varphi(\boldsymbol{y})e^{-S}.$$

As usual in Gaussian theory, one has to invert the quadratic form

$$-\Delta G(\boldsymbol{x})=4\pi\delta_R^2(\boldsymbol{x})\quad\text{where}\quad \delta_R^2(\boldsymbol{x})=\delta(x_1)\sum_{n=-\infty}^{\infty}\delta(x_2+2\pi nR),$$

Clearly

$$G(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} K\big(|z - 2i\pi nR|\big) \quad \text{where} \quad -\Delta K(|z|) = 4\pi\delta^2(\boldsymbol{x}).$$

Integrating last equation over the disk of radius r, we obtain

$$-rK'(r) = 2 \qquad \Longrightarrow \qquad K(r) = -2\log r + \text{const} = -\log|z|^2 + \text{const},$$

which implies

$$G(\boldsymbol{x}) = -\log\left(4\sinh\frac{z}{2R}\sinh\frac{\bar{z}}{2R}\right) = -\log\frac{z\bar{z}}{R^2} + O\left(\frac{1}{R^2}\right) \quad \text{at} \quad R \to \infty. \tag{81}$$

We treat R as an infrared cut-off: it is assumed to be infinite, but we keep it large in the intermediate calculations and then send $R \to \infty$ in the final answer.

Multipoint correlation functions are computed by the Wick rules:

$$\begin{aligned}
&\langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\rangle = \\
&\quad = \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_2)\rangle\langle\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\rangle + \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_3)\rangle\langle\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_4)\rangle + \langle\varphi(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_4)\rangle\langle\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_3)\rangle = \\
&\qquad = G(\boldsymbol{x}_1 - \boldsymbol{x}_2)G(\boldsymbol{x}_3 - \boldsymbol{x}_4) + G(\boldsymbol{x}_1 - \boldsymbol{x}_3)G(\boldsymbol{x}_2 - \boldsymbol{x}_4) + G(\boldsymbol{x}_1 - \boldsymbol{x}_4)G(\boldsymbol{x}_2 - \boldsymbol{x}_3) \quad \text{etc}
\end{aligned}$$

We note that the field φ does not look like a conformal field, its correlation functions behave logarithmically rather than power-like. Conformal fields in the theory (80) are represented by the exponential fields

$$e^{i\alpha\varphi(\boldsymbol{x})} \qquad \alpha \in \mathbb{R}. \tag{82}$$

We are interested in multipoint correlation functions

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)} \dots e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle$$

One can compute these correlation functions by expanding exponents in series, then using the Wick theorem and then resuming again. But it is better and much easier to use the following general fact, that for any Φ functional linear in fundamental field φ: $\Phi = \int J(\boldsymbol{x})\varphi(\boldsymbol{x})d^2\boldsymbol{x}$ we have

$$\langle e^{\Phi}\rangle = e^{\frac{1}{2}\langle\Phi^2\rangle}. \tag{83}$$

In our case

$$\Phi = i\sum_{k=1}^{n}\alpha_k\varphi(\boldsymbol{x}_k) = \int J(\boldsymbol{x})\varphi(\boldsymbol{x})d^2\boldsymbol{x} \quad \text{where} \quad J(\boldsymbol{x}) = i\sum_{k=1}^{n}\alpha_k\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_k).$$

Then we have

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)} \dots e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\langle\varphi(\boldsymbol{x}_k)\varphi(\boldsymbol{x}_k)\rangle - \sum_{i<j}\alpha_i\alpha_j\langle\varphi(\boldsymbol{x}_i)\varphi(\boldsymbol{x}_j)\rangle\right).$$

At this point we have a UV problem, since

$$\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = G(0) = \infty.$$

A standard way to deal with it is to introduce the UV cut-off. It s not universal. There many ways to do it, or as one says, there are many regularization schemes. In renormalizable QFT physically observable quantities must be independent on regularization scheme used for their computation. We define the scheme as follows

$$\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x})\rangle = -\log\frac{r_0^2}{R^2},$$

where $r_0 \ll 1$. Then, according to (83), the correlation function has the form

$$\langle e^{i\alpha_1\varphi(\boldsymbol{x}_1)}\dots e^{i\alpha_n\varphi(\boldsymbol{x}_n)}\rangle = \frac{r_0^{\sum\alpha_k^2}}{R^{(\sum\alpha_k)^2}}\prod_{i<j}|z_i - z_j|^{2\alpha_i\alpha_j}. \tag{84}$$

Observables should be independent on the UV cut-off. We define the new field

$$V_\alpha \overset{\text{def}}{=} r_0^{-\alpha^2}e^{i\alpha\varphi} = z_0^{-\frac{\alpha^2}{2}}\bar{z}_0^{-\frac{\alpha^2}{2}}e^{i\alpha\varphi} \tag{85}$$

We note that the operator V_α depends explicitly on a scale and hence has an anomalous conformal dimension $\Delta(\alpha) = \bar{\Delta}(\alpha) = \frac{\alpha^2}{2}$. Even for renormalized operators we see that the correlation function (84) vanishes in the limit $R \to \infty$ unless the neutrality condition

$$\sum_{k=1}^{n}\alpha_k = 0 \tag{86}$$

is satisfied.

It is instructive to derive the anomalous dimension of the operator V_α in different, but equivalent way. We expand

$$e^{a\varphi} = \sum_{k=0}^{\infty}\frac{a^k}{k!}\varphi^k, \tag{87}$$

and express it in terms of Wick ordered quantities and then resum back. The field φ^k is not Wick ordered. Namely, consider the correlation function

$$\langle\varphi(\boldsymbol{x})^k\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle.$$

If one knows how to compute these correlation functions for any n, one knows (in principle) how to compute everything, like correlation functions of exponential operators (82) for example. While computing (87) one meets two types of contractions: either $\varphi(\boldsymbol{x})$'s are contracted among themselves, or with some of the $\varphi(\boldsymbol{x}_j)$'s. For example,

$$\begin{aligned}\langle\varphi(\boldsymbol{x})^2\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle &= \langle\varphi(\boldsymbol{x})^2\rangle\langle\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle+\\ &+\sum_{i\neq j}\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_i)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_j)\rangle\langle\varphi(\boldsymbol{x}_1)\dots\cancel{\varphi(\boldsymbol{x}_i)}\dots\cancel{\varphi(\boldsymbol{x}_j)}\dots\varphi(\boldsymbol{x}_n)\rangle,\end{aligned}$$

or

$$\begin{aligned}\langle\varphi(\boldsymbol{x})^4\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle &= 3\langle\varphi(\boldsymbol{x})^2\rangle\langle\varphi(\boldsymbol{x})^2\rangle\langle\varphi(\boldsymbol{x}_1)\dots\varphi(\boldsymbol{x}_n)\rangle+\\ &+6\langle\varphi(\boldsymbol{x})^2\rangle\sum_{i\neq j}\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_i)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_j)\rangle\langle\varphi(\boldsymbol{x}_1)\dots\cancel{\varphi(\boldsymbol{x}_i)}\dots\cancel{\varphi(\boldsymbol{x}_j)}\dots\varphi(\boldsymbol{x}_n)\rangle+\\ &+\sum_{i\neq j\neq k\neq l}\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_i)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_j)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_k)\rangle\langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{x}_l)\rangle\times\\ &\times\langle\varphi(\boldsymbol{x}_1)\dots\cancel{\varphi(\boldsymbol{x}_i)}\dots\cancel{\varphi(\boldsymbol{x}_j)}\dots\cancel{\varphi(\boldsymbol{x}_k)}\dots\cancel{\varphi(\boldsymbol{x}_l)}\dots\varphi(\boldsymbol{x}_n)\rangle.\end{aligned}$$

According to the formulae above one can define what is called Wick ordered fields (with our choice of UV regularization scheme)

$$\varphi(\boldsymbol{x})^2 =: \varphi(\boldsymbol{x})^2 : +G_0, \quad \varphi(\boldsymbol{x})^4 =: \varphi(\boldsymbol{x})^4 : +6G_0 : \varphi(\boldsymbol{x})^2 : +3G_0^2 \quad \text{where} \quad G_0 = -\log \frac{r_0^2}{R^2}.$$

The field $: \mathcal{O} :$ is ordered, meaning that in correlation function it is contracted only with other fields, not with the one entering the symbol of $\mathcal{O}$. In general, it is clear that

$$\varphi^k(\boldsymbol{x}) = \sum_{l=0}^{[k/2]} \frac{k!}{l!(k-2l)!} \frac{G_0^l}{2^l} : \varphi^{k-2l}(\boldsymbol{x}) : .$$

Substituting this in (87), we have

$$\sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{a^k}{k!} \frac{k!}{l!(k-2l)!} \frac{G_0^l}{2^l} : \varphi^{k-2l}(\boldsymbol{x}) := \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^{n+2l}}{l!n!} \frac{G_0^l}{2^l} : \varphi^n(\boldsymbol{x}) := e^{\frac{a^2 G_0}{2}} : e^{a\varphi(\boldsymbol{x})} : .$$

Applying this to the field $e^{i\alpha\varphi(\boldsymbol{x})}$ we obtain

$$e^{i\alpha\varphi(\boldsymbol{x})} = \left(\frac{r_0^2}{R^2}\right)^{\frac{\alpha^2}{2}} : e^{i\alpha\varphi(\boldsymbol{x})} : \quad \text{or} \quad V_\alpha(\boldsymbol{x}) = \frac{1}{R^{\alpha^2}} : e^{i\alpha\varphi(\boldsymbol{x})} : \tag{88}$$

Now, let us check the conformal Ward identities and find that $V_\alpha(\boldsymbol{x})$ is actually a primary field. First we note that while $\varphi(\boldsymbol{x})$ itself is not a conformal field, its derivative is. The two-point functions have the form

$$\langle \partial\varphi(z)\varphi(w,\bar{w})\rangle = -\frac{1}{(z-w)}, \quad \langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{(z-w)^2}.$$

In multipoint correlation functions we can use

$$\begin{aligned}\partial\varphi(z)\partial\varphi(w) &= -\frac{1}{(z-w)^2} + : \partial\varphi(z)\partial\varphi(w) := \\ &= -\frac{1}{(z-w)^2} + : (\partial\varphi(w))^2 : + : \partial^2\varphi(w)\partial\phi(w) : (z-w) + \frac{1}{2} : \partial^3\varphi(w)\partial\phi(w) : (z-w)^2 + \ldots,\end{aligned}$$

where we expanded the right hand side at $z \to w$.

Now, let us compute the OPE of $T(\zeta)$:

$$T(\zeta) = -\frac{1}{2} : (\partial\varphi(\zeta))^2 : .$$

with $V_\alpha(z)$ (we hide the dependence on $\bar{z}$)

$$T(\zeta)V_\alpha(z) = \frac{\frac{\alpha^2}{2}V_\alpha(z)}{(\zeta-z)^2} + \frac{i\alpha : \partial\varphi(\zeta)V_\alpha(z) :}{\zeta - z} + : T(\zeta)V_\alpha(z) := \frac{\frac{\alpha^2}{2}V_\alpha(z)}{(\zeta-z)^2} + \frac{\partial V_\alpha(z)}{\zeta - z} + \ldots \tag{89}$$

We see that $V_\alpha(z)$ is a primary field with the conformal dimension $\Delta(\alpha) = \frac{\alpha^2}{2}$. As we learned, this should imply the conformal Ward identity

$$\langle T(\zeta)V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n)\rangle = \sum_{k=1}^{n} \left(\frac{\Delta(\alpha_k)}{(\zeta-z_k)^2} + \frac{\partial_k}{\zeta - z_k}\right) \langle V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n)\rangle, \tag{90}$$

where we assumed that the neutrality condition (86) is fulfilled. Similarly one can show that

$$T(\zeta)T(z) = \frac{1}{2(\zeta - z)^4} - \frac{:\partial\varphi(\zeta)\partial\varphi(z):}{(\zeta - z)^2} + :T(\zeta)T(z): = \frac{1}{2(\zeta - z)^4} + \frac{2T(z)}{(\zeta - z)^2} + \frac{\partial T(z)}{\zeta - z} + \ldots, \quad (91)$$

and hence the stress-energy tensor $T(z) = -\frac{1}{2} : (\partial\varphi(z))^2 :$ defines the Virasoro algebra with the central charge

$$c = 1.$$

As we learned before, two identities (89) and (91) are enough to express any correlation function of descendant fields through the correlation function of the primary fields only. In the free theory the last one is pretty trivial

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) \ldots V_{\alpha_n}(z_n, \bar{z}_n) \rangle = \begin{cases} \prod_{i<j} |z_i - z_j|^{2\alpha_i\alpha_j} & \text{if} \quad \sum_{k=1}^n \alpha_k = 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

We note however, that the Ward identities or the OPE's are universal. They do not depend on an actual theory, being just the constraints imposed by the conformal symmetry.

We mention here the following important point. What we constructed in this lecture is the map from the Verma module to the free fields, usually referred as bosonization. Namely, we have a Verma module

$$\mathcal{V}_\Delta = \{\Phi_\Delta, L_{-1}\Phi_\Delta, L_{-1}^2\Phi_\Delta, L_{-2}\Phi_\Delta, L_{-1}^3\Phi_\Delta, L_{-1}L_{-2}\Phi_\Delta, L_{-3}\Phi_\Delta, \ldots\}$$

and a *Fock* module (here all fields are assumed to be Wick ordered)

$$\mathcal{F}_\alpha = \{V_\alpha, (\partial\varphi)V_\alpha, (\partial\varphi)^2V_\alpha, (\partial^2\varphi)V_\alpha, (\partial\varphi)^3V_\alpha, (\partial\varphi)(\partial^2\varphi)V_\alpha, (\partial^3\varphi)V_\alpha, \ldots\}$$

The map $\pi : \mathcal{V}_\Delta \xrightarrow{\pi} \mathcal{F}_\alpha$ goes as follows ($\Delta = \alpha^2$)

$$\Phi_\Delta \xrightarrow{\pi} V_\alpha, \quad L_{-1}\Phi_\Delta \xrightarrow{\pi} i\alpha(\partial\varphi)V_\alpha,$$
$$L_{-1}^2\Phi_\Delta \xrightarrow{\pi} \big(i\alpha\partial^2\varphi - \alpha^2(\partial\varphi)^2\big)V_\alpha, \quad L_{-2}\Phi_\Delta \xrightarrow{\pi} \big(i\alpha\partial^2\varphi - \frac{1}{2}(\partial\varphi)^2\big)V_\alpha, \quad \ldots$$

For generic values of $\Delta = \frac{\alpha^2}{2}$ this map provides an isomorphism between the two modules. However, for special values of α it has a kernel. For example, for $\alpha = 0$ all fields of the form

$$\ldots L_{-n_2}L_{-n_1}L_{-1}\Phi_\Delta$$

are mapped to zero. We interpret this as a fact that the field V_0 is a degenerate field with $\Delta = \Delta_{1,1}$: it has a null-vector at the first level. In the language of bosonization it implies that this field together with all its descendants vanishes identically. Next example of the kernel occurs at the level 2. There are two fields

$$L_{-1}^2\Phi_\Delta \sim \big(i\alpha\partial^2\varphi - \alpha^2(\partial\varphi)^2\big)V_\alpha \quad \text{and} \quad L_{-2}\Phi_\Delta \sim \big(i\alpha\partial^2\varphi - \frac{1}{2}(\partial\varphi)^2\big)V_\alpha.$$

They are linearly dependent provided that either $\alpha = 0$ or $\alpha^2 = \frac{1}{2}$. First possibility corresponds to the one we already know, the descendant of the null-vector for the degenerate field $\Phi_{1,1}$. Second possibility corresponds to the degenerate field $\Phi_{2,1}$ or $\Phi_{1,2}$ for the special value of the central charge that we have

$c = 1$. We note that it follows from (72) that $\Delta_{2,1} = \Delta_{1,2}$ for $c = 1$. This condition can be relaxed if one consider the bosonization map for imprived stress-energy tensor (see exercise 2).

Last, consider the product $V_\alpha(z,\bar{z})V_\beta(w,\bar{w})$. Let us bring it to the Wick ordered form. Using

$$: \varphi(z,\bar{z})^k :: e^{i\beta\varphi(w,\bar{w})} := \sum_{l=0}^{k} \frac{(i\beta)^l k! G^l(z-w)}{(k-l)!\, l!} : \varphi(z,\bar{z})^{l-k} e^{i\beta\varphi(w,\bar{w})} :,$$

we find that

$$: e^{i\alpha\varphi(z,\bar{z})} :: e^{i\beta\varphi(w,\bar{w})} := \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(i\alpha)^k}{k!} \frac{(i\beta)^l k! G^l(z-w)}{(k-l)!\, l!} : \varphi(z,\bar{z})^{l-k} e^{i\beta\varphi(w,\bar{w})} :=$$

$$= \frac{|z-w|^{2\alpha\beta}}{R^{2\alpha\beta}} : e^{i\alpha\varphi(z,\bar{z})} e^{i\beta\varphi(w,\bar{w})} : . \quad (92)$$

Using the relation (88) we can rewrite this in the form of OPE

$$V_\alpha(z,\bar{z})V_\beta(w,\bar{w}) = |z-w|^{2\alpha\beta} V_{\alpha+\beta}(w,\bar{w}) \left(1 + (z-w)\frac{\alpha}{\alpha+\beta} L_{-1} V_{\alpha+\beta}(w,\bar{w}) + \dots\right) \quad \text{at} \quad z \to w. \quad (93)$$

We note that the degree $2\alpha\beta$ in (93) has a natural interpretation $2\alpha\beta = 2\big(\Delta(\alpha+\beta) - \Delta(\alpha) - \Delta(\beta)\big)$, which follows from dimensional analysis of both sides of the OPE.

Probs:

1. Show by explicit free-field calculations that (90) is satisfied.
2. Consider the bosonization map π from Verma module realized by the improved stress-energy tensor
$$T(z) = -\frac{1}{2}(\partial\varphi)^2 + \frac{Q}{\sqrt{2}}\partial^2\varphi \quad \text{where} \quad Q = b + \frac{1}{b}.$$
Find the values of α at which this map has a kernel at levels 1, 2 and 3.

Lecture 7: Free bosonic CFT II: Hamiltonian approach

It is instructive to rederive the results obtained in the previous lecture starting from the $U(1)$ current algebra. Namely, we have a current

$$J(z) = i\partial\varphi(z), \tag{94}$$

which satisfies an OPE

$$J(z)J(w) = \frac{1}{(z-w)^2} + \text{reg} \tag{95}$$

We define the mode of the current $J(z)$ applied to the local field $\mathcal{O}$ by

$$a_n\mathcal{O}(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\mathcal{C}_z} (\xi - z)^n J(\xi)\mathcal{O}(z)dz. \tag{96}$$

Repeating the same calculations which lead us to the Virasoro algebra (62) we obtain commutation relation for the modes (96)

$$[a_m, a_n] = m\delta_{m,-n}, \tag{97}$$

known as Heisenberg algebra. Among other fields there are $U(1)$-primary ones, which have the simplest OPE with $J(\xi)$

$$J(\xi)V_\alpha(z) = \frac{\alpha V_\alpha(z)}{(\xi - z)} + \ldots \quad \text{at} \quad \xi \to z,$$

which implies the following Ward identity

$$\langle J(\xi)V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\rangle = \sum_{k=1}^{N} \frac{\alpha_k}{\xi - z_k}\langle V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\rangle. \tag{98}$$

Now we define the stress-energy tensor via Sugawara formula

$$T(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\mathcal{C}_z} \frac{J(\xi)J(z)}{2(\xi - z)}d\xi, \tag{99}$$

where the components of $T(z)$ satisfy (91). Then in terms of modes Sugawara formula (99) takes the form

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k a_{n-k}, \qquad L_0 = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_{-k}a_k. \tag{100}$$

Then one can easily see that the field $J(z)$ is primary field with $\Delta = 1$

$$T(\xi)J(z) = \frac{J(z)}{(\xi - z)^2} + \frac{J'(z)}{\xi - z} + \ldots$$

It implies that

$$J(\xi) \sim \frac{1}{\xi^2} \quad \text{at} \quad \xi \to \infty.$$

Substituting this asymptotic into the $U(1)$ Ward identity (98), one obtains a $U(1)$ global Ward identity

$$\left(\sum_{k=1}^{N} \alpha_k\right)\langle V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\rangle = 0.$$

Now we notice that the Sugawara construction (99) of $T(\xi)$ leads to additional differential equations on correlation functions. We have

$$\partial V_\alpha = L_{-1} V_\alpha = a_{-1} a_0 V_\alpha = \frac{\alpha}{2\pi i} \oint_{\mathcal{C}_z} (\xi - z)^{-1} J(\xi) V_\alpha(z) d\xi.$$

Applying this to correlation function, we get

$$\left(\partial_k + \sum_{j \neq k} \frac{\alpha_j \alpha_k}{z_j - z_k} \right) \langle V_{\alpha_1}(z_1) \dots V_{\alpha_N}(z_N) \rangle = 0.$$

This implies

$$\langle V_{\alpha_1}(z_1) \dots V_{\alpha_N}(z_N) \rangle \sim \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j}.$$

In Hamiltonian approach an exponential fields V_α corresponds to the highest weight state $|\alpha\rangle$

$$a_n |\alpha\rangle = 0 \quad \text{for} \quad n > 0, \qquad a_0 |\alpha\rangle = \alpha |\alpha\rangle,$$

which generates a representation of the Heisenberg algebra (97) known as Fock module

$$\mathcal{F}_P = \text{span}\Big(a_{\boldsymbol{\lambda}} |P\rangle \overset{\text{def}}{=} a_{-\lambda_1} a_{-\lambda_2} \dots |P\rangle \Big)$$

Then according to Sugawara formula (100) one can define an action of the Virasoro algebra with $c = 1$ on $\mathcal{F}_P$.

To be more precise, in radial quantization picture our bosonic field $\varphi(z, \bar{z})$ admits the following mode expansion

$$\varphi(z, \bar{z}) = -i\hat{q} - i\hat{p} \log \left(\frac{z\bar{z}}{R^2} \right) - i \sum_{k \neq 0} \left(\frac{a_k}{k} z^{-k} + \frac{\bar{a}_k}{k} \bar{z}^{-k} \right),$$

where the modes satisfy the commutation relations

$$[\hat{p}, \hat{q}] = 1, \quad [a_m, a_n] = [\bar{a}_m, \bar{a}_n] = m\delta_{m,-n}, \quad [a_m, \bar{a}_n] = 0.$$

The absolute vacuum state $|0\rangle$ is defined as follows

$$\hat{p}|0\rangle = 0, \quad a_n|0\rangle = \bar{a}_n|0\rangle = 0 \quad \text{for} \quad n > 0.$$

One can also define excited vacuum $|\alpha\rangle$

$$|\alpha\rangle \overset{\text{def}}{=} \lim_{z \to 0} : e^{i\alpha\varphi(z,\bar{z})} : |0\rangle = e^{\alpha\hat{q}} |0\rangle \implies \hat{p}|\alpha\rangle = \alpha|\alpha\rangle.$$

where the normal ordered exponent called the *verter operator* has the form

$$: e^{i\alpha\varphi(z,\bar{z})} := e^{\alpha\hat{q}} \left(\frac{z\bar{z}}{R^2} \right)^{\alpha\hat{p}} \exp \left(-\alpha \sum_{k>0} \left(\frac{a_{-k}}{k} z^k + \frac{\bar{a}_{-k}}{k} \bar{z}^k \right) \right) \exp \left(\alpha \sum_{k>0} \left(\frac{a_k}{k} z^{-k} + \frac{\bar{a}_k}{k} \bar{z}^{-k} \right) \right),$$

i.e. we placed all the creation operators to the left of annihilation ones. We also define the Hermitian conjugation by $a_n^+ = a_{-n}$ and $\bar{a}_n^+ = \bar{a}_{-n}$ and hence the conjugated vacuum satisfies

$$\langle 0| a_n = \langle 0| \bar{a}_n = 0 \quad \text{for} \quad n < 0.$$

Let us compute the two-point Green function (we assume that $|z| > |w|$)

$$\langle\varphi(z,\bar z)\varphi(w,\bar w)\rangle = \langle 0|\varphi(z,\bar z)\varphi(w,\bar w)|0\rangle = -\langle 0|\hat q^2|0\rangle - \langle 0|\hat q\hat p\log\left(\frac{w\bar w}{R^2}\right) + \hat p\hat q\log\left(\frac{z\bar z}{R^2}\right)|0\rangle +$$
$$+\sum_{k>0}\frac{1}{k^2}\Big\langle 0\Big|a_k a_{-k}\left(\frac{w}{z}\right)^k + \bar a_k\bar a_{-k}\left(\frac{\bar w}{\bar z}\right)^k\Big|0\Big\rangle = -\langle 0|\hat q^2|0\rangle - \log\left(\frac{z\bar z}{R^2}\right) + \sum_{k>0}\frac{1}{k}\Big(\left(\frac{w}{z}\right)^k + \left(\frac{\bar w}{\bar z}\right)^k\Big) =$$
$$= -\langle 0|\hat q^2|0\rangle - \log\left(\frac{z\bar z}{R^2}\right) - \log\left(1-\frac{w}{z}\right)\left(1-\frac{\bar w}{\bar z}\right) = -\langle 0|\hat q^2|0\rangle - \log\frac{|z-w|^2}{R^2}.$$

Comparing to (81) we find that the average $\langle 0|\hat q^2|0\rangle$ can be identified to the IR cut-off R as

$$\langle 0|\hat q^2|0\rangle = 0. \tag{101}$$

Now consider the product of two vertex operators

$$:e^{i\alpha\varphi(z,\bar z)}::e^{i\beta\varphi(w,\bar w)}:= e^{\alpha\hat q}\left(\frac{z\bar z}{R^2}\right)^{\alpha\hat p}\exp\left(-\alpha\sum_{k>0}\left(\frac{a_{-k}}{k}z^k + \frac{\bar a_{-k}}{k}\bar z^k\right)\right)\exp\left(\alpha\sum_{k>0}\left(\frac{a_k}{k}z^{-k} + \frac{\bar a_k}{k}\bar z^{-k}\right)\right)\times$$
$$\times e^{\beta\hat q}\left(\frac{w\bar w}{R^2}\right)^{\beta\hat p}\exp\left(-\beta\sum_{k>0}\left(\frac{a_{-k}}{k}w^k + \frac{\bar a_{-k}}{k}\bar w^k\right)\right)\exp\left(\beta\sum_{k>0}\left(\frac{a_k}{k}w^{-k} + \frac{\bar a_k}{k}\bar w^{-k}\right)\right).$$

We note that this expression is not normal ordered. To make it normal ordered, one has to flip red terms with red and blue ones with blue. Using the Baker–Campbell–Hausdorff formula we know that if the commutator of two values $[A,B]$ is a c number, then one has

$$e^A e^B = e^{[A,B]}e^B e^A.$$

In our case we have

$$[A,B] = \alpha\beta\left(\log\left(\frac{z\bar z}{R^2}\right) + \log\left(1-\frac{w}{z}\right)\left(1-\frac{\bar w}{\bar z}\right)\right) = \log\left(\frac{|z-w|^2}{R^2}\right)^{\alpha\beta},$$

and hence we obtain already familiar expression (92)

$$:e^{i\alpha\varphi(z,\bar z)}::e^{i\beta\varphi(w,\bar w)}:= \frac{|z-w|^{2\alpha\beta}}{R^{2\alpha\beta}}:e^{i\alpha\varphi(z,\bar z)}e^{i\beta\varphi(w,\bar w)}:. \tag{102}$$

Using (102) one can show that (here $|z_1| > |z_2| > \cdots > |z_n|$)

$$\langle 0|:e^{i\alpha_1\varphi(z_1,\bar z_1)}:\cdots:e^{i\alpha_n\varphi(z_n,\bar z_n)}:|0\rangle = \prod_{i<j}\frac{|z_i-z_j|^{2\alpha_i\alpha_j}}{R^{2\alpha_i\alpha_j}}\langle 0|e^{\sum\alpha_k\hat q}|0\rangle = \prod_{i<j}\frac{|z_i-z_j|^{2\alpha_i\alpha_j}}{R^{2\alpha_i\alpha_j}} \tag{103}$$

where in the last equality we have used (101). We note that the average in (103) is non-zero in the IR limit $R\to\infty$ only if the neutrality condition $\sum_k\alpha_k = 0$ holds. In fact it is convenient to set $R=1$ from the very beginning holding in mind the neutrality condition.

We note that formally one can consider holomorphic bosonic field

$$\varphi(z) \stackrel{\text{def}}{=} -i\hat q - i\hat p\log(z) - i\sum_{k\neq 0}\frac{a_k}{k}z^{-k},$$

which is intrinsically non-local. This non-locality manifests itself as

$$\langle 0|\varphi(z)\varphi(w)|0\rangle = -\log(z-w)$$

and

$$: e^{i\alpha\varphi(z)} :: e^{i\beta\varphi(w)} := (z-w)^{\alpha\beta} : e^{i\alpha\varphi(z)} e^{i\beta\varphi(w)} : \Longrightarrow : e^{i\alpha\varphi(z)} :: e^{i\beta\varphi(w)} := e^{i\pi\alpha\beta} : e^{i\beta\varphi(w)} :: e^{i\alpha\varphi(z)} : . \quad (104)$$

So, in general holomorphic vertex operators commute on a phase.

Lecture 8: Free fermionic CFT, boson-fermion correspondence

We consider Euclidean theory of massless Dirac fermions

$$S=\frac{1}{8\pi}\int\Psi^{+}\gamma_1\partial\!\!\!/\Psi d^2z=\frac{1}{4\pi}\int\left(\bar{\psi}^*\partial\bar{\psi}+\psi^*\bar{\partial}\psi\right)d^2z, \tag{105}$$

where

$$\gamma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad\gamma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad\Psi=\begin{pmatrix}\psi\\\bar{\psi}\end{pmatrix},\qquad\Psi^{+}=\begin{pmatrix}\psi^*&\bar{\psi}^*\end{pmatrix}$$

Classical equations of motion following from the action (105) are

$$\partial\bar{\psi}=\partial\bar{\psi}^*=0,\qquad\bar{\partial}\psi=\bar{\partial}\psi^*=0.$$

The non-trivial two-point functions are

$$\langle\psi^*(z)\psi(w)\rangle=\langle\psi(z)\psi^*(w)\rangle=\frac{1}{z-w},\qquad\langle\bar{\psi}^*(\bar{z})\bar{\psi}(\bar{w})\rangle=\langle\bar{\psi}(\bar{z})\bar{\psi}^*(\bar{w})\rangle=\frac{1}{\bar{z}-\bar{w}}$$

Since the theory (105) is Gaussian its correlation functions are computed via Wick rules. For example (here the minus sign between the two terms is due to the Grasmanian nature of the field Ψ)

$$\langle\psi^*(z_1)\psi(w_1)\psi^*(z_2)\psi(w_2)\rangle=\frac{1}{z_1-w_1}\frac{1}{z_2-w_2}-\frac{1}{z_1-w_2}\frac{1}{z_2-w_1}$$

In general, the correlation function is given by Cauchy determinant

$$\langle\psi^*(z_1)\psi(w_1)\dots\psi^*(z_n)\psi(w_n)\rangle=\det\left(\frac{1}{z_k-w_l}\right). \tag{106}$$

We will treat the theory (105) as a representation of fermionic algebra. We have two holomorphic current $\psi(z)$ and $\psi^*(z)$ (and two antiholomorphic), which satisfy the OPE

$$\psi(z)\psi^*(w)=\frac{1}{z-w}+\dots,\qquad\psi(z)\psi(w)=\text{reg},\qquad\psi^*(z)\psi^*(w)=\text{reg} \tag{107}$$

One can defined their modes as

$$\psi_r\mathcal{O}(z)\overset{\text{def}}{=}\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{r-\frac{1}{2}}\psi(\xi)\mathcal{O}(z)d\xi,\qquad\psi^*_r\mathcal{O}(z)\overset{\text{def}}{=}\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{r-\frac{1}{2}}\psi^*(\xi)\mathcal{O}(z)d\xi.$$

Then one can computes their commutation relations. The only non-trivial one is

$$\{\psi_r,\psi^*_s\}\mathcal{O}(z)\overset{\text{def}}{=}(\psi_r\psi^*_s+\psi^*_s\psi_r)\mathcal{O}(z)=\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_z}\oint_{\mathcal{C}_\xi}(\xi-z)^{r-\frac{1}{2}}(\eta-z)^{s-\frac{1}{2}}\psi^*(\eta)\psi(\xi)\mathcal{O}(z)d\xi d\eta=$$
$$=\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_z}\oint_{\mathcal{C}_\xi}(\xi-z)^{r-\frac{1}{2}}(\eta-z)^{s-\frac{1}{2}}\left(\frac{1}{\eta-\xi}+\dots\right)\mathcal{O}(z)d\xi d\eta=\delta_{r,-s}\mathcal{O}(z).$$

We have

$$\{\psi_r,\psi_s\}=\{\psi^*_r,\psi^*_s\}=0,\qquad\{\psi_r,\psi^*_s\}=\delta_{r,-s} \tag{108}$$

We note that if one requires the locality of an operator $\mathcal{O}$ with respect to currents $\psi(z)$ and $\psi^*(z)$, the indexes in (108) must be half integer $r, s \in \mathbb{Z} + \frac{1}{2}$. We call such fields Neveu-Schwarz fields (NS). One can also argue that it is natural to assume the existence of the fields wich are semi-local with respect to the currents $\psi(z)$ and $\psi^*(z)$. In this case the modes of ψ_r and ψ_r^* take integer values.

We define the fermionic Wick ordering as

$$: \psi(z)\psi^*(w) := \psi(z)\psi^*(w) - \frac{1}{z-w}.$$

Then one can define the $U(1)$ current algebra inside the fermionic algebra

$$J(z) \stackrel{\text{def}}{=} : \psi^*(z)\psi(z) : . \tag{109}$$

Using (107) one finds

$$J(z)J(w) = \frac{1}{(z-w)^2} + \dots \tag{110}$$

which coincides with (95). Regular term in (110) can be associated with the stress-energy tensor. Explicitly, one has

$$T = \frac{1}{2} : \big(\partial\psi^*\psi - \psi^*\partial\psi\big) :,$$

which defines the Virasoro algebra with the central charge $c = 1$, as it should be. One can also check the following OPE's

$$T(\xi)\psi(z) = \frac{\frac{1}{2}\psi(z)}{(\xi - z)^2} + \frac{\partial\psi(z)}{\xi - z} + \dots, \qquad T(\xi)\psi^*(z) = \frac{\frac{1}{2}\psi^*(z)}{(\xi - z)^2} + \frac{\partial\psi^*(z)}{\xi - z} + \dots,$$

which means that the fields $\psi(z)$ and $\psi^*(z)$ are both primary fields with conformal dimensions $\Delta_\psi = \Delta_{\psi^*} = \frac{1}{2}$.

We have two realizations of the same $U(1)$ current algebra (94) and (109), which implies

$$i\partial\varphi =: \psi^*(z)\psi(z) : \tag{111}$$

Formula (111) is known as bosonization. It terms of the modes, it reads

$$a_n = \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \psi_r^* \psi_{-r+n} :$$

The bosonization formula (111) can be inverted

$$\psi(z)\bar{\psi}(z) =: e^{i\varphi(z,\bar{z})}, \qquad \psi^*(z)\bar{\psi}^*(z) =: e^{-i\varphi(z,\bar{z})}$$

Actually, it will be more convenient to work in terms of holomorphic bosonic field $\varphi(z)$ and holomorphic vertex operators $: e^{i\alpha\varphi} :$ with commutation relations (104). We note that (104) implies that for $\alpha\beta \in 2\mathbb{Z}$ these fields commute, while for $\alpha\beta \in \mathbb{Z}$ they anti-commute i.e. behave as fermions. We identify

$$\psi =: e^{i\varphi(z)} :, \qquad \psi^* =: e^{-i\varphi(z)} :, \tag{112}$$

which has correct OPE (107). In terms of correlation functions, the relation (112) is equivalent to Cauchy determinant identity

$$\det\left(\frac{1}{z_k - w_l}\right) = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{i<j}(z_i - z_j)(w_i - w_j)}{\prod_{k,l}(z_k - w_l)}. \tag{113}$$

We note that the bosonization map identifies more vertex operators $: e^{ik\varphi(z)} :$ with $k \in \mathbb{Z}$ with fermionic operators. In particular,

$$: e^{2i\varphi(z)} :=: \partial\psi\psi :, \; : e^{-2i\varphi(z)} :=: \partial\psi^*\psi^* :, \; : e^{3i\varphi(z)} := \frac{1}{2} : \partial^2\psi\partial\psi\psi :, \; : e^{-3i\varphi(z)} := \frac{1}{2} : \partial^2\psi^*\partial\psi^*\psi^* : \quad \text{etc}$$

Consider highest weight representation $\mathfrak{F}$ of the fermion algebra (108). It is defined by the highest weight state $|0\rangle$ (an image of an identity operator)

$$\psi_r|0\rangle = \psi_r^*|0\rangle = 0 \quad \text{for} \quad r > 0.$$

Then the module $\mathfrak{F}$ is spanned by the vectors of the form

$$\psi_{-\boldsymbol{r}}\psi^*_{-\boldsymbol{s}}|0\rangle \stackrel{\text{def}}{=} \big(\psi_{-r_1}\psi_{-r_2}\dots\big)\big(\psi^*_{-s_1}\psi^*_{-s_2}\dots\big)|0\rangle, \tag{114}$$

where $\boldsymbol{r} = \{r_1 > r_2 > \dots\}$ and $\boldsymbol{s} = \{s_1 > s_2 > \dots\}$ are two strictly decreasing sequences.

It is convenient to think about representation $\mathfrak{F}$ in terms of particles/holes and Dirac sees. Namely, we introduce absolute vacua state $|\varnothing\rangle$ by

$$\psi_r^*|\varnothing\rangle = 0 \quad \text{for all} \quad r.$$

Then the vacua state $|0\rangle$ corresponds to the semi-infinite product state

$$|0\rangle = \psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\psi_{\frac{5}{2}}\dots|\varnothing\rangle.$$

It can be interpreted as follows. We have an infinite line $\mathbb{R}$, where $\mathbb{R}^+$ is all filled by particles at positions $\frac{1}{2}$, $\frac{3}{2}$ etc and $\mathbb{R}^-$ is empty (or filled by holes). Then ψ_{-r} creates a particle at position $-r$ and ψ^*_{-s} deletes a particle at position s (creates a hole at position s). The corresponding sequence of particles and holes is usually referred as Maya diagram. We also define dual absolute vacua state $\langle\varnothing|$ by

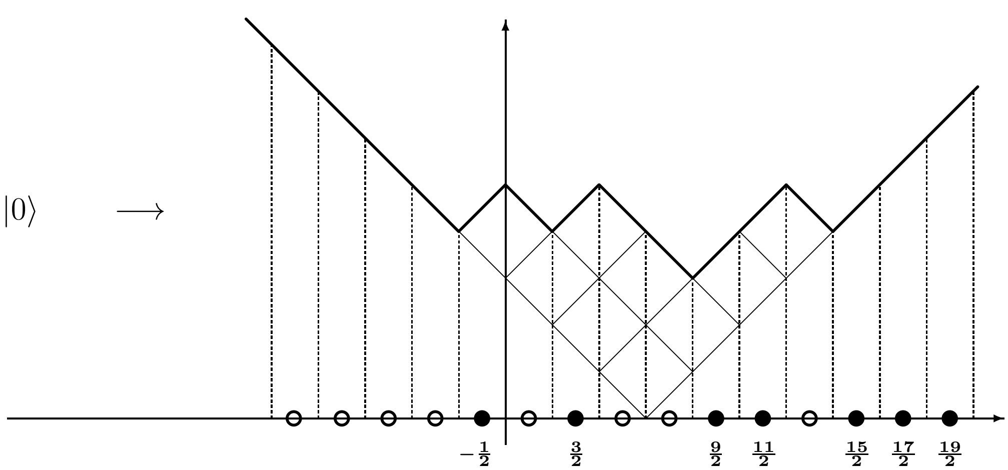
$$\langle\varnothing|\psi_r = 0 \quad \text{for all} \quad r \implies \langle 0| = \langle\varnothing|\dots\psi^*_{\frac{5}{2}}\psi^*_{\frac{3}{2}}\psi^*_{\frac{1}{2}}.$$

Then the state conjugated to (114) can be represented as

$$\langle 0|\psi_{\boldsymbol{s}}\psi^*_{\boldsymbol{r}} \stackrel{\text{def}}{=} \langle 0|\big(\dots\psi_{s_2}\psi_{s_1}\big)\big(\dots\psi^*_{r_2}\psi^*_{r_1}\big).$$

There is a nice bijection between Maya diagrams and charged partitions, which can be explained by the following picture

$$\psi_{-\frac{1}{2}}\psi^*_{-\frac{1}{2}}\psi^*_{-\frac{5}{2}}\psi^*_{-\frac{7}{2}}\psi^*_{-\frac{13}{2}}|0\rangle \qquad \longrightarrow$$



where the "charge" of Maya diagram is a distance between the origin and the bottom corner of the Young diagram. Any Maya diagram has its charge given by an eigenvalue of the operator

$$\hat{c} = a_0 = \sum_{r\in\mathbb{Z}+\frac{1}{2}} :\psi_r^*\psi_{-r}: \Longrightarrow [\hat{c},\psi_r] = -\psi_r, \qquad [\hat{c},\psi_r^*] = -\psi_r^*,$$

and an energy given by an eigenvalue of the operator

$$L_0 = \sum_{r\in\mathbb{Z}+\frac{1}{2}} r:\psi_{-r}\psi_r^*: \Longrightarrow [L_0,\psi_r] = -r\psi_r, \qquad [L_0,\psi_r^*] = -r\psi_r^*.$$

It is interesting to compute the character of the fermionic module $\mathfrak{F}$ (the partition function)

$$Z(q,t) \overset{\text{def}}{=} \operatorname{tr}\left(q^{L_0}t^{\hat{c}}\right)\Big|_{\mathfrak{F}}. \tag{115}$$

According to boson/fermion correspondence, there are two ways to compute this character.

1. Bosonic way: at any value of c we have the bosonic Fock module $\mathcal{F}_c$, which implies the character formula
$$Z(q,t) = \sum_{c=-\infty}^{\infty} t^c \sum_{\boldsymbol{\lambda}} q^{\frac{c^2}{2}+|\boldsymbol{\lambda}|} = \left(\prod_{k=1}^{\infty}\frac{1}{1-q^k}\right)\sum_{c=-\infty}^{\infty} q^{\frac{c^2}{2}}t^c. \tag{116}$$
is the Jacobi theta function.

2. Fermionic way. At position $(k+\frac{1}{2})$ for $k\geq 0$ we have two options: a hole with weight 1 and a particle with weight $t^{-1}q^{k+\frac{1}{2}}$. Similarly, at position $-(k+\frac{1}{2})$ for $k\geq 0$ we could have a hole with the weight $tq^{k+\frac{1}{2}}$ and a particle with the weight 1. Since all the positions are independent we have for partition function
$$Z(q,t) = \prod_{k=1}^{\infty}\left(1+tq^{k+\frac{1}{2}}\right)\left(1+t^{-1}q^{k+\frac{1}{2}}\right). \tag{117}$$

Comparing (116) and (117) we arrive to the Jacobi triple product identity

$$\prod_{k=0}^{\infty}\left(1+tq^{k+\frac{1}{2}}\right)\left(1+t^{-1}q^{k+\frac{1}{2}}\right)(1-q^{k+1}) = \sum_{c=-\infty}^{\infty} q^{\frac{c^2}{2}}t^c. \tag{118}$$

Probs:

1. Consider generalized stress-energy tensor
$$T(z) = \lambda_1 :\partial\psi^*\psi: + \lambda_2 :\partial\psi\psi^*: .$$
Find the conformal dimensions of ψ and ψ^* under this $T(z)$. Compute the central charge.

Lecture 9: $\beta-\gamma$ system, free-field representation of $\mathfrak{sl}_2$ current algebra

We consider Euclidean theory of massless $\beta-\gamma$ system

$$S=\frac{1}{4\pi}\int\left(\bar{\beta}\partial\bar{\gamma}+\beta\bar{\partial}\gamma\right)d^2z, \tag{119}$$

where now, compared to (105), fundamental fields $(\beta,\gamma,\bar{\beta},\bar{\gamma})$ are considered as bosonic variables in the path integral formalism. We have two holomorphic current $\beta(z)$ and $\gamma(z)$, which satisfy the OPE

$$\gamma(z)\beta(w)=\frac{1}{z-w}+\dots,\qquad \beta(z)\beta(w)=\text{reg},\qquad \gamma(z)\gamma(w)=\text{reg} \tag{120}$$

One can defined their modes as (where β and γ have conformal weights 1 and 0 respectively)

$$\beta_r\mathcal{O}(z)\overset{\text{def}}{=}\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^r\beta(\xi)\mathcal{O}(z)d\xi,\qquad \gamma_r\mathcal{O}(z)\overset{\text{def}}{=}\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\xi-z)^{r-1}\gamma(\xi)\mathcal{O}(z)d\xi.$$

Then one can compute the commutation relations

$$[\beta_r,\beta_s]=[\gamma_r,\gamma_s]=0,\qquad [\beta_r,\gamma_s]=\delta_{r,-s}$$

Correlation functions of $\beta-\gamma$ fields are computed by the Wick rules

$$\langle\beta(z_1)\gamma(w_1)\dots\beta(z_n)\gamma(w_n)\rangle=\frac{1}{z_1-w_1}\frac{1}{z_2-w_2}\dots\frac{1}{z_n-w_n}+\text{Perms}=\text{perm}\left(\frac{1}{z_i-w_j}\right) \tag{121}$$

We note that compared to fermionic case (106) there are no signs in (121) and this correlation function can not be written as a determinant but rather as permanent.

One can try to find a representation for the algebra (120) similar to the one in fermionic case (112). A naive attempt $\beta\sim e^{i\alpha\varphi}$, $\gamma\sim e^{i\beta\varphi}$ would fail since the $\beta-\gamma$ fields are bosonic and in particular

$$\gamma(z)\gamma(w)=\gamma^2(w)+\dots$$

This field can be bosonized by two holomorphic bosonic fields u and v as

$$\langle u(z)u(w)\rangle=\langle v(z)v(w)\rangle=-\log(z-w)$$

as (we note that here normal ordering is not needed)

$$\gamma=e^{-u-iv}.$$

Then the β field should be a level one descendant of e^{u+iv}. This follows from charge conservation and dimensional arguments. We have

$$\beta=:(\lambda_1\partial u+\lambda_2\partial v)e^{u+iv}:.$$

Computing the OPE one has

$$\beta(z)\beta(w)=-\frac{2\lambda_1(\lambda_1+i\lambda_2)}{(z-w)^2}e^{u(z)+u(w)+iv(z)+iv(w)}+\text{reg},\qquad \beta(z)\gamma(w)=\frac{\lambda_1+i\lambda_2}{z-w}+\text{reg}$$

i.e. we have $\lambda_1 = 0$, $\lambda_2 = i$, which leads to Friedan, Matrinec and Shenker bosonization of $\beta\gamma$ system [2]

$$\beta = i : \partial v e^{u+iv} :, \qquad \gamma = e^{-u-iv}. \tag{122}$$

Using the bosonization formula (122) one can compute correlation functions. The result should be the same as the one coming from $\beta\gamma-$system Wick rules

$$\begin{aligned}
&\langle \beta(z_1)\gamma(w_1) \dots \beta(z_n)\gamma(w_n) \rangle = \\
&\quad = \langle e^{u(z_1)} \dots e^{u(z_n)} e^{-u(w_1)} \dots e^{-u(w_n)} \rangle \, \partial_{z_1} \dots \partial_{z_n} \langle e^{iv(z_1)} \dots e^{iv(z_n)} e^{-iv(w_1)} \dots e^{-iv(w_n)} \rangle = \\
&\qquad = \frac{\prod (z_i - w_j)}{\prod (z_i - z_j)(w_i - w_j)} \, \partial_{z_1} \dots \partial_{z_n} \frac{\prod (z_i - z_j)(w_i - w_j)}{\prod (z_i - w_j)}.
\end{aligned} \tag{123}$$

The equality of both representations (121) and (123) can be viewed as a bosonic version of the Cauchy determinant identity (113). It is basically equivalent to Borchardt's identity

$$\det \left(\frac{1}{z_i - w_j} \right) \operatorname{perm} \left(\frac{1}{z_i - w_j} \right) = \det \left(\frac{1}{(z_i - w_j)^2} \right).$$

Among other things $\beta\gamma$ system plays a role in free-field representation of current algebras. Let us consider an example of $\mathfrak{sl}(2)$ current algebra

$$\begin{gathered}
h(z)h(w) = \frac{2k}{(z-w)^2} + \text{reg}, \quad h(z)e(w) = \frac{2e(w)}{z-w} + \text{reg}, \quad h(z)f(w) = \frac{-2f(w)}{z-w} + \text{reg} \\
e(z)e(w) = \text{reg}, \quad f(z)f(w) = \text{reg}, \quad e(z)f(w) = \frac{k}{(z-w)^2} + \frac{h(w)}{z-w} + \text{reg},
\end{gathered} \tag{124}$$

where k is the parameter called the level. In terms of modes

$$h_m \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint \xi^m h(\xi) d\xi, \qquad e_m \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint \xi^m e(\xi) d\xi, \qquad f_m \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint \xi^m f(\xi) d\xi.$$

the OPE expansions (124) read as

$$\begin{gathered}
[e_m, e_n] = [f_m, f_n] = 0, \qquad [h_m, e_n] = 2e_{m+n}, \qquad [h_m, f_n] = -2f_{m+n}, \\
[e_m, f_n] = h_{m+n} + mk\delta_{m,-n}, \quad [h_m, h_n] = 2km\delta_{m,-n}
\end{gathered}$$

The relations (124) can be written in the more general form [3] valid for any semi-simple Lie algebra $\mathfrak{g}$

$$J^a(z)J^b(w) = \frac{kK^{ab}}{(z-w)^2} + \frac{f_c^{ab} J^c(w)}{z-w} + \dots, \tag{125}$$

where K^{ab} is the Killing form and f_c^{ab} are the structure constants of $\mathfrak{g}$. In components one has

$$[J_m^a, J_n^b] = f_c^{ab} J_{m+n}^c + mkK^{ab}\delta_{m,-n}.$$

Before describing the usage of the $\beta\gamma$-systems, let us present short review of representation theory of the current algebras (125). First, is the Sugawara construction of the stress-energy tensor. Similarly to the $U(1)$ case one has[4]

$$T(z) = \frac{1}{2(k+g)} K_{ab} \left(J^a J^b \right) \tag{126}$$

where g is the dual Coxeter number, in particular $g = n$ for $\mathfrak{sl}(n)$. For $\mathfrak{g} = \mathfrak{sl}(2)$ we have $g = 2$ and equation (126) reduces to

$$T(z) = \frac{1}{2(k+2)} \left(\frac{1}{2} hh(z) + ef(z) + fe(z) \right).$$

Let us check that (126) indeed satisfies the properties of the stress-energy tensor. Consider the product

$$\begin{aligned}
J^c(w) K_{ab} (J^a J^b)(z) &= \frac{1}{2\pi i} \oint \frac{1}{\xi - z} J^c(w) K_{ab} J^a(\xi) J^b(z) d\xi = \\
&= \frac{1}{2\pi i} \oint \frac{K_{ab}}{\xi - z} \left[\left(\frac{k K^{ca}}{(w-\xi)^2} + \frac{f^{ca}_d J^d(\xi)}{w - \xi} \right) J^b(z) + J^a(\xi) \left(\frac{k K^{cb}}{(w-z)^2} + \frac{f^{cb}_d J^d(z)}{w-z} \right) + \dots \right] d\xi = \\
&= \frac{1}{2\pi i} \oint \frac{1}{\xi - z} \left[\frac{k J^c(z)}{(w-\xi)^2} + \frac{k J^c(\xi)}{(w-z)^2} + f^{cad} J_d(\xi) J_a(z) \left(\frac{1}{w-\xi} - \frac{1}{w-z} \right) + \dots \right] d\xi = \\
&= \frac{2k J^c(z)}{(z-w)^2} + \frac{f^{cad} f_{dae} J^e(z)}{(z-w)^2} + \text{regular}
\end{aligned}$$

where in the second line we have used the OPE (125) and retained only the terms which will contribute to the singular behavior, in the third line we have raised the indexes $f^{abc} = K^{ad} f^{ab}_d$ and used antisymmetry $f^{abc} = -f^{acb}$, while in the fourth line we have used OPE (125) again. Now, using

$$f^{cad} f_{dae} = 2g \delta^c_e,$$

one finds that

$$T(z) J^c(w) = \frac{J^c(z)}{(z-w)^2} + \text{regular} = \frac{J^c(w)}{(z-w)^2} + \frac{\partial J^c(w)}{(z-w)^2} + \dots = \frac{\partial}{\partial w} \left(\frac{J^c(w)}{z-w} \right) + \dots, \tag{127}$$

that is all $J^c(z)$'s are primary fields with conformal dimensions $\Delta = 1$.

Now consider the product

$$\begin{aligned}
T(w) K_{ab} (J^a J^b)(z) &= \frac{1}{2\pi i} \oint \frac{1}{\xi - z} K_{ab} \left[\frac{\partial}{\partial \xi} \left(\frac{J^a(\xi) J^b(z)}{w - \xi} \right) + \frac{\partial}{\partial z} \left(\frac{J^a(\xi) J^b(z)}{w-z} \right) + \dots \right] d\xi = \\
&= \frac{1}{2\pi i} \oint \frac{K_{ab} J^a(\xi) J^b(z)}{(\xi - z)(w-\xi)(w-z)} d\xi + \frac{\partial}{\partial z} \frac{1}{2\pi i} \oint_{\mathcal{C}_z} \frac{K_{ab} J^a(\xi) J^b(w)}{(\xi - z)(w-z)} d\xi + \dots = \\
&= \frac{2 K_{ab} K^{ab}}{(w-z)^4} + \frac{K_{ab} (J^a J^b)(z)}{(w-z)^2} + \frac{\partial}{\partial z} \left(\frac{K_{ab} (J^a J^b)(z)}{(w-z)} \right) + \dots,
\end{aligned} \tag{128}$$

[4]Here we use the following regularization prescription

$$(AB)(w) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_{\mathcal{C}_w} \frac{A(z) B(w)}{z - w} dz$$

We note that in general $(AB)(z) \neq (BA)(z)$.

where in the first line we have used (127), integrated by parts in the second line, and used OPE (125) in the third line. Now it is immediate to see that (128) implies that $T(z)$ satisfies commutation of the Virasoro algebra with the central charge

$$c_{\mathfrak{g}} = \frac{k \dim \mathfrak{g}}{k+g} \qquad \left(\text{in particular} \quad c = \frac{3k}{k+2} \quad \text{for} \quad \mathfrak{sl}(2)\right).$$

In terms of modes the Sugawara formula (126) has the form

$$L_m = \frac{K_{ab}}{2(k+g)} \sum_{n \neq \mathbb{Z}} J^a_{m-n} J^b_n \quad \text{for} \quad m \neq 0, \qquad L_0 = \frac{K_{ab}}{2(k+g)} \left(2 \sum_{n=1}^{\infty} J^a_{-n} J^b_n + J^a_0 J^b_0 \right). \tag{129}$$

Now we come to the representations of the $\mathfrak{sl}(2)$ current algebra. We note that it contains ordinary $\mathfrak{sl}(2)$ subalgebra spanned by the zero modes

$$[h_0, e_0] = 2e_0, \qquad [h_0, f_0] = -2f_0, \qquad [e_0, f_0] = h_0.$$

Thus it looks natural to study representations of the $\mathfrak{sl}(2)$ current algebra generated from the vacuum vector $|j\rangle$

$$h_0|j\rangle = j|j\rangle, \qquad e_0|j\rangle = 0, \qquad f_0 \quad \text{creates new states.} \tag{130}$$

We do not necessarily require that our representation of (h_0, e_0, f_0) is finite dimensional, so j is not supposed to take half-integer values. In terms of fields, one can say that the primary field belongs to some representation of $\mathfrak{g}$ and one has the OPE

$$J^a(\xi)\boldsymbol{\Phi}(z) = \frac{t^a \boldsymbol{\Phi}(z)}{\xi - z} + \ldots,$$

where t^a are the Lie algebra matrices in that particular representation. Having (130) we define the Verma module $\mathcal{V}_j$ as

$$\mathcal{V}_j = \mathrm{Span}\Big(h_{-\boldsymbol{\lambda}} e_{-\boldsymbol{\mu}} f_{-\boldsymbol{\nu}} |j\rangle\Big),$$
$$\boldsymbol{\lambda} = \lambda_1 \geq \lambda_2 \geq \cdots > 0, \quad \boldsymbol{\mu} = \mu_1 \geq \mu_2 \geq \cdots > 0, \quad \boldsymbol{\nu} = \nu_1 \geq \nu_2 \geq \ldots \overset{!}{\geq} 0.$$

We note that it follows from (129) that

$$L_0|j\rangle = \Delta|j\rangle \quad \text{with} \quad \Delta = \frac{j(j+2)}{4(k+2)}$$

We can define the character of $\mathcal{V}_j$ as (which is very similar to the fermionic character (115))

$$\chi_j(q,t) \overset{\text{def}}{=} \mathrm{Tr}\left(q^{L_0} t^{h_0}\right)\Big|_{\mathcal{V}_j}$$

It is clear that

$$h_0 h_{-\boldsymbol{\lambda}} e_{-\boldsymbol{\mu}} f_{-\boldsymbol{\nu}} |j\rangle = \big(j + 2\big(l(\boldsymbol{\mu}) - l(\boldsymbol{\nu})\big)\big)\, h_{-\boldsymbol{\lambda}} e_{-\boldsymbol{\mu}} f_{-\boldsymbol{\nu}} |j\rangle,$$
$$L_0 h_{-\boldsymbol{\lambda}} e_{-\boldsymbol{\mu}} f_{-\boldsymbol{\nu}} |j\rangle = \left(\Delta + |\boldsymbol{\lambda}| + |\boldsymbol{\mu}| + |\boldsymbol{\nu}|\right) h_{-\boldsymbol{\lambda}} e_{-\boldsymbol{\mu}} f_{-\boldsymbol{\nu}} |j\rangle,$$

where $l(\boldsymbol{\lambda})$ denotes the length of the partition $\boldsymbol{\lambda}$. Then the following expression for the character immediately follows

$$\chi_j(q,t) = \frac{q^{\Delta}t^j}{\prod_{k=1}^{\infty}(1-q^j)(1-t^2q^j)(1-t^{-2}q^{j-1})},$$

where the factors $(1-q^j)$, $(1-t^2q^j)$ and $(1-t^{-2}q^{j-1})$ are responsible for $h_{-\boldsymbol{\lambda}}$, $e_{-\boldsymbol{\mu}}$ and $f_{-\boldsymbol{\nu}}$ respectively.

Wakimoto suggested to bosonize $\mathfrak{sl}(2)$ current algebra (124) by $\beta\gamma$ system and another chiral boson φ as [4]

$$e(z) = \beta(z)$$

Probs:

1.

Lecture 10: Operator algebra, conformal properties of OPE, conformal blocks

We introduce the notion of operator algebra. Suppose we have a theory and a complete set of fields $\{\mathcal{O}\} = \{\mathcal{O}_1, \mathcal{O}_2, \dots\}$. Inspired by the intuition learned from free field CFT, we formulate the hypothesis of the operator algebra. Namely, we assume that the product of fields can be expanded in neighboring points

$$\mathcal{O}_i(\boldsymbol{x})\mathcal{O}_j(\boldsymbol{y}) = \sum_k C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})\mathcal{O}_k(\boldsymbol{y}) \quad \text{at} \quad \boldsymbol{x} \to \boldsymbol{y}, \tag{131}$$

with $C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})$ known as structure constants of operator algebra, which are the functions depending only on difference of points (due to translation invariance). As we already saw many times, the relation (131) should be understood as a series of relations on correlation functions

$$\langle \mathcal{O}_i(\boldsymbol{x})\mathcal{O}_j(\boldsymbol{y})X\rangle = \sum_k C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})\langle \mathcal{O}_k(\boldsymbol{y})X\rangle \quad \text{where} \quad X = \mathcal{O}_{j_1}(\boldsymbol{x}_1)\dots\mathcal{O}_{j_n}(\boldsymbol{x}_n).$$

By performing OPE, any N-point correlation can be reduced to the sum of two-point ones, which are universal in CFT and considered as known quantities. The set of structure constants satisfy the condition of associativity

$$\sum_\sigma C_{i_1 i_2}^{i_\sigma}(\boldsymbol{x}_1-\boldsymbol{x}_2)C_{i_\sigma i_3}^{i_4}(\boldsymbol{x}_2-\boldsymbol{x}_3) = \sum_\tau C_{i_1 i_\tau}^{i_4}(\boldsymbol{x}_1-\boldsymbol{x}_3)C_{i_2 i_3}^{i_\tau}(\boldsymbol{x}_2-\boldsymbol{x}_3), \tag{132}$$

also known as *bootstrap* equations in CFT. It can be thought as an infinite system of quadratic equations. In general, this is the task which is hardly believed to be accomplished. However, as we will see in 2D CFT the conformal symmetry puts strong constraints of the coefficients $C_{ij}^k(\boldsymbol{x}-\boldsymbol{y})$. So strong, that in some cases (132) can be solved.

Let us consider 2D CFT. As we learned it is enough to study the correlation functions of primary fields. Consider the OPE of two primary fields

$$\Phi_1(z,\bar{z})\Phi_2(w,\bar{w}) = \sum_{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}} C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(z-w)^{\Delta_k-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}(\bar{z}-\bar{w})^{\Delta_k-\Delta_1-\Delta_2+|\bar{\boldsymbol{\lambda}}|}\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w}), \tag{133}$$

here $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ are the partitions

$$\boldsymbol{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \dots\}, \qquad \bar{\boldsymbol{\lambda}} = \{\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \dots\}, \quad |\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \dots, \quad |\bar{\boldsymbol{\lambda}}| = \bar{\lambda_1} + \bar{\lambda_2} + \dots$$

and

$$\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w}) \overset{\text{def}}{=} (L_{-\lambda_1}L_{-\lambda_2}\dots)\left(\bar{L}_{-\bar{\lambda}_1}\bar{L}_{-\bar{\lambda}_2}\dots\right)\Phi_k(w,\bar{w})$$

is a descendant of the primary field $\Phi_k(w,\bar{w})$. The coordinate dependence of the OPE (133) is completely fixed by scaling properties. The coordinate independent coefficients $C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}$ is what is remained to be determined. In fact there are infinitely many constraints on them following from $2D$ conformal invariance.

Let us "act" on (133) by

$$\frac{1}{2\pi i}\oint_{\mathcal{C}}(\zeta-w)^{n+1}T(\zeta)d\zeta \quad n > 0,$$

where the contour $\mathcal{C}$ encircles both z and w in counterclockwise direction: $\mathcal{C}=\mathcal{C}_z+\mathcal{C}_w$. On the left hand side of (133) it acts only on Φ_1

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\zeta-w)^{n+1}T(\zeta)\Phi_1(z)d\zeta=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}(\zeta-w)^{n+1}\left(\frac{\Delta_1\Phi_1(z)}{(\zeta-z)^2}+\frac{\partial_z\Phi_1(z)}{\zeta-z}+\dots\right)d\zeta=\mathcal{L}_n \tag{134}$$

where

$$\mathcal{L}_n=\Big((z-w)^{n+1}\partial_z+(n+1)\Delta_1(z-w)^n\Big)\Phi_1(z).$$

While on the right hand side simply by

$$\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w})\to L_n\Phi_k^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w,\bar{w})=\sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|-n}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}}\Phi_k^{\boldsymbol{\mu},\bar{\boldsymbol{\lambda}}}(w,\bar{w}). \tag{135}$$

Applying (134) and (135) (and similar antiholomorphic equations) to the OPE (133) one finds

$$\begin{gathered}C_{12}^{k,\boldsymbol{\mu},\bar{\boldsymbol{\lambda}}}\left(\Delta_k+n\Delta_1-\Delta_2+|\boldsymbol{\mu}|\right)=\sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|+n}C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}},\\ C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\mu}}}\left(\Delta_k+n\Delta_1-\Delta_2+|\bar{\boldsymbol{\mu}}|\right)=\sum_{|\bar{\boldsymbol{\lambda}}|=|\bar{\boldsymbol{\mu}}|+n}C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\mu}}}\Lambda_{\bar{\boldsymbol{\mu}}}^{\bar{\boldsymbol{\lambda}}}\end{gathered} \tag{136}$$

These relations are enough to find them uniquely (for generic values of the parameters). We note that because of commutation relations

$$[L_m,L_n]=(m-n)L_{m+n},\qquad[\mathcal{L}_m,\mathcal{L}_n]\overset{!}{=}-(m-n)\mathcal{L}_{m+n},$$

it is enough to impose (136) for $n=1$ and $n=2$ and the rest will follow.

We note that the partitions $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ enter (136) completely independent. It is clear that the solution can be represented in the form

$$C_{12}^{k,\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}=C_{12}^{k}\beta_{\boldsymbol{\lambda}}\beta_{\bar{\boldsymbol{\lambda}}}\quad\text{where by definition}\quad\beta_{\varnothing}=1.$$

Herte C_{12}^k is the structure constant which gives the contribution of the primary field Φ_k in the OPE of Φ_1 with Φ_2. The constants $\beta_{\boldsymbol{\lambda}}$ encode the relative contribution of the descendant fields. The structure constant C_{12}^k factors out of (136) and we have

$$\begin{gathered}\beta_{\boldsymbol{\mu}}\left(\Delta_k+n\Delta_1-\Delta_2+|\boldsymbol{\mu}|\right)=\sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|+n}\beta_{\boldsymbol{\lambda}}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}},\qquad\beta_{\varnothing}=1,\\ L_n\Phi_k^{\boldsymbol{\lambda}}=\sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|-n}\Lambda_{\boldsymbol{\mu}}^{\boldsymbol{\lambda}}\Phi_k^{\boldsymbol{\mu}}.\end{gathered} \tag{137}$$

As we already mentioned, it is enough to consider (137) for $n=1$ and $n=2$ only. It is convenient to imagine, that we are computing the following function

$$\begin{aligned}\boldsymbol{\Phi}_k(w)\overset{\text{def}}{=}\Phi_k(w)+(z-w)\beta_{\{1\}}L_{-1}\Phi(w)+(z-w)^2\left(\beta_{\{1,1\}}L_{-1}^2\Phi_k(w)+\beta_{\{2\}}L_{-2}\Phi_k(w)\right)+\\ +(z-w)^2\left(\beta_{\{1,1,1\}}L_{-1}^3\Phi_k(w)+\beta_{\{2,1\}}L_{-2}L_{-1}\Phi_k(w)+\beta_{\{3\}}L_{-3}\Phi_k(w)\right)+\dots\end{aligned} \tag{138}$$

All the coefficients $\beta_{\boldsymbol{\lambda}}$ in (138) are computed recursively by (137). Consider first examples

Level 1**:**

$$(\Delta_k + \Delta_1 - \Delta_2) = 2\Delta_k \beta_{\{1\}} \implies \beta_{\{1\}} = \frac{\Delta_k + \Delta_1 - \Delta_2}{2\Delta_k}, \tag{139}$$

provided that $\Delta_k \neq 0$, which we assume.

Level 2**:** We have two states

$$(\beta_{\{1,1\}} L_{-1}^2 \Phi_k + \beta_{\{2\}} L_{-2} \Phi_k)$$

From level $\varnothing$ with $n = 2$ we obtain

$$(\Delta_k + 2\Delta_1 - \Delta_2) = \beta_{\{1,1\}} \Lambda_{\varnothing}^{\{1,1\}} + \beta_{\{2\}} \Lambda_{\varnothing}^{\{2\}} \quad \text{where} \quad \Lambda_{\varnothing}^{\{1,1\}} = 6\Delta_k, \quad \Lambda_{\varnothing}^{\{2\}} = 4\Delta_k + \frac{c}{2}.$$

From level 1 with $n = 1$ we obtain

$$\beta_{\{1\}}(\Delta_k + \Delta_1 - \Delta_2 + 1) = \beta_{\{1,1\}} \Lambda_{\{1\}}^{\{1,1\}} + \beta_{\{2\}} \Lambda_{\{1\}}^{\{2\}} \quad \text{where} \quad \Lambda_{\{1\}}^{\{1,1\}} = 2(2\Delta_k + 1), \quad \Lambda_{\{1\}}^{\{2\}} = 3.$$

Altogether we have a system of equations

$$\begin{aligned} \frac{(\Delta_k + \Delta_1 - \Delta_2)(\Delta_k + \Delta_1 - \Delta_2 + 1)}{2\Delta_k} &= 2(2\Delta_k + 1)\beta_{\{1,1\}} + 3\beta_{\{2\}} \\ (\Delta_k + 2\Delta_1 - \Delta_2) &= 6\Delta_k \beta_{\{1,1\}} + \left(4\Delta_k + \frac{c}{2}\right) \beta_{\{2\}}. \end{aligned}$$

This system is non-degenerate, provided that the determinant

$$\det \begin{pmatrix} 2(2\Delta_k + 1) & 3 \\ 6\Delta_k & \left(4\Delta_k + \frac{c}{2}\right) \end{pmatrix} = 2\left(8\Delta_k^2 + (c-5)\Delta_k + \frac{c}{2}\right)$$

does not vanish. We note, that the determinant actually vanishes at the values

$$\Delta_k = \frac{5 - c \pm \sqrt{(c-1)(c-25)}}{16},$$

which are exactly the values of conformal dimensions of the degenerate fields $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$. Similar phenomenon holds at level 1: the coefficient $\beta_{\{1\}}$ has a pole at $\Delta = 0$, i.e. at $\Delta = \Delta_{1,1}$ (see (139)).

In general, at level N we have $p(N)$ constants $\beta_{\boldsymbol{\lambda}}$ with $|\boldsymbol{\lambda}| = N$ subject to $p(N-1) + p(N-2)$ relations, provided that the coefficients $\beta_{\boldsymbol{\mu}}$ with $|\boldsymbol{\mu}| = N - 1$ and $|\boldsymbol{\mu}| = N - 2$ are known. In fact

$$p(N) < p(N-1) + p(N-2), \tag{140}$$

so we have an overdetermined system of equations and it looks puzzled that we have a solution. The resolution of this puzzle is hidden in the fact that the equations followed from (137) for $n = 1$ and $n = 2$ are not all algebraically independent. First example of an algebraic relation is

$$[L_1, [L_1, [L_1, L_2]]] + 6[L_2, [L_1, L_2]] \equiv 0,$$

and hence we have to correct (140) by subtracting an auxiliary term

$$p(N) < p(N-1) + p(N-2) - p(N-5). \tag{141}$$

In fact, there are more algebraic relations. If we take them all into account, we will correct the inequalities (140), (141) to equality known as pentagonal number identity

$$p(N) = p(N-1) + p(N-2) - p(N-5) - p(N-7) + p(N-12) + p(N-15) + \dots . \tag{142}$$

It follows from the identity

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots \tag{143}$$

Indeed from (143) one has

$$1 = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)}\prod_{k=1}^{\infty}(1-q^k) = \left(\sum_{N=0}^{\infty} p(N)q^N\right)\left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots\right),$$

which implies (142). We note that the pentagonal number identity is a special case of Jacobi triple identity (118) with

$$q \to q^3, \qquad t \to -q^{\frac{1}{2}}.$$

The calculation above can be formalized with the help of a dual "basis". Consider a generic descendant

$$\Phi_k^{\boldsymbol{\lambda}} = L_{-\lambda_1}L_{-\lambda_2}\dots\Phi_k.$$

Suppose we found a generator $\chi_{\boldsymbol{\lambda}} \in \mathrm{Vir}_+$

$$\chi_{\boldsymbol{\lambda}} = \sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|} a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} L_{\boldsymbol{\mu}} \quad \text{where} \quad L_{\boldsymbol{\mu}} \overset{\text{def}}{=} L_{\mu_1}L_{\mu_2}\dots,$$

such that

$$\chi_{\boldsymbol{\lambda}}\Phi_k^{\boldsymbol{\lambda}} = \Phi_k, \qquad \chi_{\boldsymbol{\lambda}}\Phi_k^{\boldsymbol{\lambda}'} = 0 \quad \text{for all} \quad \boldsymbol{\lambda}' \neq \boldsymbol{\lambda} \quad |\boldsymbol{\lambda}'| = |\boldsymbol{\lambda}|.$$

For example

$$\dots$$

Give the dual basis constructed, it is easy to show that

$$\beta_{\boldsymbol{\lambda}} = \frac{1}{(z-w)^{\Delta_k-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}}\sum_{|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|} a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\mathcal{L}_{\boldsymbol{\mu}}\cdot(z-w)^{\Delta_k-\Delta_1-\Delta_2} \quad \text{where} \quad \mathcal{L}_{\boldsymbol{\mu}} \overset{\text{def}}{=} \dots\mathcal{L}_{\mu_2}\mathcal{L}_{\mu_1}.$$

We note that there is another in a sense more transparent way to compute the same expansion. Namely, we can rearrange the states by derivatives of quasiprimary fields

$$\boldsymbol{\Phi}_k(w) = \Phi_k(w) + (z-w)\rho_1 L_{-1}\Phi_k + (z-w)^2\left(\rho_2 L_{-1}^2 + \nu_0\left(L_{-2} + \frac{3}{2(2\Delta+1)}L_{-1}^2\right)\right)\Phi_k + \dots \tag{144}$$

By definition, L_1 kills quasiprimary fields. It is clear, that acting by L_1 only one stays within given quasiprimary family. For example, for coefficients ρ_k in

$$\Phi_k(w) + (z-w)\rho_1 L_{-1}\Phi_k + (z-w)^2\rho_2 L_{-1}^2\Phi_k(w) + (z-w)^3\rho_3 L_{-1}^3\Phi_k(w) + \dots,$$

we have a recursive system

$$\begin{aligned}
&2\Delta_k\rho_1 = (\Delta_k + \Delta_1 - \Delta_2),\\
&2(2\Delta_k + 1)\rho_2 = (\Delta_k + \Delta_1 - \Delta_2 + 1)\rho_1,\\
&3(2\Delta_k + 2)\rho_3 = (\Delta_k + \Delta_1 - \Delta_2 + 2)\rho_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

which can be explicitly solved

$$\rho_N = \frac{1}{N!}\prod_{j=1}^{N}\frac{\Delta_k + \Delta_1 - \Delta_2 + j - 1}{2\Delta_k + j - 1}$$

We can proceed further and collect the derivatives of the next quasiprimary field in (144)

$$(z-w)^2\left(\nu_0\left(L_{-2} + \frac{3}{2(2\Delta+1)}L_{-1}^2\right)\Phi_k + (z-w)\nu_1 L_{-1}\left(L_{-2} + \frac{3}{2(2\Delta+1)}L_{-1}^2\right)\Phi_k + \dots\right).$$

Clearly, we have

$$\nu_N = \frac{\nu_0}{N!}\prod_{j=1}^{N}\frac{\Delta_k + 2 + \Delta_1 - \Delta_2 + j - 1}{2(\Delta_k + 2) + j - 1}.$$

The coefficient ν_0 can not be determined from commutation relations with L_1 only, since the quasiprimary state $L_{-2} + \frac{3}{2(2\Delta_k+1)}L_{-1}^2$ belongs to its kernel. One has to use L_2 as well. In fact, we know that

$$\nu_0 = \beta_{\{2\}},$$

which was found before.

We note here a strange singularities of the coefficients ρ_k of the form

$$\rho_2 \sim \frac{1}{2\Delta_k + 1}.$$

This fake singularity is cancelled by the term $\frac{3}{2(2\Delta_k+1)}L_{-1}^2$ in

$$\left(L_{-2} + \frac{3}{2(2\Delta_k+1)}L_{-1}^2\right)\Phi_k.$$

It can be shown that the only singularities of $\beta_{\boldsymbol{\lambda}}$'s are located at the values $\Delta = \Delta_{m,n}$.

Applying the OPE (133) to the 4-point correlation function, we obtain

$$\begin{aligned}
&\langle\Phi_1(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)\Phi_3(z_3,\bar z_3)\Phi_4(z_4,\bar z_4)\rangle = \sum_k C_{12}^k|z_1 - z_2|^{2(\Delta_k - \Delta_1 - \Delta_2)}\times\\
&\times\sum_{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}\beta_{\boldsymbol{\lambda}}\beta_{\bar{\boldsymbol{\lambda}}}(z_1 - z_2)^{|\boldsymbol{\lambda}|}(\bar z_1 - \bar z_2)^{|\bar{\boldsymbol{\lambda}}|}\langle\Big((L_{-\lambda_1}L_{-\lambda_2}\dots)(\bar L_{-\bar\lambda_1}\bar L_{-\bar\lambda_2}\dots)\Phi_k(z_2,\bar z_2)\Big)\Phi_3(z_3,\bar z_3)\Phi_4(z_4,\bar z_4)\rangle =\\
&= \sum_k C_{12}^k C_{k34}\left|\sum_{\boldsymbol{\lambda}}(z_1 - z_2)^{\Delta_k - \Delta_1 - \Delta_2 + |\boldsymbol{\lambda}|}\beta_{\boldsymbol{\lambda}}\Big(\hat{\mathcal{L}}_{-\boldsymbol{\lambda}}\cdot(z_2 - z_3)^{\gamma_{23}}(z_2 - z_4)^{\gamma_{24}}(z_3 - z_4)^{\gamma_{34}}\Big)\right|^2 =\\
&= \sum_k C_{12}^k C_{k34}\left|\sum_{\boldsymbol{\lambda},\boldsymbol{\mu}:|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|}a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\Big(\mathcal{L}_{\boldsymbol{\mu}}\cdot(z_1 - z_2)^{\Delta_k - \Delta_1 - \Delta_2}\Big)\Big(\hat{\mathcal{L}}_{-\boldsymbol{\lambda}}\cdot(z_2 - z_3)^{\gamma_{23}}(z_2 - z_4)^{\gamma_{24}}(z_3 - z_4)^{\gamma_{34}}\Big)\right|^2,
\end{aligned}$$

where $\gamma_{23} = \Delta_4 - \Delta_k - \Delta_3$, $\gamma_{24} = \Delta_3 - \Delta_k - \Delta_4$ and $\gamma_{34} = \Delta_k - \Delta_3 - \Delta_4$ and

$$\hat{\mathcal{L}}_{-\boldsymbol{\lambda}} \stackrel{\text{def}}{=} \hat{\mathcal{L}}_{-\lambda_1}\hat{\mathcal{L}}_{-\lambda_2}\ldots, \quad \hat{\mathcal{L}}_{-n} = \sum_{j=3,4}\left(\frac{(n-1)\Delta_j}{(z_j-z_2)^n} - \frac{\partial_j}{(z_j-z_2)^{n-1}}\right).$$

In the third line we used the explicit form of the three-point function. We see that the 4-point function has split into a sum of modulus squared of holomorphic functions

$$\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\Phi_3(z_3,\bar{z}_3)\Phi_4(z_4,\bar{z}_4)\rangle = \sum_k C_{12}^k C_{k34}\Big|\mathcal{F}_{\Delta_k}(\Delta_1,\Delta_2,\Delta_3,\Delta_4|z_1,z_2,z_3,z_4)\Big|^2,$$

where

$$\mathcal{F}_{\Delta_k}(\Delta_1,\Delta_2,\Delta_3,\Delta_4|z_1,z_2,z_3,z_4) \stackrel{\text{def}}{=} \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}:|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\Big(\mathcal{L}_{\boldsymbol{\mu}}\cdot(z_1-z_2)^{\Delta_k-\Delta_1-\Delta_2}\Big)\times$$
$$\times\Big(\hat{\mathcal{L}}_{-\boldsymbol{\lambda}}\cdot(z_2-z_3)^{\Delta_4-\Delta_k-\Delta_3}(z_2-z_4)^{\Delta_3-\Delta_k-\Delta_4}(z_3-z_4)^{\gamma_{34}}\Big) \quad (145)$$

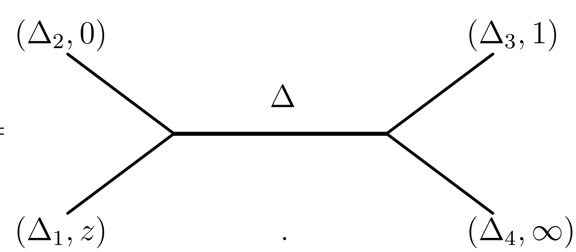
is known as a conformal block. It sums explicitly the contribution of entire conformal family. And this sum is universal in a sense that it does not depend on dynamics of the theory.

In fact, it is rather inconvenient to work with the definition (145). The problem is with the action of the operator $\hat{\mathcal{L}}_{-n}$ which produces many terms inconvenient for "logarithmization". The computation can be facilitated by remembering the projective invariance of correlation functions. Namely, the following formula (65)

$$\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\Phi_3(z_3,\bar{z}_3)\Phi_4(z_4,\bar{z}_4)\rangle =$$
$$= |z_1-z_4|^{-2\Delta_1}|z_2-z_3|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}|z_2-z_4|^{2(\Delta_1+\Delta_3-\Delta_2-\Delta_4)}|z_3-z_4|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}\times$$
$$\times\lim_{\zeta\to\infty}\zeta^{2\Delta_4}\langle\Phi_1(z,\bar{z})\Phi_2(0)\Phi_3(1)\Phi_4(\zeta,\bar{\zeta})\rangle \quad \text{where} \quad z = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}.$$

It means that it is enough to find $z_4\to\infty$ limit of the conformal block

$$\mathfrak{F}_\Delta\begin{pmatrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{pmatrix}\Big|z\Big) =$$



which we define as follows

$$\mathfrak{F}_\Delta\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right) = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\Big(\mathcal{L}_{\boldsymbol{\mu}}\cdot z^{\Delta-\Delta_1-\Delta_2}\Big)\Big(\hat{\mathcal{L}}_{-\boldsymbol{\lambda}}\cdot x^{\Delta_4-\Delta-\Delta_3}\Big)\Bigg|_{x=1},$$

where

$$\mathcal{L}_{\boldsymbol{\mu}} = \ldots\mathcal{L}_{\mu_2}\mathcal{L}_{\mu_1}, \quad \mathcal{L}_n = z^{n+1}\partial_z + (n+1)\Delta_1 z^n,$$
$$\hat{\mathcal{L}}_{-\boldsymbol{\lambda}} = \ldots(\mathcal{L}_{-\lambda_2})(-\mathcal{L}_{-\lambda_1}), \quad \mathcal{L}_{-n} = x^{-n+1}\partial_x + (-n+1)\Delta_3 x^{-n}.$$

All this can be formalized as follows. We introduce the matrix element

$$\langle\Delta'|L_{\boldsymbol{\mu}}\Phi_k(z)L_{-\boldsymbol{\lambda}}|\Delta\rangle \overset{\text{def}}{=} \lim_{\zeta\to\infty}|\zeta|^{2\Delta'}\langle\Phi_{\Delta}^{\boldsymbol{\lambda}}(0)\Phi_{\Delta_k}(z,\bar{z})\Phi_{\Delta'}^{\boldsymbol{\mu}}(\zeta,\bar{\zeta})\rangle,$$
$$L_{-\boldsymbol{\lambda}} = L_{-\lambda_1}L_{-\lambda_2}\ldots, \quad L_{\boldsymbol{\mu}} = \ldots L_{\mu_2}L_{\mu_1}.$$

This matrix element is computed with the help of Virasoro commutation relations and

$$[L_n,\Phi_k(z)] = \left(z^{n+1}\partial_z + \Delta_k(n+1)z^n\right)\Phi_k(z) = \mathcal{L}_n\cdot\Phi_k(z). \tag{146}$$

Note that

$$[L_m,[L_n,\Phi_k(z)]] \overset{!}{=} \mathcal{L}_n\cdot\mathcal{L}_m\cdot\Phi_k(z), \qquad \Phi_k(z)\sim z^{\Delta'-\Delta-\Delta_k}.$$

Using these commutation relations, one can compute any matrix element

$$\frac{\langle\Delta'|L_{\boldsymbol{\mu}}\Phi_kL_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta'|\Phi_k|\Delta\rangle} \overset{\text{def}}{=} \lim_{z\to1}\frac{\langle\Delta'|L_{\boldsymbol{\mu}}\Phi_k(z)L_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta'|\Phi_k(z)|\Delta\rangle},$$

which is some polynomial in Δ, Δ' and Δ_k. We note that the matrix $a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}$ is nothing else, but the inverse Gram matrix

$$a_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} = \left(\Gamma^{-1}\right)_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \quad \text{where} \quad \Gamma_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} = \frac{\langle\Delta|L_{\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta|\Delta\rangle}$$

In these terms the conformal block is given by

$$\mathfrak{F}_{\Delta}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right) = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} z^{\Delta-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}\left(\Gamma^{-1}\right)_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\frac{\langle\Delta_4|\Phi_3L_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta_4|\Phi_3|\Delta\rangle}\frac{\langle\Delta|L_{\boldsymbol{\mu}}\Phi_1|\Delta_2\rangle}{\langle\Delta|\Phi_1|\Delta_2\rangle}. \tag{147}$$

Explicitly, we have

$$\mathfrak{F}_{\Delta}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right) = z^{\Delta-\Delta_1-\Delta_2}\left(1+\frac{(\Delta+\Delta_3-\Delta_4)(\Delta+\Delta_1-\Delta_2)}{2\Delta}z+\ldots\right) \tag{148}$$

In Hamiltonian language the expansion (147) can be viewed as an "insertion of a complete set of states"

$$\mathbf{1} \text{ "} = \text{"} \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} z^{\Delta_k-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}\left(\Gamma^{-1}\right)_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\Big(L_{-\boldsymbol{\lambda}}|\Delta\rangle\langle\Delta|L_{\boldsymbol{\mu}}\Big),$$

and the z dependence in this formula is due to the fact that operators are taken at different time slices.

Using the conformal block decomposition, one can rewrite the associativity condition (132) as

$$\sum_k C_{12}^kC_{k34}\left|\mathfrak{F}_{\Delta_k}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right)\right|^2 = |z|^{-4\Delta_1}\sum_l C_{14}^lC_{l34}\left|\mathfrak{F}_{\Delta_l}\left(\begin{matrix}\Delta_4 & \Delta_3\\ \Delta_1 & \Delta_2\end{matrix}\Big|\frac{1}{z}\right)\right|^2 \tag{149}$$

Probs:

1. Consider the case $\Delta_1 = \Delta(\alpha)$, $\Delta_2 = \Delta(\beta)$ and $\Delta_k = \Delta(\alpha+\beta)$, where

$$\Delta(\alpha) = \alpha(Q-\alpha), \qquad c = 1+6Q^2,$$

and show by explicit calculations on first two levels that the OPE coefficients $\beta_{\boldsymbol{\lambda}}$'s coincide with those following form the free-field formula

$$: e^{2\alpha\varphi(z,\bar{z})} :: e^{2\beta\varphi(w,\bar{w})} :=\frac{R^{4\alpha\beta}}{|z-w|^{4\alpha\beta}} : e^{2\alpha\varphi(z,\bar{z})}e^{2\beta\varphi(w,\bar{w})} : .$$

Lecture 11: BPZ differential equation and three-point function

In this lecture we will study the associativity condition (149) for the special case with one of the fields being degenerate. We will consider the case of $\Phi_{2,1}$ field. Consider 4-point correlation function

$$\Psi(z,\bar{z}) \overset{\text{def}}{=} \langle \Phi_{-\frac{b}{2}}(z,\bar{z})\Phi_{\alpha_1}(z_1,\bar{z}_1)\Phi_{\alpha_2}(z_2,\bar{z}_2)\Phi_{\alpha_3}(z_3,\bar{z}_3)\rangle.$$

which satisfies BPZ differential equation (77) and similar anti-holomorphic equation. Using the projective invariance, one can set $z_1 = 0$, $z_2 = \infty$ and $z_3 = 1$

$$\left[z(1-z)\partial^2 + b^2\left((2z-1)\partial + \frac{\Delta_1}{z} + \frac{\Delta_3}{1-z} + \Delta_{2,1} - \Delta_2\right)\right]\Psi(z,\bar{z}) = 0. \tag{150}$$

It can be brought to the conventional hypergeometric form by the following change of variables

$$\Psi(z,\bar{z}) = z^{b\alpha_1}(1-z)^{b\alpha_3} f(z).$$

We obtain

$$\left[z(1-z)\partial^2 + \big(C - (A+B+1)z\big)\partial - AB\right] f(z) = 0, \tag{151}$$

where

$$A = \frac{1}{2} + b(\alpha_1+\alpha_3-Q) + b\left(\alpha_2 - \frac{Q}{2}\right), \quad B = \frac{1}{2} + b(\alpha_1+\alpha_3-Q) - b\left(\alpha_2 - \frac{Q}{2}\right),$$
$$C = 1 + b\,(2\alpha_1 - Q)\,.$$

This equation has two solutions with diagonal monodromy around $z = 0$ expressed through hypergeometric function

$$F\left(\left.{}^{A\,B}_{\;C}\right| z\right) \qquad \text{and} \qquad z^{1-C}F\left(\left.{}^{1+A-C\;1+B-C}_{\quad 2-C}\right| z\right),$$

where the second one is obtained from the first one by substitution $\alpha_1 \to Q - \alpha_1$. For the original equation (150) we have

$$\mathcal{F}^s_-(z) = z^{b\alpha_1}(1-z)^{b\alpha_3}F\left(\left.{}^{A\,B}_{\;C}\right| z\right), \qquad \mathcal{F}^s_+(z) = z^{b(Q-\alpha_1)}(1-z)^{b\alpha_3}F\left(\left.{}^{1+A-C\;1+B-C}_{\quad 2-C}\right| z\right).$$

We note that these solutions correspond to $s-$chanel conformal blocks

$$\mathcal{F}^s_\pm(z) = \mathfrak{F}_{\Delta_\pm}\left(\left.\begin{matrix}\Delta_1 & \Delta_2\\ \Delta_{2,1} & \Delta_3\end{matrix}\right| z\right),$$

so that they have simple monodromic properties at $z = 0$.

There is another basis of solutions to (151) with diagonal monodromy around $z = \infty$

$$z^{-A}F\left(\left.{}^{A\;1+A-C}_{\;1+A-B}\right| \frac{1}{z}\right), \qquad z^{-B}F\left(\left.{}^{A\;1+B-C}_{\;1+B-A}\right| \frac{1}{z}\right).$$

which correspond to $t-$channel conformal blocks.

$$\mathcal{F}^t_-(z)=z^{-2\Delta_{2,1}}\left(\frac{1}{z}\right)^{b\alpha_2}\left(1-\frac{1}{z}\right)^{b\alpha_3}F\left(\left.{}^{A\ 1+A-C}_{\ \ 1+A-B}\right|\frac{1}{z}\right),$$
$$\mathcal{F}^t_+(z)=z^{-2\Delta_{2,1}}\left(\frac{1}{z}\right)^{b(Q-\alpha_2)}\left(1-\frac{1}{z}\right)^{b\alpha_3}F\left(\left.{}^{A\ 1+B-C}_{\ \ 1+B-A}\right|\frac{1}{z}\right).$$

Of course these two bases of solutions are linearly dependent. To see this we consider Mellin-Barnes representation for hypergeometric function

$$\frac{\Gamma(A)\Gamma(B)}{\Gamma(C)}F\left(\left.{}^{A\,B}_{\ C}\right|z\right)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma(A+s)\Gamma(B+s)\Gamma(-s)}{\Gamma(C+s)}(-z)^s ds,$$

where $|z|<1$ and the contour $\mathcal{C}$ encircles the poles of $\Gamma(-s)$ in counterclockwise direction. For $|z|>1$ the contour $\mathcal{C}$ rather picks the poles of $\Gamma(A+s)$ and $\Gamma(B+s)$ and hence we have

$$\frac{\Gamma(A)\Gamma(B)}{\Gamma(C)}F\left(\left.{}^{A\,B}_{\ C}\right|z\right)=\frac{\Gamma(A)\Gamma(B-A)}{\Gamma(C-A)}(-z)^{-A}F\left(\left.{}^{A\ 1+A-C}_{\ \ 1+A-B}\right|\frac{1}{z}\right)+$$
$$+\frac{\Gamma(B)\Gamma(A-B)}{\Gamma(C-B)}(-z)^{-B}F\left(\left.{}^{A\ 1+B-C}_{\ \ 1+B-A}\right|\frac{1}{z}\right).$$

Similar transformation law we have for another solution. In terms of $s-$ and $t-$ channel conformal blocks the relation can be written as

$$\begin{aligned}\mathcal{F}^s_+&=\frac{\Gamma(A-B)\Gamma(2-C)}{\Gamma(1-B)\Gamma(1+A-C)}\mathcal{F}^t_++\frac{\Gamma(B-A)\Gamma(2-C)}{\Gamma(1-A)\Gamma(1+B-C)}\mathcal{F}^t_+,\\ \mathcal{F}^s_-&=\frac{\Gamma(A-B)\Gamma(C)}{\Gamma(A)\Gamma(C-B)}\mathcal{F}^t_++\frac{\Gamma(B-A)\Gamma(C)}{\Gamma(B)\Gamma(C-A)}\mathcal{F}^t_+.\end{aligned}\tag{152}$$

We have only two conformal blocks appearing in the $s-$channel decomposition

$$\langle\Phi_{-\frac{b}{2}}(z,\bar{z})\Phi_{\alpha_1}(0)\Phi_{\alpha_2}(\infty)\Phi_{\alpha_3}(1)\rangle=C^{\alpha_1+\frac{b}{2}}_{-\frac{b}{2},\alpha_1}C\Big(\alpha_1+\frac{b}{2},\alpha,\alpha_3\Big)|\mathcal{F}^s_+(z)|^2+C^{\alpha_1-\frac{b}{2}}_{-\frac{b}{2},\alpha_1}C\Big(\alpha_1+\frac{b}{2},\alpha,\alpha_3\Big)|\mathcal{F}^s_-(z)|^2.\tag{153}$$

At the same the $t-$channel decomposition should also hold

$$\langle\Phi_{-\frac{b}{2}}(z,\bar{z})\Phi_{\alpha_1}(0)\Phi_{\alpha_2}(\infty)\Phi_{\alpha_3}(1)\rangle=C^{\alpha_2+\frac{b}{2}}_{-\frac{b}{2},\alpha_2}C\Big(\alpha_1,\alpha_2+\frac{b}{2},\alpha_3\Big)|\mathcal{F}^t_+(z)|^2+C^{\alpha_2+\frac{b}{2}}_{-\frac{b}{2},\alpha_2}C\Big(\alpha_1,\alpha_2-\frac{b}{2},\alpha_3\Big)|\mathcal{F}^t_-(z)|^2.\tag{154}$$

The validity of both decompositions (153) and (154) guaranties the the correlation function is single-valued on the thrice punctured sphere. We note however, that applying (152) to (153) we will have unwanted terms like

$$\mathcal{F}^t_+(z)\mathcal{F}^t_-(\bar{z}),$$

which will destroy this property. The condition that unwanted terms cancel leads us to

$$\frac{C^{\alpha_1-\frac{b}{2}}_{-\frac{b}{2},\alpha_1}}{C^{\alpha_1+\frac{b}{2}}_{-\frac{b}{2},\alpha_1}}\frac{C\Big(\alpha_1-\frac{b}{2},\alpha,\alpha_3\Big)}{C\Big(\alpha_1+\frac{b}{2},\alpha,\alpha_3\Big)}=\frac{\gamma(A)\gamma(B)\gamma(C-A)\gamma(C-B)}{\gamma(C)\gamma(C-1)},\qquad\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}.\tag{155}$$

It is convenient to rewrite the relation (155) in the form

$$\frac{C\left(\alpha_1+b,\alpha_2,\alpha_3\right)}{C(\alpha_1,\alpha_2,\alpha_3)}\sim\frac{\gamma(2b\alpha_1)\gamma(2b\alpha_1-1)\gamma\big(b(\alpha_3+\alpha_2-\alpha_1)-b^2\big)}{\gamma\big(b(\alpha_1+\alpha_2-\alpha_3)\big)\gamma\big(b(\alpha_1+\alpha_3-\alpha_2)\big)\gamma\big(b(\alpha_1+\alpha_2+\alpha_3-Q)\big)},\tag{156}$$

where $\sim$ means up to a factor depending only on α_1. Similar relation should hold with b being replaced with b^{-1}

$$\frac{C\left(\alpha_1+b^{-1},\alpha_2,\alpha_3\right)}{C(\alpha_1,\alpha_2,\alpha_3)}\sim\frac{\gamma(2b^{-1}\alpha_1)\gamma(2b^{-1}\alpha_1+b^{-2})\gamma\big(b^{-1}(\alpha_3+\alpha_2-\alpha_1)-b^{-2}\big)}{\gamma\big(b^{-1}(\alpha_1+\alpha_2-\alpha_3)\big)\gamma\big(b^{-1}(\alpha_1+\alpha_3-\alpha_2)\big)\gamma\big(b^{-1}(\alpha_1+\alpha_2+\alpha_3-Q)\big)}.\tag{157}$$

In order to solve (156) and (157) it is desirable to have a function $\Upsilon(x)$, which is self dual with respect to $b\leftrightarrow b^{-1}$ and satisfies

$$\Upsilon(x+b)=b^{1-2bx}\gamma(bx)\Upsilon(x),\quad \Upsilon\left(x+\frac{1}{b}\right)=b^{\frac{2x}{b}-1}\gamma\left(\frac{x}{b}\right)\Upsilon(x).\tag{158}$$

We note that this condition is consistent with the requirement

$$\frac{\Upsilon\left(x+b+\frac{1}{b}\right)}{\Upsilon\left(x+b\right)}\frac{\Upsilon\left(x+b\right)}{\Upsilon\left(x\right)}=\frac{\Upsilon\left(x+b+\frac{1}{b}\right)}{\Upsilon\left(x+\frac{1}{b}\right)}\frac{\Upsilon\left(x+\frac{1}{b}\right)}{\Upsilon\left(x\right)}.$$

It is clear, that if the function which obeys (158) exists, it should be unique up to a constant provided that $b^2\neq p/q$. One can easily check that $\Upsilon(Q-x)$ satisfies exactly the same properties (158), which suggests

$$\Upsilon(x)=\Upsilon(Q-x),\qquad \Upsilon\left(\frac{Q}{2}\right)=1.$$

There is an integral representation

$$\log\Upsilon(x)=\int_0^{\infty}\frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2-\frac{\sinh^2\left(\left(\frac{Q}{2}-x\right)t\right)}{\sinh(bt)\sinh\left(\frac{t}{b}\right)}\right],$$

which is valid in the domain $0<x<Q$. The function $\Upsilon(x)$ does not have poles, only zeroes

$$x=-mb-\frac{n}{b},\qquad x=Q+mb+\frac{n}{b},\qquad m,n\geq 0.$$

Having the $\Upsilon(x)$ function defined, one can write the solution to (156) and (157) as

$$C(\alpha_1,\alpha_2,\alpha_3)=\frac{\mathcal{N}(\alpha_1)\mathcal{N}(\alpha_2)\mathcal{N}(\alpha_3)}{\Upsilon(\alpha-Q)\prod_{k=1}^3\Upsilon(\alpha-2\alpha_k)},\quad\text{where}\quad \alpha=\alpha_1+\alpha_2+\alpha_3,$$

where the factors $\mathcal{N}(\alpha_k)$ correspond to unknown normalization factors for primary fields.

Lecture 12: Minimal models I

Minimal models were introduced by Belavin, Polyakov and Zamolodchikov in their seminal paper [5].

We have seen many times a simplification happening for correlation functions involving degenerate fields. For example the fusion (79)

$$\Phi_{2,1}\Phi_\alpha = [\Phi_{\alpha+\frac{b}{2}}] + [\Phi_{\alpha-\frac{b}{2}}], \qquad \Phi_{1,2}\Phi_\alpha = [\Phi_{\alpha+\frac{b^{-1}}{2}}] + [\Phi_{\alpha-\frac{b^{-1}}{2}}].$$

In particular it implies that

$$\Phi_{2,1}\Phi_{m,n} = [\Phi_{m+1,n}] + [\Phi_{m-1,n}], \qquad \Phi_{1,2}\Phi_{m,n} = [\Phi_{m,n+1}] + [\Phi_{m,n-1}].$$

Both these fusion rules can be interpreted as $\mathfrak{sl}(2)$ fusion rules. Namely, the product of 2-dimensional and m-dimensional (or n-dimensional) representations of $\mathfrak{sl}(2)$ is the sum of $m+1$-dimensional and $m-1$-dimensional representations ($n+1$-dimensional and $n-1$-dimensional). Then using associativity of the OPE, one finds that

$$\Phi_{m,n}\Phi_\alpha = \sum_{r,s}\Big[\Phi_{\alpha+\frac{rb}{2}+\frac{sb^{-1}}{2}}\Big], \tag{159}$$

where the sum goes over the set

$$r = \{m-1, m-3, \ldots, 3-m, 1-m\} \quad \text{and} \quad s = \{n-1, n-3, \ldots, 3-n, 1-n\}. \tag{160}$$

But what if correlation function consists of degenerate fields only? Consider the OPE (323) were the sum goes over the set (322) and suppose that $\alpha = \alpha_{m',n'}$. Then there are two ways to rewrite (323)

$$\Phi_{m,n}\Phi_{m',n'} = \sum_{r,s}\left[\Phi_{r,s}\right], \qquad \Phi_{m,n}\Phi_{m',n'} = \sum_{r',s'}\left[\Phi_{r',s'}\right],$$

where the sums go over the sets

$$\begin{aligned} r &\in (m'-m+1, \ldots, m'+m-1), \qquad & s &\in (n'-n+1, \ldots, n'+n-1), \\ r' &\in (m-m'+1, \ldots, m'+m-1), \qquad & s' &\in (n-n'+1, \ldots, n'+n-1). \end{aligned} \tag{161}$$

The compatibility condition for validity of both expansions requires the sum go over the intersection of two sets (161). That is

$$\Phi_{m,n}\Phi_{m',n'} = \sum_{r,s}\left[\Phi_{r,s}\right], \qquad r \in \big(|m'-m|+1, \ldots, m'+m-1\big),\ s \in \big(|n'-n|+1, \ldots, n'+n-1\big), \tag{162}$$

Since negative numbers do not appear in the r.h.s. of (162), we conclude that the sum goes over the degenerate fields only. In other words the OPE is closed on degenerate fields. So, we might try to construct a CFT which will consists of degenerate fields $\Phi_{m,n}$ only, where (m,n) belong to some set. Actually, this time we will be more cautious about unitarity issues. In particular, we will require

$$\Delta_{m,n} = \frac{(b+b^{-1})^2}{4} - \frac{(mb+nb^{-1})^2}{4} \geq 0. \tag{163}$$

We see that (163) does not hold for $b \in \mathbb{R}$, for $b = e^{i\theta}$ it is in general complex, while the only hope is for $b = i\beta$ with $\beta \in \mathbb{R}$. In this case one can still find (m,n), such that $|m\beta + n\beta^{-1}| \ll 1$ and so $\Delta_{m,n} < 0$.

So, the only hope to construct an unitary CFT with degenerate fields only, would be the case where set of possible (m,n) is restricted.

This is where doubly-degenerate fields come into a play. Namely, suppose that

$$b^2 = -\frac{p}{q}, \tag{164}$$

where p and q are coprime positive integers $q > p$. In this case all the fields

$$\Phi_{m+kq,n+kp} \quad \text{and} \quad \Phi_{q-m+kq,p-n+kp} \tag{165}$$

have the same conformal dimensions. In new parametrization (164) one has

$$c = 1 - \frac{6(p-2)^2}{pq} \tag{166}$$

Consider the notion of the Kac table, that is the set $0 < m < q$, $0 < n < p$. Take basic degenerate field $\Phi_{m,n}$ with $0 < m < q$, $0 < n < p$ and its nearest partner $\Phi_{m',n'} = \Phi_{q-m,p-n}$ (also with $0 < m' < q$, $0 < n' < p$) and consider the OPE

$$\Phi_{m,n}\Phi_\alpha = \sum_{i,j}\left[\Phi_{\alpha-\frac{(m-1)b}{2}-\frac{(n-1)b^{-1}}{2}+ib+jb^{-1}}\right] = \sum_{i',j'}\left[\Phi_{\alpha-\frac{(m'-1)b}{2}-\frac{(n'-1)b^{-1}}{2}+i'b+j'b^{-1}}\right], \tag{167}$$

where $i = 0,1,\ldots,m-1$, $j = 0,1,\ldots,n-1$ etc. In the r.h.s. of (167) we should either have

$$\alpha - \frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2} + ib + jb^{-1} = \alpha - \frac{(m'-1)b}{2} - \frac{(n'-1)b^{-1}}{2} + i'b + j'b^{-1},$$

or

$$\alpha - \frac{(m-1)b}{2} - \frac{(n-1)b^{-1}}{2} + ib + jb^{-1} = Q - \alpha + \frac{(m'-1)b}{2} + \frac{(n'-1)b^{-1}}{2} - i'b - j'b^{-1}.$$

First possibility leads to the condition

$$p(m+i'-i) = q(n+j'-j),$$

which never holds, while the second one gives

$$\alpha = \alpha_{m'',n''} \quad \text{with} \quad m'' = i+i'+1, \quad n'' = j+j'+1.$$

We note that $0 < m'' < q$, $0 < n'' < p$. That is the necessary condition for the field Φ_α to have a non-trivial OPE with degenerate field $\Phi_{m,n}$ from Kac table $0 < m < q$, $0 < n < p$, is to be a degenerate field from the Kac table also. Then the associativity implies that the OPE is closed on degenerate fields from the Kac table only.

The fact noticed above opens the possibility to have a CFT for quantized values of the parameter (164), which has only finitely many degenerate fields from the Kac table with

$$\Delta_{m,n} = \frac{\left(mp-nq\right)^2 - (p-q)^2}{4pq}. \tag{168}$$

Such CFT's are known as minimal models $\mathcal{M}_{p,q}$. One can show that there are always negative numbers in (168) except the case $q = p + 1$ where all the values

$$\Delta_{m,n} = \frac{\big(mp - n(p+1)\big)^2 - 1}{4p(p+1)} \tag{169}$$

are obligatory non-negative. The models $\mathcal{M}_{p,p+1}$ are known as unitary series of minimal models.

There might be a problem since according to (165) for any field from the Kac table we have infinitely many singular vectors.

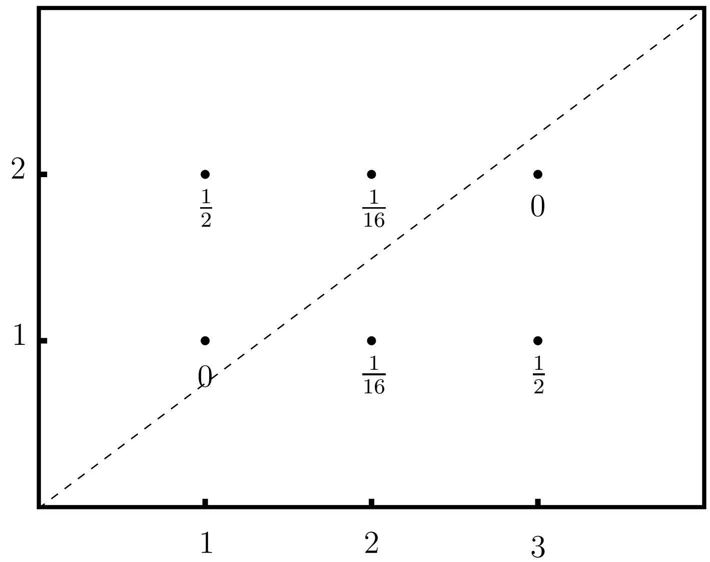
The character of the irreducible Verma module $\mathcal{V}_{m,n}$ has the form

$$\chi_{m,n}(x) = \chi(x) \sum_{k\in\mathbb{Z}} \left(x^{\Delta_{m+2kq,n}} - x^{\Delta_{-m+2kq,n}}\right) \quad \text{where} \quad \chi(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)}. \tag{170}$$

Probs:

Lecture 13: Minimal Models II: Ising model, tricritical Ising model, $N = 1$ SUSY CFT

In this lecture we will study minimal model $\mathcal{M}_{3,4}$ in details. It corresponds to the following Kac table



(171)

According to (166) the central charge of this theory is $c = \frac{1}{2}$. The conformal dimensions are given by

$$\Delta_{m,n} = \frac{(3m-4n)^2-1}{48}.$$

The fields inside the Kac table are identified by the reflection

$$\Phi_{m,n} \sim \Phi_{4-m,3-n}.$$

Thus we have three different primary families

$$\begin{aligned} I &= \Phi_{1,1} = \Phi_{3,2} \quad \text{with} \quad \Delta = \bar{\Delta} = 0, \\ \epsilon &= \Phi_{3,1} = \Phi_{1,2} \quad \text{with} \quad \Delta = \bar{\Delta} = \frac{1}{2}, \\ \sigma &= \Phi_{2,1} = \Phi_{2,2} \quad \text{with} \quad \Delta = \bar{\Delta} = \frac{1}{16}, \end{aligned}$$

the identity operator I, the "energy" operator ϵ and the "spin" operator σ.

These fields describe critical Ising model. In order to see this, we note that the field ϵ has very special OPE

$$\epsilon\epsilon = \Phi_{1,2}\Phi_{3,1} = [\Phi_{3,2}] = [I], \qquad \epsilon\sigma = \Phi_{1,2}\Phi_{2,1} = [\Phi_{2,2}] = [\sigma] \tag{172}$$

It implies that for any correlation function of ϵ one has

$$\langle \epsilon(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = \Big|F(z|z_1,\dots,z_n)\Big|^2 G(z_1,\bar{z}_1,\dots,z_n,\bar{z}_n),$$

where $\mathcal{O}_k$ stand for either ϵ, I or σ. It suggests that the field ϵ admits the holomorhic factorization

$$\epsilon = \psi\bar{\psi}.$$

The holomorphic current $\psi(z)$ has dimension $\frac{1}{2}$ and admits the OPE

$$\psi(z)\psi(w) = \frac{1}{z-w} + \ldots$$

We note that this type of OPE is only compatible with the fermionic statistic for ψ. Similar statement applies to $\bar{\psi}$. We also demand that ψ anticommutes with $\bar{\psi}$.

The pair $(\psi, \bar{\psi})$ can be treated as Maiorana (real) fermion, the real part of the complex fermion studied before

$$\psi(z) = \frac{1}{\sqrt{2}}\left(\psi(z) + \psi^*(z)\right), \qquad \bar{\psi}(z) = \frac{1}{\sqrt{2}}\left(\bar{\psi}(z) + \bar{\psi}^*(z)\right)$$

The dynamics of $(\psi, \bar{\psi})$ is described by the massless Ising action

$$S = \frac{1}{2\pi}\int \left(\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi\right) d^2z.$$

Correspondingly correlation functions of $\psi(z)$ are computed by Wick theorem

$$\langle \boldsymbol{\psi}(z_1)\ldots\boldsymbol{\psi}(z_{2n})\rangle = \frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\ldots\frac{1}{z_{2n-1} - z_{2n}} + \ldots$$

The holomorphic stress-energy tensor in this theory has the form

$$T(z) = -\frac{1}{2} : \boldsymbol{\psi}\partial\boldsymbol{\psi} : .$$

and it defines the CFT with the central charge $c = \frac{1}{2}$. The representation of Maiorana fermion is given by the fermionic Fock module

$$\mathcal{F}_{\mathrm{F}} = \mathrm{Span}\left(\psi_{-\boldsymbol{s}}|\varnothing\rangle = \psi_{-s_1}\psi_{-s_2}\ldots|\varnothing\rangle | s_1 > s_2 > \ldots\right) \tag{173}$$

where $s \in \mathbb{Z} + \frac{1}{2}$, which corresponds to NS sector. The generators ψ_s form an algebra

$$\{\psi_r, \psi_s\} = \delta_{r,-s}. \tag{174}$$

The character of (173) is given by

$$\chi_{\mathrm{F}}(x) = \prod_{k=1}^{\infty}(1 + x^{k-\frac{1}{2}})$$

From the point of view of Minimal Model the Fock module $\mathcal{F}_{\mathrm{F}}$ corresponds to direct sum of irreducible Verma modules

$$\mathcal{F}_{\mathrm{F}} = \mathcal{V}_{1,1} \oplus \mathcal{V}_{3,1}$$

In particular, it implies the character identity (see (170))

$$\chi_{1,1}(x) + \chi_{3,1}(x) = \chi_{\mathrm{F}}(x) \tag{175}$$

which can be thought as an additional confirmation of the coincidence of two theories. Indeed, from (170) one has

$$\chi_{1,1}(x) + \chi_{3,1}(x) = \prod_{l=1}^{\infty}\frac{1}{(1-x^l)}\sum_{k\in\mathbb{Z}}\left(x^{\Delta_{1+8k,1}} - x^{\Delta_{-1+8k,1}} + x^{\Delta_{3+8k,1}} - x^{\Delta_{-3+8k,1}}\right).$$

We note that

$$\Delta_{1+8k,1} = \frac{3(4k)^2 - 4k}{4}, \qquad \Delta_{-1+8k,1} = \frac{3(4k-1)^2 - (4k-1)}{4},$$
$$\Delta_{3+8k,1} = \frac{3(4k+1)^2 - (4k+1)}{4}, \qquad \Delta_{-3+8k,1} = \frac{3(4k-2)^2 - (4k-2)}{4},$$

that is

$$\chi_{1,1}(x) + \chi_{3,1}(x) = \prod_{l=1}^{\infty} \frac{1}{(1-x^l)} \sum_{k\in\mathbb{Z}} \left(x^{\frac{3(4k)^2-4k}{4}} - x^{\frac{3(4k+3)^2-(4k+3)}{4}} + x^{\frac{3(4k+1)^2-(4k+1)}{4}} - x^{\frac{3(4k+2)^2-(4k+2)}{4}} \right). \tag{176}$$

We note that in (176) the sum goes over $4k+s$ where $s = 0,1,2,3$, that is over all integers. Then one can apply the pentagonal identity (143) with $q = -x^{\frac{1}{2}}$ which gives

$$\chi_{1,1}(x) + \chi_{3,1}(x) = \prod_{l=1}^{\infty} \frac{1}{(1-x^l)} \prod_{k=1}^{\infty} \left(1 - (-x^{\frac{1}{2}})^k\right) = \prod_{k=1}^{\infty} \left(1 + x^{k-\frac{1}{2}}\right)$$

Similarly, one obtains the dual identity

$$\chi_{1,1}(x) - \chi_{3,1}(x) = \prod_{k=1}^{\infty} \left(1 - x^{k-\frac{1}{2}}\right).$$

Now we come to the $\sigma(z,\bar{z})$ field. According to the OPE rules (172), one has

$$\epsilon(z,\bar{z})\sigma(w,\bar{w}) = -\frac{i}{2|z-w|}\Big(\sigma(w,\bar{w}) + \dots\Big), \tag{177}$$

where we have chosen the normalization of the field $\epsilon(z,\bar{z})$ to have a convenient factor of $i/2$ in the right hand side of (177). But can we conclude from (177) that

$$\psi(z)\sigma(w,\bar{w}) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\Big(\sigma(w,\bar{w}) + \dots\Big)\,? \tag{178}$$

In fact no. The reason for that is the following. The OPE of the the form (178) correspond to Ramond field which is semi-local with $\psi(z)$. It means that the indexes of ψ_s are integer. Then it follows from (174) that the zero modes ψ_0 and $\bar{\psi}_0$ form an algebra

$$\psi_0^2 = \bar{\psi}_0^2 = \frac{1}{2}, \quad \{\psi_0, \bar{\psi}_0\} = 0. \tag{179}$$

This algebra does not have a one-dimensional representation. In other words the fields $\psi_0\sigma(z,\bar{z})$ and $\bar{\psi}_0\sigma(z,\bar{z})$ can not be proportional to the field $\sigma(z,\bar{z})$. The best we can do is the two-dimensional representation of the algebra (179)

$$\begin{aligned} \psi_0\sigma(z,\bar{z}) &= \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\mu(z,\bar{z}), \qquad & \psi_0\mu(z,\bar{z}) &= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\sigma(z,\bar{z}), \\ \bar{\psi}_0\sigma(z,\bar{z}) &= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\mu(z,\bar{z}), \qquad & \bar{\psi}_0\mu(z,\bar{z}) &= \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\sigma(z,\bar{z}). \end{aligned} \tag{180}$$

Equation (180) can be taken as a definition of the spin field in Ising CFT. It means that the spin field is rather a doublet and the OPE (177) is also supplemented by the dual OPE

$$\epsilon(z,\bar{z})\mu(w,\bar{w}) = -\frac{i}{2|z-w|}\Big(\mu(w,\bar{w}) + \dots\Big),$$

From representation point of view one has a Ramond representation of the fermionic algebra

$$\mathcal{F}_{\mathrm{F}}^{\mathrm{R}} = \mathrm{Span}\Big(\psi_{-\boldsymbol{s}}|\pm\rangle, \quad \boldsymbol{s} = \{s_1 > s_2 > \dots > 0\},\ |-\rangle = \psi_0|+\rangle\Big).$$

Clearly the character of this module is given by $2\chi_{\mathrm{F}}^{\mathrm{R}}(x)$ where

$$\chi_{\mathrm{F}}^{\mathrm{R}}(x) = x^{\frac{1}{16}}\prod_{k=1}^{\infty}(1+q^k).$$

Again, it can be checked that

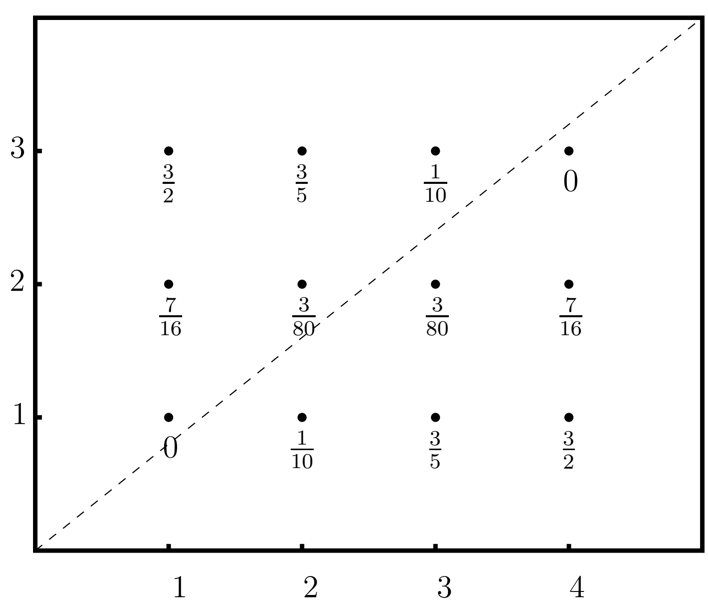
$$\chi_{2,1}(x) = \chi_{\mathrm{F}}^{\mathrm{R}}(x), \tag{181}$$

which confirms the coincidence. Indeed, one has

$$\Delta_{2+8k,1} = \frac{1}{16} + \big(3(2k)^2 + 2k\big), \quad \Delta_{-2+8k,1} = \frac{1}{16} + \big(3(2k-1)^2 + 2k - 1\big),$$

which implies via (143) the desired identity (181).

The conformal symmetry can be extended in many ways. In this lecture we consider supersymmetric extension. As a motivation we consider unitary minimal model $\mathcal{M}_{4,5}$ which has been identified with the scaling limit of tricritical Ising model in [6]. It corresponds to the following Kac table



(182)

The central charge of this theory is $c = \frac{7}{10}$. The conformal dimensions are given according to (169) by

$$\Delta_{m,n} = \frac{(4m-5n)^2 - 1}{80}.$$

The fields inside the Kac table are identified by the reflection

$$\Phi_{m,n} \sim \Phi_{5-m,4-n}.$$

We note that the field $\Phi_{4,1} = \Phi_{1,3}$ ia special: the operator product expansion of $\Phi_{4,1}$ contains only contribution of $[I] = [\Phi_{1,1}]$, as follows from the identity

$$[\Phi_{1,3}][\Phi_{4,1}] = [\Phi_{4,3}] + \cancel{[\Phi_{4,1}]} = \cancel{[\Phi_{2,3}]} + [\Phi_{4,3}] = [\Phi_{4,3}] = [\Phi_{1,1}].$$

It implies that we can construct local fields G and $\bar{G}$ of dimension $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ respectively, subject to the constraints

$$\bar{\partial} G = \partial \bar{G} = 0,$$

such that $\Phi_{1,3} = G\bar{G}$. Similarly we have

$$\left.\begin{aligned} [\Phi_{4,1}][\Phi_{2,1}] = [\Phi_{3,1}] \\ [\Phi_{4,1}][\Phi_{3,1}] = [\Phi_{2,1}] \end{aligned}\right\} \qquad \text{Neveu-Schwarz sector,}$$

$$\left.\begin{aligned} [\Phi_{4,1}][\Phi_{3,2}] = [\Phi_{3,2}] \\ [\Phi_{4,1}][\Phi_{4,2}] = [\Phi_{4,2}] \end{aligned}\right\} \qquad \text{Ramond sector.}$$

The fields $G(z)$ and $T(z) = L_{-2}I(z)$ can be regarded as generators of extended chiral symmetry[5]

$$\begin{aligned} T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots = \frac{3\hat{c}}{4(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots, \\ T(Z)G(w) &= \frac{3G(w)}{2(z-w)^2} + \frac{G'(w)}{z-w} + \dots, \\ G(z)G(w) &= \frac{2c}{3(z-w)^3} + \frac{2T(w)}{z-w} + \dots = \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w} + \dots \end{aligned} \tag{183}$$

The algebra (183) is known as Neveu-Schwarz-Ramond algebra (NSR algebra) and appeared first in superstring theory[6]. We note that the last OPE only make sense if $G(z)$ is a grassmann variable. An OPE of $G(z)$ with generic field must have the form

$$G(z)\mathcal{O}(w) = \sum_r \frac{G_r\mathcal{O}(w)}{(z-w)^{r+\frac{3}{2}}},$$

where $G_r\mathcal{O}(w)$ is just the notation for the new field. Then the "generators" G_r together with L_n's form an algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{8}(m^3 - m)\delta_m, -n, \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}. \end{aligned} \tag{184}$$

[5]Last equation follows from general formula

$$\Phi_\Delta(z)\Phi_\Delta(w) \sim (z-w)^{-2\Delta}\left(1 + \frac{2\Delta(z-w)^2}{c}T(w) + \dots\right)$$

[6]We note that due to historical reasons it is customary to use the parameter $\hat{c} = \frac{2}{3}c$.

The space of fields decomposes onto the space of NS fields $\mathcal{O}_{\mathrm{NS}}$ local with respect to $S(z)$, and the space of the Ramond fields $\mathcal{O}_{\mathrm{R}}$ such that the correlation function

$$\langle G(z)\mathcal{O}_{\mathrm{R}}(w,\bar{w})\dots\rangle$$

changes sign when z goes around w. In particular the fields $\Phi_{2,1}$ and $\Phi_{3,1}$ in minimal model $\mathcal{M}_{4,5}$ are NS fields and $\Phi_{3,2}$ and $\Phi_{4,2}$ are Ramond ones. We see that the indexes r, s are half-integer in NS sector and integer in R sector. One defines NS primary field by

$$T(z)\Phi(w) = \frac{\Delta\Phi(w)}{(z-w)^2} + \frac{\Phi'(w)}{z-w} + \dots \qquad G(z)\Phi(w) = \frac{G_{-\frac{1}{2}}\Phi(w)}{z-w} + \dots$$

where $\Psi = G_{-\frac{1}{2}}\Phi(w)$ is a new field. We note also that

$$G(z)\Psi(w) = \frac{2\Delta\Phi(w)}{(z-w)^2} + \frac{\Phi'(w)}{z-w} + \dots$$

Correspondingly the primary field in Ramond sector form a doublet

$$R = \begin{pmatrix} R^+ \\ R^- \end{pmatrix}, \qquad G_0 \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - \frac{\hat{c}}{16} & 0 \end{pmatrix} \begin{pmatrix} R^+ \\ R^- \end{pmatrix}.$$

One can use OPE to constraint correlation functions. Consider two-point Ward identities in NS sector

$$\begin{aligned} \langle G(\xi)\Phi_1(z_1)\Phi_2(z_2)\rangle &= \frac{\langle\Psi_1(z_1)\Phi_2(z_2)\rangle}{\xi - z_1} + \frac{\langle\Phi_1(z_1)\Psi_2(z_2)\rangle}{\xi - z_2}, \\ \langle G(\xi)\Phi_1(z_1)\Psi_2(z_2)\rangle &= \frac{\langle\Psi_1(z_1)\Psi_2(z_2)\rangle}{\xi - z_1} + \frac{2\Delta_2}{(\xi - z_2)^2}\langle\Phi_1(z_1)\Phi_2(z_2)\rangle + \frac{1}{\xi - z_2}\langle\Phi_1(z_1)\Phi_2'(z_2)\rangle. \end{aligned}$$

Using the fact that $G(\xi) \sim \frac{1}{\xi^3}$ at $\xi \to \infty$ one finds the constraint

$$\langle\Psi_1(z_1)\Phi_2(z_2)\rangle = 0, \quad \langle\Psi_1(z_1)\Psi_2(z_2)\rangle = -\langle\Phi_1(z_1)\Phi_2'(z_2)\rangle$$

One can generalize this for $n-$point correlation functions. Consider master Ward identity

$$\langle G(\xi)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle = \sum_{k=1}^{n}\langle\Phi_1(z_1)\dots\Psi(z_k)\dots\Phi_n(z_n)\rangle,$$

and similar ones with $\Phi_k \to \Psi_k$. In particular, there 2 independent $3-$point correlation out of 8

$$\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle \quad \text{and} \quad \langle\Phi_1(z_1)\Phi_2(z_2)\Psi_3(z_3)\rangle$$

Representation theory of NSR algebra (184) is very similar to the one of Virasoro algebra. It is convenient to introduce the following parametrization of the central charge and conformal dimensions of NS and R primary fields

$$c = 1 + 2Q^2, \qquad \Delta_{\mathrm{NS}}(\alpha) = \frac{\alpha(Q-\alpha)}{2}, \quad \Delta_{\mathrm{NS}}(\alpha) = \Delta_{\mathrm{R}}(\alpha) + \frac{1}{16}, \qquad Q = b + \frac{1}{b}.$$

The Verma module $\mathcal{V}_\Delta$ is a linear span of vectors

$$L_{-\boldsymbol{\lambda}}G_{-\boldsymbol{r}}|\Delta\rangle,$$

for ordered set $\boldsymbol{\lambda} = \lambda_1 >\geq \lambda_2 \geq \dots$ and *strictly* ordered set $\boldsymbol{r} = r_1 > r_2 > \dots$ A singular vector is by definition a state $|\chi\rangle$ in $\mathcal{V}_\Delta$ which is killed by positive part of NSR algebra

$$L_n|\chi\rangle = G_r|\chi\rangle = 0 \quad \text{for} \quad n, r > 0.$$

A supersymmetric version of Kac theorem states that for

$$\alpha_{m,n} = -\frac{mb}{2} - \frac{nb^{-1}}{2}$$

there is a singular vector at level $\frac{mn}{2}$ which appears in

$$\begin{aligned} &\text{in NS sector for} \qquad m - n \in 2\mathbb{Z}, \\ &\text{in R sector for} \qquad m - n \in 2\mathbb{Z} + 1. \end{aligned}$$

Consider first examples:

- Level $\frac{1}{2}$. The state

  $$G_{-\frac{1}{2}}|\Delta\rangle$$

  is a singular vector provided that $\Delta = 0 = \Delta_{\text{NS}}(0)$.

- Level 1. There are two null-vectors in Ramond sector

  $$\begin{aligned} &\left(L_{-1} - \frac{2b^2}{1+2b^2}G_{-1}G_0\right)|\Delta\rangle \quad \text{for} \qquad \alpha = -\frac{b}{2}, \\ &\left(L_{-1} - \frac{2b^{-2}}{1+2b^{-2}}G_{-1}G_0\right)|\Delta\rangle \quad \text{for} \qquad \alpha = -\frac{1}{2b}, \end{aligned}$$

- Level $\frac{3}{2}$. Two null-vectors in NS sector

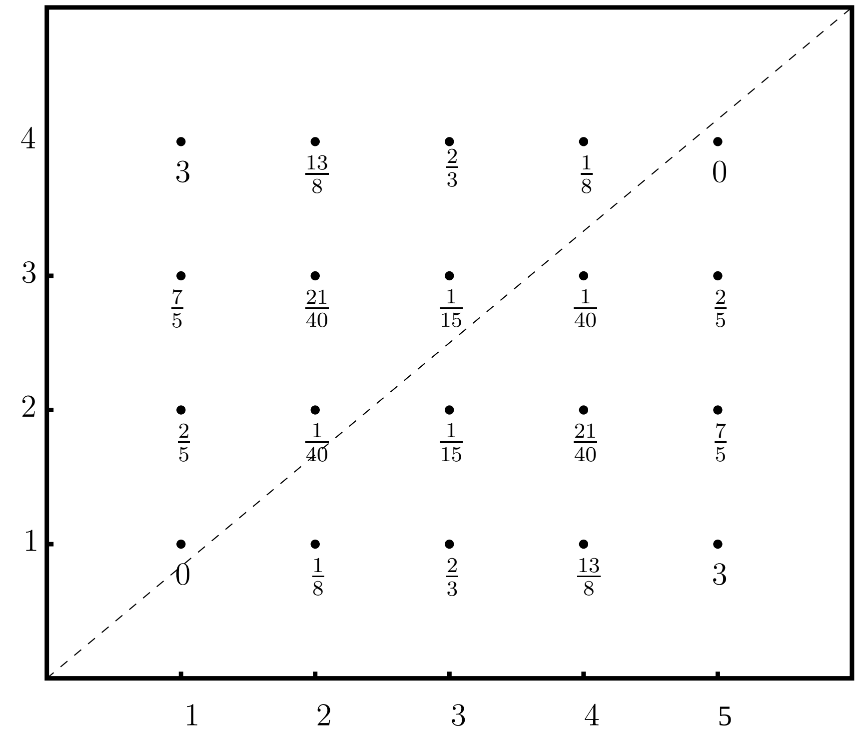
  $$\begin{aligned} &\left(L_{-1}G_{-\frac{1}{2}} + b^2 G_{-\frac{3}{2}}\right)|\Delta\rangle \quad \text{for} \quad \alpha = -b, \\ &\left(L_{-1}G_{-\frac{1}{2}} + b^{-2} G_{-\frac{3}{2}}\right)|\Delta\rangle \quad \text{for} \quad \alpha = -b^{-1}. \end{aligned}$$

Probs:

1. Prove (175), (181).

Lecture 14: Minimal models III: Potts model, $W-$algebras, parafermionic CFT

We consider next unitary minimal model $\mathcal{M}_{5,6}$ which is known to be related to $\mathbb{Z}_3$ Potts model [7]. It has the Kac table



(185)

The central charge of this theory is $c = \frac{4}{5}$ and Kac dimensions are

$$\Delta_{m,n} = \frac{(5m - 6n)^2 - 1}{120}.$$

Similarly to the previous lecture the field $\Phi_{5,1} = \Phi_{1,4}$ can be decomposed as

$$\Phi_{5,1}(z, \bar{z}) = W(z)\bar{W}(\bar{z})$$

The current $W(z)$ of spin 3 extends the Virasoro algebra[7]

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots,$$
$$T(Z)W(w) = \frac{3W(w)}{(z-w)^2} + \frac{W'(w)}{z-w} + \dots,$$

First OPE is just for Virasoro algebra, second states that $W(z)$ is a primary field of dimension 3, while the third one

$$W(z)W(w) = \frac{c}{3(z-w)^6} + \frac{\lambda_1 T(w)}{(z-w)^4} + \frac{\lambda_2 T'(w)}{(z-w)^3} + \frac{\lambda_3 T''(w) + \lambda_4 \Lambda(w)}{(z-w)^2} + \frac{\lambda_5 T'''(w) + \lambda_6 \Lambda'(w)}{z-w} + \dots \tag{186}$$

[7]We have $c = \frac{4}{5}$, but we can keep c arbitrary in discussions below

is an OPE expansion of the field with $\Delta = 3$ into identity operator. Here $\Lambda(z)$ is quasi-primary field which appears in OPE

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \left(\Lambda(w) + \frac{3}{10}T''(w)\right) + \dots$$

i.e.

$$\Lambda(w) = \left(L_{-2} - \frac{3}{10}L_{-1}^2\right)T(w) = \left(L_{-2} - \frac{3}{10}L_{-1}^2\right)L_{-2}I(w). \tag{187}$$

Let us compute the coefficients λ_k in (186). We can do it exactly as before when we studied conformal properties of OPE. Namely, we act on both sides of (186) by

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_z+\mathcal{C}_z}(\xi - w)^{n+1}T(\xi)d\xi,$$

which can be interpreted as

$$\mathcal{L}_n = (z-w)^{n+1}\partial_z + 3(n+1)(z-w)^n \quad \text{or} \quad L_n.$$

Taking $n = 1$ one obtains (we use that $L_1\Lambda = 0$)

$$\frac{2\lambda_1 T(w)}{(z-w)^3} + \frac{3\lambda_2 T'(w)}{(z-w)^2} + \frac{4\lambda_3 T''(w) + 4\lambda_4\Lambda(w)}{z-w} + \dots = \frac{4\lambda_2 T(w)}{(z-w)^3} + \frac{10\lambda_3 T'(w)}{(z-w)^2} + \frac{18\lambda_5 T''(w) + 8\lambda_6\Lambda(w)}{z-w} + \dots,$$

which implies

$$\lambda_2 = \frac{\lambda_1}{2}, \quad \lambda_3 = \frac{3\lambda_2}{10} = \frac{3\lambda_1}{20}, \quad \lambda_5 = \frac{2\lambda_3}{18} = \frac{\lambda_1}{30}, \quad \lambda_6 = \frac{\lambda_4}{2}.$$

While taking $n = 2$ we find

$$\frac{c}{(z-w)^4} + \frac{5\lambda_1 T(w)}{(z-w)^2} + \dots = \frac{\lambda_1 L_2 T(w)}{(z-w)^4} + \frac{\lambda_3 L_2 T''(w) + \lambda_4 L_2\Lambda(w)}{(z-w)^2} + \dots$$

which fixes

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_4 = \frac{32}{5c+22}.$$

Altogether we obtain [8]

$$W(z)W(w) = \frac{c}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{T'(w)}{(z-w)^3} + \frac{\frac{3}{10}T''(w) + \frac{32}{5c+22}\Lambda(w)}{(z-w)^2} + \frac{\frac{1}{15}T'''(w) + \frac{16}{5c+22}\Lambda'(w)}{z-w} + \dots$$

In terms of the modes we obtain

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n},\\
&[L_m, W_n] = (2m-n)W_{m+n},\\
&[W_m, W_n] = \frac{c}{3\cdot 5!}m(m^2-1)(m^2-4)\delta_{m,-n} + \frac{16}{5c+22}(m-n)\Lambda_{m+n} +\\
&\qquad + (m-n)\left(\frac{(m+n+2)(m+n+3)}{15} - \frac{(m+2)(n+2)}{6}\right)L_{m+n}.
\end{aligned} \tag{188}$$

We note that Λ_m is not new and expressed in terms of generator L_n as follows from the definition (187)

$$\Lambda(z) + \frac{3}{10}T''(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}_z} \frac{T(\xi)T(x)}{\xi - z}d\xi.$$

After simple calculation one obtains

$$\Lambda_m = \sum_k : L_k L_{m-k} : + \frac{1}{5}x_m L_m, \quad \text{where} \quad x_{2l} = (1+l)(1-l), \quad x_{2l+1} = (2+l)(1-l). \tag{189}$$

Now consider W primary field

$$\begin{aligned} T(\xi)\Phi(z) &= \frac{\Delta\Phi(z)}{(\xi - z)^2} + \frac{L_{-1}\Phi}{\xi - z} + \ldots, \\ W(\xi)\Phi(z) &= \frac{w\Phi(z)}{(\xi - z)^3} + \frac{W_{-1}\Phi}{(\xi - z)^2} + + \frac{W_{-2}\Phi}{\xi - z} + \ldots. \end{aligned}$$

We stress that while we have $L_1\Phi = \Phi'$ for Virasoro descendant, W-descendants $W_{-1}\Phi$ and $W_{-2}\Phi$ are new fields which do not have immediate relation to Φ. Consider Ward identity for $n-$point correlation function of primary fields

$$\begin{aligned} \langle W(\xi)\Phi_1(z_1)\ldots\Phi_n(z_n)\rangle &= \sum_{k=1}^n \Bigg(\frac{w_k}{(\xi - z_k)^3}\langle\Phi_1(z_1)\ldots\Phi_n(z_n)\rangle + \\ &+ \frac{1}{(\xi - z_k)^2}\langle\Phi_1(z_1)\ldots W_{-1}\Phi_k(z_k)\ldots\Phi_n(z_n)\rangle + \frac{1}{\xi - z_k}\langle\Phi_1(z_1)\ldots W_{-2}\Phi_k(z_k)\ldots\Phi_n(z_n)\rangle \Bigg). \end{aligned}$$

In the right hand side it involves $2n+1$ different correlation functions restricted by 5 projective Ward identities

$$\begin{aligned} W(\xi) \sim \frac{1}{\xi^6} \implies \sum_{k=1}^n \Big(w_k \frac{l(l-1)}{2} z_k^{l-2}\langle\Phi_1(z_1)\ldots\Phi_n(z_n)\rangle + l z_k^{l-1}\langle\Phi_1(z_1)\ldots W_{-1}\Phi_k(z_k)\ldots\Phi_n(z_n)\rangle + \\ + z_k^l\langle\Phi_1(z_1)\ldots W_{-2}\Phi_k(z_k)\ldots\Phi_n(z_n)\rangle \Big) = 0 \quad \text{for} \quad l = 0,1,2,3,4. \end{aligned}$$

- In the case of one-point function it immediately implies that

$$\langle W_{-1}\Phi\rangle = 0, \quad \langle W_{-2}\Phi\rangle = 0 \quad \text{and} \quad w = 0.$$

- For two-point function one has a system of equations

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & z_1 & z_2 \\ w_1 + w_2 & 2z_1 & 2z_2 & z_1^2 & z_2^2 \\ 3(w_1 z_1 + w_2 z_2) & 3z_1^2 & 3z_2^2 & z_1^3 & z_2^3 \\ 6(w_1 z_1^2 + w_2 z_2^2) & 4z_1^3 & 4z_2^3 & z_1^4 & z_2^4 \end{pmatrix} \begin{pmatrix} \langle\Phi_1\Phi_2\rangle \\ \langle W_{-1}\Phi_1\Phi_2\rangle \\ \langle\Phi_1 W_{-1}\Phi_2\rangle \\ \langle W_{-2}\Phi_1\Phi_2\rangle \\ \langle\Phi_1 W_{-2}\Phi_2\rangle \end{pmatrix} = 0,$$

  which has a solution provided that

$$\det \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & z_1 & z_2 \\ w_1 + w_2 & 2z_1 & 2z_2 & z_1^2 & z_2^2 \\ 3(w_1 z_1 + w_2 z_2) & 3z_1^2 & 3z_2^2 & z_1^3 & z_2^3 \\ 6(w_1 z_1^2 + w_2 z_2^2) & 4z_1^3 & 4z_2^3 & z_1^4 & z_2^4 \end{pmatrix} = 0 \implies (z_1 - z_2)^6(w_1 + w_2) = 0,$$

and hence the two-point function takes the form

$$\langle \Phi_1(z_1)\Phi_2(z_2)\rangle \sim \frac{\delta_{\Delta_1,\Delta_2}\delta_{w_1,-w_2}}{(z_1-z_2)^{2\Delta_1}}$$

- For three-point function one has 7 functions minus 5 constraints which means that everything can be expressed as a linear combination of

$$\langle \Phi_1\Phi_2\Phi_3\rangle \quad \text{and} \quad \langle \Phi_1\Phi_2 W_{-1}\Phi_3\rangle. \tag{190}$$

Here one comes to an important difference compared to Virasoro case. In Virasoro case we had the statement that correlation functions of descendant field can always be expressed from correlation functions of primary fields by application of certain differential operators. In W case this is no longer true. For example one has two three-point functions (190) which are not related by kinematics. One can show more generic statement that any three-point function of W descendant fields can be expressed through the basic ones[8]

$$\langle \Phi_1\Phi_2 W_{-1}^k\Phi_3\rangle$$

Some simplifications appear for degenerate fields. The structure of representation theory of W-algebra (188) is very similar to the one of Virasoro algebra. The Verma module $\mathcal{V}_{\Delta,w}$ is spanned by the vectors

$$W_{-\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}|\Delta,w\rangle: \quad L_n|\Delta,w\rangle = W_n|\Delta,w\rangle = 0 \text{ for } n>0, \quad L_0|\Delta,w\rangle = \Delta|\Delta,w\rangle,\ W_0|\Delta,w\rangle = w|\Delta,w\rangle,$$

for two independent partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. A singular vector $|\chi\rangle \in \mathcal{V}_{\Delta,w}$ is by definition as state killed by positive part of W-algebra

$$L_n|\chi\rangle = W_n|\chi\rangle = 0 \text{ for } n>0.$$

Consider the simplest example of a singular vector at level 1

$$|\chi\rangle = (W_{-1}+\xi L_{-1})\,|\Delta,w\rangle.$$

We should impose

$$\begin{aligned} &L_1|\chi\rangle = 0 \implies (3w+2\xi\Delta)|\Delta,w\rangle = 0, \\ &W_1|\chi\rangle = 0 \implies \left(\frac{32}{5c+22}\left(\Delta^2+\frac{1}{5}\Delta\right)-\frac{1}{5}\Delta+3\xi w\right)|\Delta,w\rangle = 0 \end{aligned}$$

which implies the constraint between quantum numbers Δ and w

$$9w^2 = 2\Delta^2\left(\frac{32}{5c+22}\left(\Delta+\frac{1}{5}\right)-\frac{1}{5}\right) \tag{191}$$

If such a field present in three-point correlation function then one can express correlation function of any descendant fields from the one of primary fields.

It is convenient to introduce "Toda" like notations. Let e_1 and e_2 be the simple roots of $\mathfrak{sl}(3)$, that is their Gram matrix is

$$(e_i\cdot e_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

[8] Try to argue this from Ward identities.

the Weyl vector ρ

$$\rho = e_1 + e_2,$$

fundamental weights ω_1 and ω_2

$$(\omega_i, e_j) = \delta_{ij} \implies \omega_1 = \frac{2}{3}e_1 + \frac{1}{3}e_2 \quad \omega_2 = \frac{2}{3}e_2 + \frac{1}{3}e_1,$$

and weights of fundamental representation

$$h_1 = \omega_1, \quad h_2 = \omega_1 - e_1, \quad h_3 = \omega_1 - e_1 - e_2 \implies \sum_{k=1}^{3} h_k = 0, \quad (h_i \cdot h_j) = \delta_{ij} - \frac{1}{3}.$$

Then we use parametrization of central charge and quantum numbers Δ and w

$$c = 2+12(\mathcal{Q}, \mathcal{Q}) = 2+24Q^2, \quad \Delta = \frac{(\alpha, 2\mathcal{Q} - \alpha)}{2}, \quad w(\alpha) = i\sqrt{\frac{48}{5c+22}}(\alpha-\mathcal{Q}, h_1)(\alpha-\mathcal{Q}, h_2)(\alpha-\mathcal{Q}, h_3),$$

where

$$\mathcal{Q} = Q\rho, \qquad Q = b + \frac{1}{b}.$$

This wild parametrization provides a solution to (191) if

$$\alpha = \varkappa\omega_1 \quad \text{or} \quad \alpha = \varkappa\omega_2.$$

We should call such field *semi-degenerate*. It is clear that $n-$point correlation function with $(n-2)$ semi-degenerate fields, for example

$$\langle \Phi_{\alpha_1}(z_1)\Phi_{\varkappa_1\omega_1}(z_2)\ldots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle, \tag{192}$$

can be computed using OPE as

$$\begin{aligned}
&\langle \Phi_{\alpha_1}(z_1)\Phi_{\varkappa_1\omega_1}(z_2)\ldots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle = \\
&\qquad = \sum_{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}} C^{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}}_{\alpha_1,\varkappa_1\omega_1}(z_1-z_2)\langle W_{-\boldsymbol{\mu}}L_{-\boldsymbol{\lambda}}\Phi_\alpha(z_2)\ldots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle = \\
&= \sum_{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}} C^{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu}}_{\alpha_1,\varkappa_1\omega_1}(z_1-z_2)\sum_{\beta,\boldsymbol{\nu},\boldsymbol{\sigma}} C^{\beta,\boldsymbol{\nu},\boldsymbol{\sigma}}_{\alpha,\boldsymbol{\lambda},\boldsymbol{\mu},\varkappa_2\omega_2}(z_2-z_3)\langle W_{-\boldsymbol{\sigma}}L_{-\boldsymbol{\nu}}\Phi_\beta(z_2)\ldots\Phi_{\varkappa_{n-1}\omega_1}(z_{n-1})\Phi_{\alpha_n}(z_n)\rangle = \ldots,
\end{aligned}$$

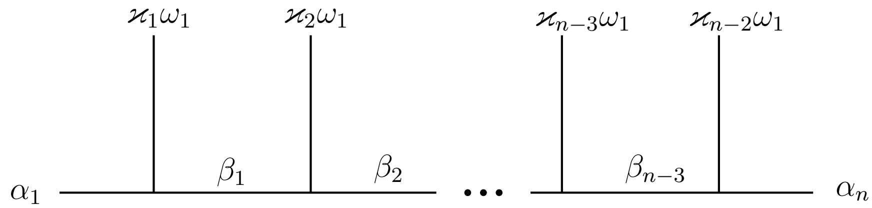
The structure constant at each step is related by lowering the index to the three-point correlation function

$$\langle \big(W_{-\boldsymbol{\sigma}}L_{-\boldsymbol{\nu}}\Phi_{\alpha_1}(z_1)\big)\big(W_{-\boldsymbol{\lambda}}L_{-\boldsymbol{\mu}}\Phi_{\alpha_2}(z_2)\Phi_{\varkappa\omega_1}(z_3)\big)\rangle.$$

Since it contains the semi-degenerate field it can be reduced to differential operator acting on correlation function of primary fields

$$\langle \Phi_{\alpha_1}(z_1,\bar{z}_1)\Phi_{\alpha_2}(z_2,\bar{z}_2)\Phi_{\varkappa\omega_1}(z_3,\bar{z}_3)\rangle = \frac{C(\alpha_1,\alpha_2,\varkappa\omega_1)}{\prod_{i<j}|z_i-z_j|^{2\gamma_{ij}}}.$$

The correlation function (192) belongs to the class of computable correlation functions, just as in Virasoro case. In particular, the conformal block



is completely determined by kinematics.

Now, we consider another aspect of the theory $\mathcal{M}_{5,6}$. Namely, consider the field $\Phi_{3,1} = \Phi_{3,4}$ with conformal dimension $\Delta = \frac{2}{3}$. Its OPE has the form

$$\Phi_{3,1}\Phi_{3,1} = [\Phi_{1,1}] + [\Phi_{3,1}] + [\Phi_{5,1}]$$

It has been noticed by Cardy [9, 10] that one can build self-consistent theory assuming that the field $\Phi_{3,1}$ enters the theory with multiplicity 2. One can choose a basis of these two fields which admits holomorphic factorization

$$\boldsymbol{\Psi}(z)\bar{\boldsymbol{\Psi}}(\bar{z}) \quad \text{and} \quad \boldsymbol{\Psi}^{+}(z)\bar{\boldsymbol{\Psi}}^{+}(\bar{z}),$$

where $\boldsymbol{\Psi}(z)$ and $\boldsymbol{\Psi}^{+}(z)$ are the so called $\mathbb{Z}_3$ *parafermionic* fields [11]

$$\begin{gathered}\boldsymbol{\Psi}(z)\boldsymbol{\Psi}(w) = \frac{C}{(z-w)^{\frac{2}{3}}}\left(\boldsymbol{\Psi}^{+}(w) + \dots\right), \qquad \boldsymbol{\Psi}^{+}(z)\boldsymbol{\Psi}^{+}(w) = \frac{C}{(z-w)^{\frac{2}{3}}}\left(\boldsymbol{\Psi}(w) + \dots\right), \\ \boldsymbol{\Psi}(z)\boldsymbol{\Psi}^{+}(w) = \frac{1}{(z-w)^{\frac{4}{3}}}\left(1 + \frac{5}{3}T(w)(z-w)^2 + \dots\right)\end{gathered} \tag{193}$$

Here the factor $\frac{5}{3} = \frac{2\Delta}{c}$ is universal and the structure constant has to be fixed from the associativity condition of the operator algebra. We note that the form of parafermionic algebra (193) is only consistent with the fractional statistic for the field $\boldsymbol{\Psi}(z)$

$$\boldsymbol{\Psi}(z)\boldsymbol{\Psi}(w) = e^{\frac{2i\pi}{3}}\boldsymbol{\Psi}(w)\boldsymbol{\Psi}(z).$$

In order to find C we consider $N-$point correlation functions of $\boldsymbol{\Psi}(z)$. We note that the algebra (193) implies that the correlation function is only non-zero if n is divisible by 3

$$\langle \boldsymbol{\Psi}(z_1)\dots\boldsymbol{\Psi}(z_{3n})\rangle = \frac{P_n^{(3)}(z_1,\dots,z_{3n})}{\prod_{i<j}(z_i-z_j)^{\frac{2}{3}}}, \tag{194}$$

where $P_n^{(3)}(z_1,\dots,z_{3n})$ is the symmetric homogeneous polynomial which satisfies the following three properties

- $P_n^{(3)}(\lambda z_1,\dots,\lambda z_{3n}) = \lambda^{3n(n-1)}P_n(z_1,\dots,z_{3n})$
- $P_n^{(3)}(z_1,\dots,z_{2n}) = z_1^{2(n-1)} + \dots$ at $z_1 \to \infty$
- $P_n^{(3)}(z_1,\dots,z_{3(n-1)},x,x,x) = C^2\prod_{k=1}^{3(n-1)}(z_k-x)^2 P_{n-1}^{(3)}(z_1,\dots,z_{3(n-1)})$

It can be proven that the polynomial with such properties exist and unique

$$P_n^{(3)}(z_1,\dots,z_{3n})=C^{2(n-1)}\mathrm{Sym}_{\boldsymbol{z}}\left[\prod_{i<j\in I}(z_i-z_j)^2\prod_{i<j\in II}(z_i-z_j)^2\prod_{i<j\in III}(z_i-z_j)^2\right],$$

where I, II and III are three groups of n points.

We note that for Maiorana fermion $\psi(z)$ we have the formula similar to (194)

$$\langle\psi(z_1)\dots\psi(z_{2n})\rangle=\frac{2^{1-n}P_n^{(2)}(z_1,\dots,z_{2n})}{\prod_{i<j}(z_i-z_j)},$$

where

$$P_n^{(2)}(z_1,\dots,z_{2n})=\mathrm{Sym}_{\boldsymbol{z}}\left[\prod_{i<j\in I}(z_i-z_j)^2\prod_{i<j\in II}(z_i-z_j)^2\right],$$

This formula is a consequence of the bosonization map

$$\psi(z)=\frac{1}{\sqrt{2}}(\psi(z)+\psi^*(z))=\frac{1}{\sqrt{2}}\left(e^{i\varphi(z)}+e^{-i\varphi(z)}\right) \tag{195}$$

Generalization of (195) is straightforward

$$\boldsymbol{\Psi}(z)=\frac{1}{\sqrt{3}}\left(e^{i(\boldsymbol{h}_1\cdot\boldsymbol{\varphi}(z))}+e^{i(\boldsymbol{h}_2\cdot\boldsymbol{\varphi}(z))}+e^{i(\boldsymbol{h}_3\cdot\boldsymbol{\varphi}(z))}\right),$$

where $\boldsymbol{\varphi}(z)=(\varphi_1(z),\varphi_2(z))$ is the two-component bosonic field and $\boldsymbol{h}_k$ are proportional to the weights of fundamental representation of $\mathfrak{sl}(3)$, that is 3 linearly dependent vectors in $\mathbb{R}^2$

$$(\boldsymbol{h}_i\cdot\boldsymbol{h}_j)=2\left(\delta_{ij}-\frac{1}{3}\right)\quad\Longrightarrow\quad\sum_{k=1}^{3}\boldsymbol{h}_k=0.$$

Probs:

1. Show (188) and (189) by explicit calculations.

Lecture 15: CFT on the torus

So far, we have discussed CFT's on the sphere, but for various reasons, especially for the purposes of string theory, it is worth to consider CFT on arbitrary Riemann surface even with boundary. In this lecture we consider the simplest case of the torus.

The easiest way to obtain the torus is to cut a piece of a cylinder and identify its ends. One can obtain the cylinder as a map from the plane (without two points). We use cylinder coordinate frame as in (68) with new coordinates t and σ related to the complex coordinate z by exponential map

$$z = Re^{-\frac{iu}{R}}, \quad u = \sigma + it \implies ds^2 = e^{\frac{2t}{R}}\left(dt^2 + d\sigma^2\right).$$

Now in fact there are two options to use the Hamitonian formalism. The one, which we already used, is known as the radial quantization. We take $t \in [-\infty, \infty]$ as a time coordinate and $\sigma \in [0, 2\pi R]$ a space one. It has to singular points $z = 0$ and $z = \infty$. Then the correlation function of local fields is related to the Green function as follows

$$\langle \mathcal{O}_1(\sigma_1, t_1) \dots \mathcal{O}_N(\sigma_N, t_N) \rangle = \frac{\langle 0 | \mathcal{T}_t \left[\mathcal{O}_1(\sigma_1, t_1) \dots \mathcal{O}_N(\sigma_N, t_N)\right] | 0 \rangle}{\langle 0 | 0 \rangle}, \tag{196}$$

where the Hamiltonian H has the form

$$H = \frac{1}{2\pi R} \int_0^{2\pi R} T_{tt} d\sigma = \frac{1}{R}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right). \tag{197}$$

We note that the momentum operator

$$P = \frac{1}{R}(L_0 - \bar{L}_0)$$

should have quantized eigenvalues n/R.

But one can also consider the same system in the framework of angular quantization. Namely, we interpret t as the spatial coordinate which spans the whole real line and σ as the time. The angular nature of σ manifests itself in different compared to (196) representation for the correlation functions

$$\langle \mathcal{O}_1(\sigma_1, t_1) \dots \mathcal{O}_N(\sigma_N, t_N) \rangle = \frac{\mathrm{Tr}\left[\mathcal{T}_\sigma \left[\mathcal{O}_1(\sigma_1, t_1) \dots \mathcal{O}_N(\sigma_N, t_N)\right] e^{-2\pi R H'}\right]_{\mathcal{H}'}}{\mathrm{Tr}\left[e^{-2\pi R H'}\right]_{\mathcal{H}'}},$$

where H' is the angular Hamiltonian

$$H' = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_{\sigma\sigma} dt,$$

and the trace goes over the Hilbert space $\mathcal{H}'$ of angular Hamiltonian.

In order to obtain the theory on the torus, one has to compactify $t \sim t + 2\pi R'$ with some radius R'. It can be interpreted as either the system of size R in radial quantization (196) at finite temperature $1/R'$

$$\langle \mathcal{O}_1(\sigma_1, t_1) \dots \mathcal{O}_N(\sigma_N, t_N) \rangle_{\text{torus}} = \frac{\mathrm{Tr}\left[\mathcal{T}_\tau \left[\mathcal{O}_1(\sigma_1, t_1) \dots \mathcal{O}_N(\sigma_N, t_N)\right] e^{-2\pi R' H}\right]_{\mathcal{H}}}{\mathrm{Tr}\left[e^{-2\pi R' H}\right]_{\mathcal{H}}},$$

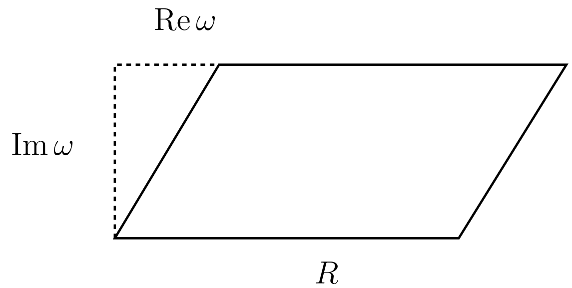
or as the system at finite temperature $1/R$ in angular quantization in finite volume R

$$\langle \mathcal{O}_1(\sigma_1,t_1)\dots\mathcal{O}_N(\sigma_N,t_N)\rangle_{\text{torus}} = \frac{\operatorname{Tr}\Big[\mathcal{T}_\sigma\left[\mathcal{O}_1(\sigma_1,t_1)\dots\mathcal{O}_N(\sigma_N,t_N)\right]e^{-2\pi R H'}\Big]_{\mathcal{H}'}}{\operatorname{Tr}\Big[e^{-2\pi R H'}\Big]_{\mathcal{H}'}}.$$

The result should be independent on a way we arrive to it, but puts non-trivial constraints on spectra and fusion coefficients. This is known under the name of modular bootstrap. We will study it for the partition function

$$Z(R,R') \stackrel{\text{def}}{=} \operatorname{Tr}\Big[e^{-2\pi R H'}\Big]_{\mathcal{H}'} = \operatorname{Tr}\Big[e^{-2\pi R' H}\Big]_{\mathcal{H}} \implies Z(R,R') = Z(R',R).$$

In fact, it will be more convenient to consider general torus with complex moduli as shown on the picture



Noticing that the operators H and P commute and that $e^{2i\pi a P}$ translates through the distance a, one can write in this case

$$\langle\dots\rangle_{\text{torus}} = \frac{\operatorname{Tr}\Big[\dots e^{-2\pi(\operatorname{Im}\omega\, H - i\operatorname{Re}\omega\, P)}\Big]_{\mathcal{H}}}{\operatorname{Tr}\Big[e^{-2\pi(\operatorname{Im}\omega\, H - i\operatorname{Re}\omega\, P)}\Big]_{\mathcal{H}}} = \frac{\operatorname{Tr}\Big[\dots q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\Big]_{\mathcal{H}}}{\operatorname{Tr}\Big[q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\Big]_{\mathcal{H}}}, \tag{198}$$

where we have used (197)-(197) and defined

$$q = e^{2i\pi\tau} \quad \text{with} \quad \tau = \frac{\omega}{R}.$$

Consider general properties of the average (198). First, since the eigenvalues of P are n/R with $n \in \mathbb{Z}$ one can shift ω by an integer amount of R. That is $\langle\dots\rangle_{\text{torus}}$ is invariant under $\mathcal{T}$ transformation

$$\mathcal{T}:\ \tau \to \tau + 1.$$

At the same time the replacement $(\omega_1,\omega_2) = (R,\omega)$ by (ω_2,ω_1) describes the same torus and hence $\langle\dots\rangle_{\text{torus}}$ should be invariant under $\mathcal{S}$ transformation

$$\mathcal{S}:\ \tau \to -\frac{1}{\tau}.$$

We note that the same transformations should be applied to $\bar{\tau}$ since they are complex conjugate of each other.

These two transformations $\mathcal{T}$ and $\mathcal{S}$ satisfy the relations

$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = 1$$

and are known to generate the $PSL(2,\mathbb{Z})$ group or the modular group

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad a,b,c,d\in\mathbb{Z}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim -\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{199}$$

We note that equivalently the torus can be regarded as the quotient of the complex plane by the lattice generated by two elements

$$z \sim z + m\omega_1 + n\omega_2, \quad m,n\in\mathbb{Z}.$$

From this point of view it is clear that the torus defined by (ω_1,ω_2) and by

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \implies \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

are equivalent. Then the corresponding modular parameters $\tau=\omega_1/\omega_2$ and $\tau=\omega_1'/\omega_2'$ are related by (199).

Let us start to consider the partition function

$$Z(\tau) \overset{\text{def}}{=} \text{Tr}\Big[q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\Big],$$

where the trace goes over some Hilbert space. Consider some simplest examples.

Free boson. In this case we take the Hilbert space which consists of all Fock spaces $\mathcal{F}_P\otimes\mathcal{F}_P$. Then one has

$$Z(\tau) = \left|\frac{q^{-\frac{1}{24}}}{\prod_k(1-q^k)}\right|^2 \int' |q|^{P^2}dP = \frac{1}{|\eta(\tau)|^2}\int' |q|^{P^2}dP, \tag{200}$$

where we introduced the so called Dedekind η-function

$$\eta(\tau) \overset{\text{def}}{=} q^{\frac{1}{24}}\prod_{k=1}^{\infty}(1-q^k).$$

From the definition of $\eta(\tau)$ function we immediately see that

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \tag{201}$$

and hence the partition function (200), which involves only absolute values squared is ultimately invariant under $\mathcal{T}$ modular transformation. Under modular transformation $\mathcal{S}$ it behaves as follows

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{202}$$

In order to see it, it is convenient to define more general objects known as theta constants

$$\vartheta_2(\tau) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \overset{(118)}{=} 2q^{\frac{1}{8}}\prod_{k=0}^{\infty}(1+q^{k+1})^2(1-q^{k+1}),$$

$$\vartheta_3(\tau) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} \overset{(118)}{=} \prod_{k=0}^{\infty}(1+q^{k+\frac{1}{2}})^2(1-q^{k+1}),$$

$$\vartheta_4(\tau) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}} \overset{(118)}{=} \prod_{k=0}^{\infty}(1-q^{k+\frac{1}{2}})^2(1-q^{k+1}),$$

where we have used the Jacobi triple identity to rewrite the sum in terms of infinite product. Using the Poisson resumation formula[9]

$$\sum_{n\in\mathbb{Z}} e^{-\pi\alpha n^2+\beta n} = \frac{1}{\sqrt{\alpha}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{\alpha}\left(n+\frac{\beta}{2i\pi}\right)^2} \tag{203}$$

one can derive modular properties of theta constants and Dedekind $\eta-$function

$$\begin{aligned}
&\vartheta_2(\tau+1) = e^{\frac{i\pi}{4}}\vartheta_2(\tau) \qquad && \vartheta_2\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\vartheta_4(\tau),\\
&\vartheta_3(\tau+1) = \vartheta_4(\tau) && \vartheta_3\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\vartheta_3(\tau),\\
&\vartheta_4(\tau+1) = \vartheta_3(\tau) && \vartheta_4\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\vartheta_2(\tau),\\
&\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau) && \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)
\end{aligned} \tag{204}$$

We note that

$$\frac{\vartheta_2\vartheta_3\vartheta_4}{2\eta^3} = 1.$$

and hence we obtain (201) and (202).

We see that if we treat the integral in (200) literally

$$\int_{-\infty}^{\infty} |q|^{P^2} dP = \int_{-\infty}^{\infty} e^{-4\pi\mathrm{Im}\tau\, P^2} dP = \frac{1}{2\sqrt{\mathrm{Im}\tau}},$$

then the partition function

$$Z(\tau) = \frac{1}{2\sqrt{\mathrm{Im}\tau}}\frac{1}{|\eta(\tau)|^2}$$

is invariant under $\mathcal{S}$ transformation as well.

Free boson on a circle. Consider the situation when our bosonic field φ takes values in a circle, that is we identify

$$\varphi(z,\bar{z}) \sim \varphi(z,\bar{z}) + 2\pi r$$

It can be interpreted as follows. Consider the mode expansion of the field

$$\varphi(z,\bar{z}) = -i\hat{q} - i\hat{p}\log\left(\frac{z}{R}\right) - i\hat{\bar{p}}\log\left(\frac{\bar{z}}{R}\right) - i\sum_{k\neq 0}\left(\frac{a_k}{k}z^{-k} + \frac{\bar{a}_k}{k}\bar{z}^{-k}\right),$$

We require that under $z \to e^{2i\pi}z$ the field φ transforms as follows

$$\varphi(z,\bar{z}) \sim \varphi(z,\bar{z}) + 2\pi n r$$

[9]It follows from application of the identity

$$\sum_{k\in\mathbb{Z}} \delta(x-k) = \sum_{k\in\mathbb{Z}} e^{2i\pi kx}$$

to the function $e^{-\pi\alpha x^2+\beta x}$.

which holds provided that

$$\hat{p} - \hat{\bar{p}} = -nr.$$

That is our Hilbert space in

$$\mathcal{H} = \oplus_P \left(\mathcal{F}_P \otimes \mathcal{F}_{P-nr}\right)$$

and the partition function takes the form

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_P q^{\frac{P^2}{2}} \bar{q}^{\frac{(P-nr)^2}{2}}.$$

The invariance under $\mathcal{T}$ transformation requires

$$\frac{1}{2}\left(P^2 - (P-nr)^2\right) = m \implies P = \frac{m}{nr} + \frac{nr}{2}$$

Then the partition function

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n} q^{\frac{1}{2}\left(\frac{m}{R}+\frac{Rn}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{m}{R}-\frac{Rn}{2}\right)^2}$$

is invariant under $\mathcal{T}$ and as a bonus under $\mathcal{S}$. It can be shown by explicit application of the Poisson formula (203).

Ising model. There are three characters in this case $\chi_{1,1}(q)$, $\chi_{3,1}(q)$ and $\chi_{2,1}(q)$ (including the factor $q^{-\frac{c}{24}} = q^{-\frac{1}{48}}$ which we dropped before)

$$\begin{aligned}
\chi_{1,1}(q) + \chi_{3,1}(q) &= q^{-\frac{1}{48}} \prod_{k=1}^{\infty}(1+q^{k+\frac{1}{2}}) = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}, \\
\chi_{1,1}(q) - \chi_{3,1}(q) &= q^{-\frac{1}{48}} \prod_{k=1}^{\infty}(1-q^{k+\frac{1}{2}}) = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}, \\
\chi_{2,1}(q) &= q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1+q^{k}) = \sqrt{\frac{\theta_2(\tau)}{2\eta(\tau)}}.
\end{aligned}$$

Using the modular properties of the theta constants (204) one can show that the combination

$$Z(\tau) = \frac{1}{2}\left(\left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| + \left|\frac{\theta_2}{\eta}\right|\right) = |\chi_{1,1}|^2 + |\chi_{3,1}|^2 + |\chi_{2,1}|^2$$

form modular invariant partition function.

Generic unitary minimal model. We consider $\mathcal{M}_{p,p+1}$ minimal model with

$$c = 1 - \frac{6}{p(p+1)}, \qquad \Delta_{m,n} = \frac{(mp - n(p+1))^2 - 1}{4p(p+1)}.$$

There are $p(p-1)/2$ primary fields in Kac table $0 < n < m < p+1$ and the character of each representation is given by Rocha-Caridi-Feigin-Fuks formula (compare to (170))

$$\chi_{m,n}(\tau) = \frac{1}{\eta(\tau)} \sum_{k\in\mathbb{Z}} \left(q^{\frac{((m+2k(p+1))p-n(p+1))^2}{4p(p+1)}} - q^{\frac{((-m+2k(p+1))p-n(p+1))^2}{4p(p+1)}} \right) =$$
$$= \frac{1}{\eta(\tau)} \left(\Theta_{pm-n(p+1),p(p+1)}(\tau) - \Theta_{pm+n(p+1),p(p+1)}(\tau)\right), \quad (205)$$

where

$$\Theta_{r,s}(\tau) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} q^{s\left(k+\frac{r}{2s}\right)^2},$$

and we used the property $\Theta_{-r,s}(\tau) = \Theta_{r,s}(\tau)$.

Using Poisson resumation formula (203) one can find modular transformation properties of $\Theta_{r,s}(\tau)$

$$\begin{aligned} &\Theta_{r,s}(\tau+1) = e^{\frac{i\pi r^2}{4s}}\Theta(\tau), \\ &\Theta_{r,s}(-\frac{1}{\tau}) = \sqrt{-i\tau}\sum_{r'=-s+1}^{s}\frac{1}{\sqrt{2s}}e^{-\frac{i\pi r r'}{s}}\Theta_{r',s}(\tau). \end{aligned} \quad (206)$$

First equation is obvious. For the second one we use

$$\Theta_{r,s}\left(-\frac{1}{\tau}\right) = \sum_{k\in\mathbb{Z}} e^{-\frac{2i\pi s}{\tau}\left(k+\frac{r}{2s}\right)^2} \stackrel{(203)}{=} \frac{\sqrt{-i\tau}}{\sqrt{2s}} \sum_{k'\in\mathbb{Z}} e^{2i\pi\tau s\left(\frac{k'}{2s}\right)^2 + \frac{i\pi r k'}{s}}.$$

It is convenient to represent

$$k' = -2sk - r' \quad \text{with} \quad k\in\mathbb{Z} \quad \text{and} \quad r' = -s+1,\ldots,s. \quad (207)$$

Then the last sum in (207) can be rewritten as

$$\sum_{k'\in\mathbb{Z}} e^{2i\pi\tau s\left(\frac{k'}{2s}\right)^2 + \frac{i\pi r k'}{s}} = \sum_{r'=-s+1}^{s}\sum_{k\in\mathbb{Z}} e^{2i\pi\tau s\left(k+\frac{r'}{2s}\right)^2 - \frac{i\pi r r'}{s}} = \sum_{r'=-s+1}^{s} e^{-\frac{i\pi r r'}{s}}\Theta_{r',s}(\tau).$$

We note that using the symmetry $\Theta_{r,s}(\tau) = \Theta_{-r,s}(\tau)$ one can rewrite

$$\Theta_{r,s}(-\frac{1}{\tau}) = \frac{\sqrt{-i\tau}}{\sqrt{2s}}\left(\Theta_{0,s}(\tau) + \sum_{r'=1}^{s-1}\left(e^{\frac{i\pi r r'}{s}} + e^{-\frac{i\pi r r'}{s}}\right)\Theta_{r',s}(\tau) + (-1)^r\Theta_{s,s}(\tau)\right)$$

Plugging (206) into the character formula (205), one finds

$$\chi_{m,n}(\tau+1) = e^{2i\pi\left(\Delta_{m,n}-\frac{c}{24}\right)}\chi_{m,n}(\tau)$$

and

$$\chi_{m,n}\left(-\frac{1}{\tau}\right) = \frac{1}{\eta(\tau)}\frac{1}{\sqrt{2p(p+1)}}\sum_{r=1}^{p(p+1)-1} 4\sin\frac{\pi m r}{p+1}\sin\frac{\pi n r}{p}\,\Theta_{r,p(p+1)}(\tau).$$

Now, we use the following lemma which belongs to Cardy [9]. Namely, one can notice that the set

$$pm' \pm (p+1)n' \quad \text{with} \quad 0 < n' < m' < p+1,$$

spans all integers r' from 1 to $p(p+1)$ not divisible by p and $p+1$ modulo $2p(p+1)$ and $r' \to -r'$. Thus we obtain[10]

$$\chi_{m,n}\left(-\frac{1}{\tau}\right) = \mathcal{S}_{m,n}^{m',n'}\chi_{m',n'}(\tau), \qquad \mathcal{S}_{m,n}^{m',n'} = \frac{4}{\sqrt{2p(p+1)}}(-1)^{mn'+m'n+1}\sin\frac{\pi pmm'}{p+1}\sin\frac{\pi(p+1)nn'}{p}$$

We see that the matrix $\mathcal{S}$ is symmetric and real. Moreover from its definition it follows that

$$\mathcal{S}\mathcal{S} = I \implies \mathcal{S}^{-1} = \mathcal{S}.$$

The expression for the partition function is

$$Z(\tau) = \sum_{m,n,m',n'} \mathcal{N}_{m,n,m',n'}\chi_{m,n}(\tau)\chi_{m',n'}(\bar{\tau}), \tag{208}$$

where $\mathcal{N}_{m,n,m',n'}$ is an integer number called the multiplicity, that is the number of times the representation with the highest weights $(\Delta,\bar{\Delta}) = (\Delta_{m,n},\Delta_{m',n'})$ is present. The modular of the partition function $Z(\tau)$ is equivalent to the set of conditions

$$\mathcal{T}\mathcal{N}\mathcal{T}^{-1} = \mathcal{S}\mathcal{N}\mathcal{S}^{-1} = 1,$$

where $\mathcal{T}$ and $\mathcal{S}$ are the matrices of elementary modular transformations. In addition we require

$$\mathcal{N}_{1,1,1,1} = 1,$$

that is we assume that the identity operator I is unique.

One solution to (208) which may always be found is

$$\mathcal{N}_{m,n,m',n'} = \delta_{m,m'}\delta_{n,n'},$$

which corresponds to the situation where all primary fields are scalar, that is $\Delta = \bar{\Delta}$, and all of them are taken just ones (with multiplicity 1). However, as was noticed by Cardy, there are other solutions. Necessary condition comes from $\mathcal{T}$ symmetry, which demands that only the operators with the integer spin may occur

$$\Delta_{m,n} - \Delta_{m',n'} = s \in \mathbb{Z}.$$

Inspecting the Kac tables for $\mathcal{M}_{3,4}$ and $\mathcal{M}_{4,5}$ theories ((171) and (182) respectively), one finds that this does not happen. However for the model $\mathcal{M}_{5,6}$ (see (185)) we see that

$$\Delta_{5,1} - \Delta_{1,1} = 3, \qquad \Delta_{5,2} - \Delta_{1,2} = 1,$$

[10]For generic $\mathcal{M}_p, q$ model one has

$$\mathcal{S}_{m,n}^{m',n'} = \frac{4}{\sqrt{2pq}}(-1)^{mn'+m'n+1}\sin\frac{\pi pmm'}{q}\sin\frac{\pi qnn'}{p}$$

and hence $\mathcal{N}_{5,1,1,1}$ and $\mathcal{N}_{5,2,1,2}$ might be non-zero. It can be shown that

$$Z(\tau) = |\chi_{1,1}(\tau) + \chi_{5,1}(\tau)|^2 + |\chi_{1,2}(\tau) + \chi_{5,2}(\tau)|^2 + 2|\chi_{3,1}(\tau)|^2 + 2|\chi_{3,2}(\tau)|^2$$

is invariant under $\mathcal{S}$ transformation. We note that in this non-diagonal solution the fields $\Phi_{4,r}$ are not present, while the fields $\Phi_{3,r}$ enter with multiplicity 2. This model corresponds to Z_3 parafermionic CFT [11]. Two copies of the field $\Phi_{3,1}$ correspond to parafermionic currents (193), while two fields $\Phi_{3,2}$ with $\Delta = \frac{1}{15}$ to the "energy operators" σ_1 and $\sigma_2 = \sigma_1^+$ from [11].

General classification of modular invariant partition functions has been done in [12].

Probs:

1. By explicit calculations show that there are exactly two solutions to modular bootstrap equations for $\mathcal{M}_{5,6}$ model.

Lecture 16: Friedan Qiu and Shenker theorem

The theorem [13] states that the Verma module $\mathcal{V}_\Delta$ does not have vectors of negative norm only in two cases:

- For $\Delta \geq 0$ and $c \geq 1$
- For unitary minimal model

$$c = 1 - \frac{6}{p(p+1)}, \quad \Delta_{m,n} = \frac{(mp - n(p+1))^2 - 1}{4p(p+1)} \quad \text{for} \quad 0 < n < m < p+1.$$

The proof substantially uses the Kac determinant formula

$$\det \Gamma^{(N)} \sim \prod_{m,n} (\Delta - \Delta_{m,n})^{p(N-mn)} \tag{209}$$

We remind the meaning of (209). Consider generic state in the Verma module $\mathcal{V}_\Delta$ at level N

$$|\rho\rangle = \sum_{|\boldsymbol{\lambda}|=\rho} C_{\boldsymbol{\lambda}} L_{-\boldsymbol{\lambda}} |\Delta\rangle.$$

Then its norm is

$$\langle\rho|\rho\rangle = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} \langle\Delta|L_{\boldsymbol{\mu}} L_{-\boldsymbol{\lambda}}|\Delta\rangle C_{\boldsymbol{\lambda}} C_{\boldsymbol{\mu}} = \Gamma^{(N)}_{\boldsymbol{\lambda},\boldsymbol{\mu}} C_{\boldsymbol{\lambda}} C_{\boldsymbol{\mu}}. \tag{210}$$

We know that at $\Delta = \Delta_{m,n}$ there is a singular vector at level mn

$$|\chi_{m,n}\rangle = D_{m,n}|\Delta_{m,n}\rangle \quad \text{where} \quad D_{m,n} = L_{-1}^{mn} + c_1(b) L_{-2} L_{-1}^{mn-2} + c_2(b) L_{-3} L_{-1}^{mn-2} + \dots,$$

with

$$c_1 = \frac{mn}{6}\left((m^2-1)b^2 + (n^2-1)b^{-2}\right) \quad \text{etc.}$$

Moreover, for any two partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ the following holds $|\boldsymbol{\lambda}| = |\boldsymbol{\nu}| + mn$

$$\langle\Delta|L_{\boldsymbol{\lambda}} L_{-\boldsymbol{\nu}} D_{m,n}|\Delta\rangle \sim (\Delta - \Delta_{m,n}),$$

which implies that

$$\det \Gamma^{(mn+|\boldsymbol{\mu}|)} \sim (\Delta - \Delta_{m,n})^{|\boldsymbol{\nu}|}.$$

Thus the product in (209) exhausts all the required zeroes with the correct multiplicities. It remains to show that it gives the correct asymptotic at $\Delta \to \infty$. It is clear that the degree of $\langle\Delta|L_{\boldsymbol{\mu}} L_{-\boldsymbol{\lambda}}|\Delta\rangle$ in Δ is not greater than $l(\boldsymbol{\lambda})$ and $l(\boldsymbol{\mu})$ and that $\langle\Delta|L_{\boldsymbol{\lambda}} L_{-\boldsymbol{\lambda}}|\Delta\rangle \sim \Delta^{l(\boldsymbol{\lambda})}$. It implies that

$$\det \Gamma^{(N)} \sim \Delta^{\sum_{|\boldsymbol{\lambda}|=N} l(\boldsymbol{\lambda})}$$

and hence we expect the combinatorial fact

$$\sum_{|\boldsymbol{\lambda}|=N} l(\boldsymbol{\lambda}) = \sum_{m,n} p(N - mn),$$

which can be proven by elementary methods. Namely representing

$$\boldsymbol{\lambda}=\{\underbrace{1,\dots,1}_{n_1},\{\underbrace{2,\dots,2}_{n_2}\},\dots\}$$

we have

$$\sum_{|\boldsymbol{\lambda}|=N} l(|\boldsymbol{\lambda}|)=\sum_{\sum_k kn_k=N}\sum_k n_k$$

Now we come to the first part of the theorem. It follows from three simple facts:

Fact 1:

$$\begin{aligned}&\langle\Delta|L_1L_{-1}|\Delta\rangle=2\Delta\langle\Delta|\Delta\rangle \implies \Delta\geq 0,\\ &\langle\Delta|L_nL_{-n}|\Delta\rangle=\Big(2n\Delta+\frac{c}{12}(n^3-n)\Big)\langle\Delta|\Delta\rangle \implies c\geq 0.\end{aligned}$$

Fact 2: The Kac determinant $\det G_N$ is positive for $\Delta>0$ and $c\geq 1$. Indeed, for $c>25$ all Kac values are negative, while for $1<c<25$ we have $\Delta_{m,n}=\Delta^*_{n,m}$ for $m\neq n$ and $\Delta_{m,m}<0$.

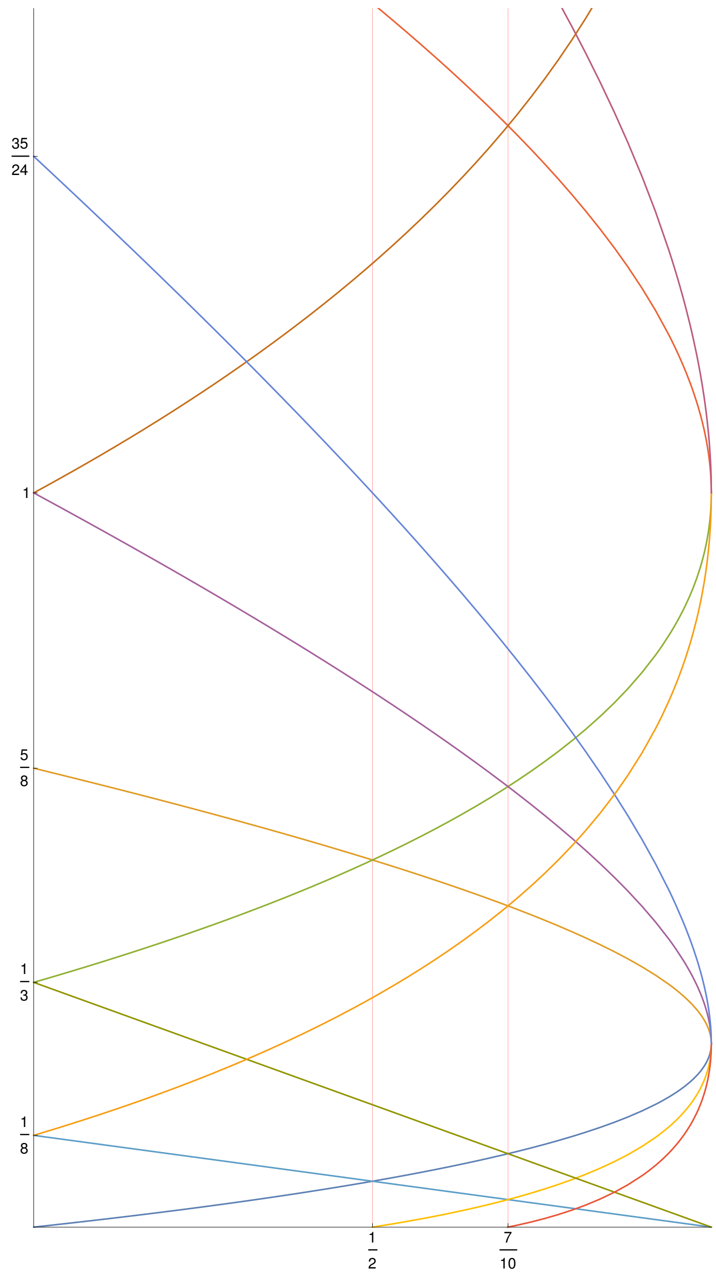
Fact 3: The Shapovalov form is positive in the limit $\Delta\to\infty$. Indeed, consider the generic state $|\rho\rangle$. Its norm is given by (210). In the limit $\Delta\to\infty$ only the states with maximal $l(\boldsymbol{\lambda})$ will contribute, but for these states we have

$$\Gamma_{\boldsymbol{\mu},\boldsymbol{\lambda}}=\Delta^{l(\boldsymbol{\lambda})}\left(\xi\delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}+O\Big(\frac{1}{\Delta}\Big)\right),$$

for some positive ξ.

From these three facts we understand that the Gramm matrix Γ is positive definite for large $\Delta>0$ and for $c>1$ its determinant is strictly positive. It means that it can not become negative since in that case it should cross 0, but this does not happen.

Now we come to the second part of the theorem. The idea is similar, using Kac determinant formula (209) we will eliminate domains in the semi-strip $(\Delta\geq 0, 0\leq c\leq 1)$ level by level.


35/24
1
5/8
1/3
1/8
1/2
7/10

Lecture 17: CFT in curved space and conformal anomaly

One way to define the stress-energy tensor is to place a system into a curved background. Then $T_{\mu\nu}$ is defined as a response to infinitesimal variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

$$\delta S = \frac{1}{4\pi} \int \delta g_{\mu\nu} T^{\mu\nu} \sqrt{g}\, d^2\boldsymbol{x}, \tag{211}$$

and then computed in flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. We remind that there are many ways to embed original theory into background metric. This leads to intrinsic ambiguity in the definition of $T_{\mu\nu}$ (see discussions around (11)). In this lecture we will study CFT in generic curved background.

For coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(\boldsymbol{x})$ we have

$$\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu, \qquad \nabla_\mu \epsilon_\nu = \partial_\mu \epsilon_\nu - \Gamma^\lambda_{\mu\nu} \epsilon_\lambda,$$

where $\Gamma^\lambda_{\mu\nu}$ is the Christoffel connection

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).$$

Under such variation (according to the definition (211)) we have

$$\delta_{\boldsymbol{\epsilon}} S = \frac{1}{2\pi} \int \nabla_\mu \epsilon_\nu T^{\mu\nu} \sqrt{g}\, d^2\boldsymbol{x} = -\frac{1}{2\pi} \int \epsilon_\nu \nabla_\mu T^{\mu\nu} \sqrt{g}\, d^2\boldsymbol{x}$$

and hence on-shell we find that the energy-momentum tensor satisfies the covariant continuity equation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{212}$$

We note that (212) leads to the conservation law, only if the metric $g_{\mu\nu}$ admits isometries

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0,$$

then we have

$$\nabla_\mu \left(T^{\mu\nu} K_\nu \right) = 0 \implies \nabla_\mu \left(T^{\mu\nu} K_\nu \right) = \partial_\mu \left(T^{\mu\nu} K_\nu \right) + \Gamma^\lambda_{\lambda\mu} \left(T^{\mu\nu} K_\nu \right) = \frac{1}{\sqrt{g}} \partial_\mu \left(T^{\mu\nu} K_\nu \sqrt{g} \right).$$

In QFT we take the identity

$$\sum_{k=1}^N \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \delta_{\boldsymbol{\epsilon}} \mathcal{O}_k(\boldsymbol{x}_k) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle = \frac{1}{2\pi} \int \nabla_\mu \epsilon_\nu(\boldsymbol{x}) \langle T_{\mu\nu}(\boldsymbol{x}) \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_N(\boldsymbol{x}_N) \rangle \sqrt{g(\boldsymbol{x})} d^2\boldsymbol{x}, \tag{213}$$

which holds for any local fields $\mathcal{O}_k$ as definition of $T_{\mu\nu}$. Then standard argument shows that (212) holds up to contact terms. More precisely, in (213) only vicinities of $\boldsymbol{x}_k$'s contribute to the integral over $d^2\boldsymbol{x}$, which means that one can write a covariant form of local relation (41)

$$\delta_{\boldsymbol{\epsilon}} \mathcal{O}(\boldsymbol{x}) = \frac{1}{2\pi} \int_{\mathcal{D}_{\boldsymbol{x}}} \nabla_\mu \epsilon_\nu(\boldsymbol{y}) T^{\mu\nu}(\boldsymbol{y}) \mathcal{O}(\boldsymbol{x}) \sqrt{g} d^2\boldsymbol{y} - \frac{1}{2\pi} \oint_{\mathcal{C}_{\boldsymbol{x}}} \epsilon_\nu(\boldsymbol{y}) \left(\varepsilon_{\mu\lambda} T^{\lambda\nu}(\boldsymbol{y}) \sqrt{g} \right) \mathcal{O}(\boldsymbol{x}) dy^\mu.$$

Locally one can choose conformal complex coordinates $(z, \bar{z})$

$$ds^2 = e^{\sigma} dz d\bar{z}, \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}e^{\sigma}, \quad g^{z\bar{z}} = 2e^{-\sigma}.$$

In these coordinates one has

$$\Gamma^{z}_{zz} = \partial\sigma, \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \bar{\partial}\sigma, \quad R_{z\bar{z}} = -\partial\bar{\partial}\sigma \implies R = -4e^{-\sigma}\partial\bar{\partial}\sigma \tag{214}$$

The conformal transformations are actually the same as in flat space $z \to w(z)$

$$e^{\sigma(z,\bar{z})} = \left|\frac{dw}{dz}\right|^2 e^{\sigma(w,\bar{w})}, \quad \text{or} \quad \sigma(z,\bar{z}) = \sigma(w,\bar{w}) - \log\left|\frac{dw}{dz}\right|^2,$$

while the continuity equation (212) takes the form ($\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda}$)

$$0 = \nabla_z T^{z\bar{z}} + \nabla_{\bar{z}} T^{\bar{z}\bar{z}} = \partial T^{z\bar{z}} + \bar{\partial} T^{\bar{z}\bar{z}} + 2\Gamma^{\bar{z}}_{\bar{z}\bar{z}} T^{\bar{z}\bar{z}} + \Gamma^{z}_{zz} T^{z\bar{z}},$$

and similar equation with $z \leftrightarrow \bar{z}$. Using (214), one finds

$$e^{-2\sigma}\bar{\partial}\big(e^{2\sigma}T^{\bar{z}\bar{z}}\big) + e^{-\sigma}\partial\big(e^{\sigma}T^{z\bar{z}}\big) = 0 \implies \bar{\partial}T_{zz} + e^{\sigma}\partial\big(e^{-\sigma}T_{z\bar{z}}\big) = 0$$

In flat metric, conformal invariance is equivalent to vanishing of the trace of $T_{\mu\nu}$. This condition is known to generalize in curved background to the so called conformal anomaly condition (in differential form)

$$T^{\mu}_{\mu} = \alpha R. \tag{215}$$

In conformal frame (215) reads

$$T_{z\bar{z}} = -\alpha\partial\bar{\partial}\sigma \implies \bar{\partial}T_{zz} - \alpha\big(-(\partial\bar{\partial}\sigma)\partial\sigma + \partial^2\bar{\partial}\sigma\big) = 0,$$

but the second term is a total $\bar{\partial}$ derivative, which implies

$$\bar{\partial}T = 0 \quad \text{where} \quad T \stackrel{\text{def}}{=} T_{zz} - \frac{\alpha}{2}\left(-\big(\partial\sigma\big)^2 + 2\partial^2\sigma\right). \tag{216}$$

The object T is known as holomorphic stress-energy tensor. However, in view of (216) it is not actually a tensor, but rather a pseudo-tensor. Indeed the additional term $t = -\big(\partial\sigma\big)^2 + 2\partial^2\sigma$ in (216) transforms as

$$t(z) = \left(\frac{dw}{dz}\right)^2 t(w) - 2\{w, z\} \quad \text{under conformal transformation} \quad z \to w(z).$$

Since T_{zz} is a true tensor, the holomorphic object $T(z)$ should transform anomalously as well

$$T(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \alpha\{w, z\}. \tag{217}$$

It is natural to interpret $T(z)$ as a holomorphic stress-energy tensor in flat metric. Then, comparing (217) to (56), one finds

$$\alpha = -\frac{c}{12} \implies T^{\mu}_{\mu} = -\frac{c}{12}R. \tag{218}$$

Similarly to the stress-energy tensor, other fields in CFT can be dressed by the metric to invert them into covariant objects. For example, having spinless primary field $\Phi_\Delta(z,\bar{z})$ one can construct a scalar field

$$\Phi^{(g)}_\Delta(z,\bar{z}) = e^{-\Delta\sigma(z,\bar{z})}\Phi_\Delta(z,\bar{z}). \tag{219}$$

Similarly a covariant extension of the descendant field $L_{-1}\Phi_\Delta(z,\bar{z})$ is

$$L_{-1}\Phi_\Delta(z,\bar{z}) \to \partial\left(e^{-\Delta\sigma(z,\bar{z})}\Phi_\Delta(z,\bar{z})\right).$$

It is instructive to derive integral form of the conformal (Weyl) anomaly (218). By definition (211) the trace of $T_{\mu\nu}$ describes the response to the variation of the conformal factor of the background metric $g_{\mu\nu} \to (1+\delta\sigma)g_{\mu\nu}$

$$\delta S = \frac{1}{4\pi}\int \delta\sigma\, T^\mu_\mu\, \sqrt{g}\, d^2\boldsymbol{x}.$$

For partition function in CFT it gives the variation formula

$$\delta \log Z = \frac{c}{48\pi}\int \delta\sigma\, R\, \sqrt{g}\, d^2\boldsymbol{x}. \tag{220}$$

It will prove convenient to decompose the metric as

$$g_{\mu\nu}(\boldsymbol{x}) = e^{\sigma(\boldsymbol{x})}\hat{g}_{\mu\nu}(\boldsymbol{x}),$$

with some reference metric $\hat{g}_{\mu\nu}(\boldsymbol{x})$ and use the transformation formula for the scalar curvature

$$\sqrt{g}R = \sqrt{\hat{g}}\left(\hat{R} - \Delta_{\hat{g}}\sigma\right),$$

where $\Delta_{\hat{g}}$ is the covariant Laplacian in reference metric $\hat{g}_{\mu\nu}(\boldsymbol{x})$. Then the variation formula (220) converts to

$$\delta \log Z = \frac{c}{48\pi}\int \delta\sigma\,\left(\hat{R} - \Delta_{\hat{g}}\sigma\right)\,\sqrt{\hat{g}}\, d^2\boldsymbol{x}.$$

This equation can be easily integrated producing the integral form of the conformal anomaly equation

$$Z[e^\sigma \hat{g}] = \exp\left[\frac{c}{48\pi}\int \sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \hat{R}\sigma\right)\right] Z[\hat{g}]. \tag{221}$$

Similar analysis can recover σ dependence of correlation functions of covariant fields. In particular for scalar fields (219) one has

$$\langle \Phi^{(g)}_{\Delta_1}(z_1,\bar{z}_1)\ldots\Phi^{(g)}_{\Delta_n}(z_1,\bar{z}_1)\rangle = \left[\prod_{k=1}^{n} e^{-\Delta_k\sigma(z_k,\bar{z}_k)}\right]\langle \Phi^{(\hat{g})}_{\Delta_1}(z_1,\bar{z}_1)\ldots\Phi^{(\hat{g})}_{\Delta_n}(z_1,\bar{z}_1)\rangle \tag{222}$$

Equations (221), (222) and similar equations for descendant fields allow to study CFT in curved background, since the metric dependence of correlation functions is universal.

Now we are going to derive the Weyl anomaly equation (221) for Gaussian theories. We consider generalized $B-C$ system (compare to (105) and (119))

$$S = \frac{1}{4\pi}\int \left(B\bar{\partial}C + \bar{B}\partial\bar{C}\right) d^2\boldsymbol{x}. \tag{223}$$

The fields B and C can be either Grassmann numbers or not. This is not important at the moment. We are working in the conformal gauge: $g_{\mu\nu}(z,\bar{z}) = e^{\sigma(z,\bar{z})}\delta_{\mu\nu}$ and the conformal factor is already cancelled in (223) due to conformal invariance of the model. But it does not cancel in the measure in the path integral and this this the reason for σ-dependence of the partition function called the conformal anomaly.

We note that the action (223) is invariant under the holomorphic change of coordinates $z \to w(z)$, provided the the fields transform as j- and $(1-j)$-differentials

$$B \to \left(\frac{dw}{dz}\right)^{1-j} B, \quad C \to \left(\frac{dw}{dz}\right)^{j} C, \qquad \bar{B} \to \left(\frac{d\bar{w}}{d\bar{z}}\right)^{1-j} \bar{B}, \quad \bar{C} \to \left(\frac{d\bar{w}}{d\bar{z}}\right)^{j} \bar{C},$$

where j is an arbitrary number. For example, it is 0 for $\beta - \gamma$ system (119) and $\frac{1}{2}$ for Dirac fermion QFT (119). The measure in the path integral over the fundamental fields $(B, C, \bar{B}, \bar{C})$ should be defined according to the invariant distance in the space of fields (here $\rho = e^{\sigma}$)

$$||\delta B||^2 = \int \rho^j \delta B \delta \bar{B} d^2\boldsymbol{x}, \quad ||\delta C||^2 = \int \rho^{1-j} \delta C \delta \bar{C} d^2\boldsymbol{x}. \tag{224}$$

In practice it is more convenient to redefine fundamental fields

$$B \to \rho^{-j} B, \quad \bar{B} \to \bar{B}, \qquad C \to C, \quad \bar{C} \to \rho^{j-1}\bar{C}.$$

After this redefinition and integration by parts (here we treat our fields as bosons), the action and the interval will have the form

$$S = \frac{1}{4\pi}\int \left(B(\rho^{-j}\bar{\partial})C - \bar{C}(\rho^{j-1}\partial)\bar{B}\right) d^2\boldsymbol{x}, \quad ||\delta B||^2 = \int \delta B \delta \bar{B} d^2\boldsymbol{x}, \quad ||\delta C||^2 = \int \delta C \delta \bar{C} d^2\boldsymbol{x}.$$

In other words, we just moved ρ dependence from the measure to the action. Then it is clear that the partition function is equal to the inverse of the detemimant

$$Z^{-1} = \det(2\rho^{-j}\bar{\partial})\det(-2\rho^{j-1}\partial) = \det \boldsymbol{\Delta}_j, \boldsymbol{\Delta}_{-1} = -4\rho^{-2}\partial\rho\bar{\partial}.$$

where

$$\boldsymbol{\Delta}_j \overset{\text{def}}{=} -4\rho^{j-1}\partial\rho^{-j}\bar{\partial}.$$

First of all, we note that under the transformation

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z}), \qquad \rho \to |w'|^2\rho \tag{225}$$

the operator $\boldsymbol{\Delta}_j$ transforms as follows

$$\boldsymbol{\Delta}_j \to (w')^j \boldsymbol{\Delta}_j (w')^{-j},$$

and hence the eigenfunctions transform as

$$\Psi_n^{(j)} \to (w')^j \Psi_n^{(j)}.$$

It means that the covariant metric on the eigenspace of the operator $\boldsymbol{\Delta}_j$ is (compare to (224))

$$(\Psi_m^{(j)}, \Psi_n^{(j)}) = \int \rho^{1-j}\bar{\Psi}_m^{(j)}\Psi_n^{(j)} d^2\boldsymbol{x}.$$

It is easy to show that

$$(\Psi_m^{(j)}, \mathbf{\Delta}_j \Psi_n^{(j)}) = (\mathcal{D}_j \Psi_m^{(j)}, \mathcal{D}_j \Psi_n^{(j)}) \quad \text{where} \quad \mathcal{D}_j = 2\rho^{-j}\bar{\partial},$$

and hence $\mathbf{\Delta}_j$ has non-negative eigenvalues. In fact, having a zero eigenvalue is a subtlety. Formally, any holomorphic function $\epsilon(z)$ is a zero mode of $\mathcal{D}_j$, but we need only those which are normalizable. Consider for example a sphere with Fubini-Studi metric

$$ds^2 = \frac{dzd\bar{z}}{(1+|z|^2)^2}, \implies \rho = \frac{1}{(1+|z|^2)^2}.$$

Then we must have

$$\int \frac{|\epsilon(z)|^2 d^2\boldsymbol{x}}{(1+|z|^2)^{2(1-j)}} < \infty.$$

For $j=0$ only $\epsilon = \text{const}$ satisfies this bound. It corresponds to the zero mode of the bosonic field. We can either throw it or compactify on a circle. In another case, which will be interesting to us, $j=-1$, any quadratic function

$$\epsilon(z) = a + bz + cz^2$$

is a zero mode of the operator $\mathbf{\Delta}_{-1}$. They correspond to the global conformal transformations.

Our goal is to compute the determinant of the operator $\mathbf{\Delta}_j$. Clearly the spectrum $\lambda_j^{(n)}$ is unbounded and hence the determinant is divergent and requires regularization. It is convenient to use the so called proper time regularization[11]

$$\log\det \mathbf{\Delta}_j = -\text{tr}\int_\epsilon^\infty \frac{dt}{t}\left(e^{-\mathbf{\Delta}_j t} - e^{-t}\right) = -\int_\epsilon^\infty \frac{dt}{t}\sum_{n=1}^\infty \left(e^{-\lambda_j^{(n)}t} - e^{-t}\right) \tag{226}$$

It is hard to compute (226) by it self, but it is relatively easy to compute its variation with respect to Weyl transformations $\rho \to \rho + \delta\rho$

$$\begin{aligned}
\delta\log\det \mathbf{\Delta}_j &= \int_\epsilon^\infty \text{tr}\left(\delta\mathbf{\Delta}_j e^{-\mathbf{\Delta}_j t}\right) dt = \int_\epsilon^\infty \text{tr}\left((j-1)\frac{\delta\rho}{\rho}\mathbf{\Delta}_j e^{-\mathbf{\Delta}_j t} - j\rho^{j-1}\partial\cdot\rho^{j-1}\frac{\delta\rho}{\rho}e^{-\mathbf{\Delta}_j t}\right) dt = \\
&= \int_\epsilon^\infty \text{tr}\left((j-1)\frac{\delta\rho}{\rho}\mathbf{\Delta}_j e^{-\mathbf{\Delta}_j t} + 4j\frac{\delta\rho}{\rho}\rho^{-j}\bar{\partial}\cdot e^{-\mathbf{\Delta}_j t}\cdot\rho^{j-1}\partial\right) dt = \\
&= \int_\epsilon^\infty \text{tr}\left((j-1)\frac{\delta\rho}{\rho}\mathbf{\Delta}_j e^{-\mathbf{\Delta}_j t} - j\frac{\delta\rho}{\rho}\mathbf{\Delta}_{1-j} e^{-\mathbf{\Delta}_{1-j} t}\right) dt,
\end{aligned}$$

where in the second line we have used cyclic property of the trace and in the third an identity

$$Ae^{BAt}B = ABe^{ABt} \quad \text{with} \quad A = 2\rho^{-j}\bar{\partial}, \quad B = 2\rho^{j-1}\partial, \quad AB = \mathbf{\Delta}_{1-j}.$$

The integral can be explicitly taken and we arrive to

$$\delta\log\det \mathbf{\Delta}_j = \text{tr}\left[\frac{\delta\rho}{\rho}\Big((j-1)e^{-\mathbf{\Delta}_j\epsilon} - je^{-\mathbf{\Delta}_{1-j}\epsilon}\Big)\right]. \tag{227}$$

[11] Here we use integral representation formula for $\log\lambda$

$$\log\lambda = -\int_0^\infty \frac{dt}{t}\left(e^{-\lambda t} - e^{-t}\right)$$

It is convenient to study (227) in coordinate representation, i.e. we represent

$$e^{-\boldsymbol{\Delta}_j t} f(\boldsymbol{x}) = \int K_j(t|\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y}) d^2\boldsymbol{y}.$$

Clearly $K_j(t|\boldsymbol{x},\boldsymbol{y})$ satisfies heat equation

$$\left(\partial_t + \boldsymbol{\Delta}_j^{\boldsymbol{x}}\right) K_j(t|\boldsymbol{x},\boldsymbol{y}) = 0 \quad \text{with} \quad K_j(0|\boldsymbol{x},\boldsymbol{y}) = \delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}).$$

In terms of this heat kernel the variation (227) takes the form

$$\delta \log\det \boldsymbol{\Delta}_j = \int \frac{\delta\rho(\boldsymbol{x})}{\rho(\boldsymbol{x})}\Big((j-1)K_j(\epsilon|\boldsymbol{x},\boldsymbol{x}) - jK_{1-j}(\epsilon|\boldsymbol{x},\boldsymbol{x})\Big) d^2\boldsymbol{x}. \tag{228}$$

In order to find the variation (228) we have to compute small time expansion of the heat kernel $K_j(t|\boldsymbol{x},\boldsymbol{y})$. It is convenient to consider more general operator

$$\boldsymbol{D} = -g^{\mu\nu}(\boldsymbol{x})\partial_\mu\partial_\nu + \xi^\mu(\boldsymbol{x})\partial_\mu$$

One may try to solve the heat kernel equation

$$\left(\partial_t + \boldsymbol{D}_{\boldsymbol{x}}\right) K(t|\boldsymbol{x},\boldsymbol{y}) = 0 \quad \text{with} \quad K(0|\boldsymbol{x},\boldsymbol{y}) = \delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}).$$

by the following anzatz[12]

$$K(t|\boldsymbol{x},\boldsymbol{y}) = \Sigma(\boldsymbol{x},\boldsymbol{y}) e^{-\frac{s(\boldsymbol{x},\boldsymbol{y})}{t}}\left(\frac{1}{t} + \mathrm{a}_0(\boldsymbol{x},\boldsymbol{y}) + \mathrm{a}_1(\boldsymbol{x},\boldsymbol{y})t + \dots\right) \quad \text{at} \quad t\to 0.$$

In the leading order one obtains

$$g^{\mu\nu}(\boldsymbol{x})\frac{\partial s(\boldsymbol{x},\boldsymbol{y})}{\partial x^\mu}\frac{\partial s(\boldsymbol{x},\boldsymbol{y})}{\partial x^\nu} - s(\boldsymbol{x},\boldsymbol{y}) = 0$$

This equation is equivalent to

$$g^{\mu\nu}(\boldsymbol{x})\frac{\partial S(\boldsymbol{x},\boldsymbol{y})}{\partial x^\mu}\frac{\partial S(\boldsymbol{x},\boldsymbol{y})}{\partial x^\nu} = 1, \quad \text{where} \quad s(\boldsymbol{x},\boldsymbol{y}) = \frac{S^2(\boldsymbol{x},\boldsymbol{y})}{4},$$

which is the Hamilton-Jacobi equation for geodesic length

$$S(\boldsymbol{x},\boldsymbol{y}) = \int\limits_{\boldsymbol{\xi}(0)=\boldsymbol{y}}^{\boldsymbol{\xi}(t)=\boldsymbol{x}} \sqrt{g_{\mu\nu}(\boldsymbol{\xi})\dot{\xi}^\mu\dot{\xi}^\nu}\, d\tau$$

[12]We remind that ordinary heat equation

$$(\partial_t - 4\partial\bar{\partial})K(t|\boldsymbol{x},\boldsymbol{y}) = 0 \quad \text{with} \quad K(0|\boldsymbol{x},\boldsymbol{y}) = \delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})$$

has a solution

$$K_j(t|\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi t} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}}.$$

Further calculations become even more involved, but for our purposes we need only the expansion of the diagonal part of the heat kernel $K_j(\epsilon|\boldsymbol{x},\boldsymbol{x})$. For given $\boldsymbol{x}=(z_0,\bar{z}_0)$ we perform global conformal transformation

$$z=\frac{aw+b}{cw+d},\quad ad-bc=1$$

such that in w coordinate system the point z_0 is mapped to 0 and moreover the following holds

$$\rho(0,0)=1,\quad \partial\rho(0,0)=0,\quad \bar{\partial}\rho(0,0)=0. \tag{229}$$

Then from (225) we find

$$b=z_0\rho(0,0)^{\frac{1}{4}},\quad c=\frac{\partial\rho(0,0)}{2\rho(0,0)^{\frac{3}{4}}},\quad d=\rho(0,0)^{\frac{1}{4}}.$$

Then we represent

$$\boldsymbol{\Delta}_j=-4\partial\bar{\partial}+\boldsymbol{V}_j(w,\bar{w})$$

and compute the kernel $K_j(\epsilon|\boldsymbol{x},\boldsymbol{x})$ by perturbation theory. The key observation is that since we are interested only in first few terms at $\epsilon\to 0$, only few orders are needed. Indeed, one can write

$$K_j=(\partial_t-4\partial\bar{\partial}+\boldsymbol{V}_j)^{-1}=(K^{-1}+V_j)^{-1}=(1+KV_j)^{-1}K=K-KV_jK+\dots \tag{230}$$

where K is the Green function of $\partial_t-4\partial\bar{\partial}$

$$K(t|z,z')=\frac{1}{4\pi t}e^{-\frac{|z-z'|^2}{4t}}.$$

In coordinate representation (230) can be rewritten as

$$K_j(\epsilon|0,0)=K(\epsilon|0,0)-\int_0^\epsilon dt'\int d^2z'K(\epsilon-t'|0,z')V_j(z')K(t'|z',0)+\dots \tag{231}$$

Since we are interested in small ϵ expansion, it is convenient to rescale all integrals in (231)

$$t\to\epsilon t,\quad z\to\epsilon^{\frac{1}{2}}z\implies K(t|z,z')\to\frac{1}{\epsilon}K(t|z,z'),\quad \int dt\int d^2z\to\epsilon^2\int dt\int d^2z. \tag{232}$$

From (232) it is clear that the integral of the k-th order in perturbation theory in (231) is proportional to

$$\epsilon^{k-1}=\frac{(\epsilon^2)^{\text{number of}\int dt\int d^2z}}{\epsilon^{\text{number of}\,K}}.$$

Thus if the operator $\boldsymbol{V}_j$ is of order ϵ^0, as we will show in a moment, the non-vanishing contribution will be given by zeroth and first order terms in (231). For $\boldsymbol{V}_j(\boldsymbol{x})$ one has

$$\begin{aligned}\boldsymbol{V}_j(\boldsymbol{x})=4\partial\bar{\partial}-4\rho^{j-1}(\boldsymbol{x})\partial\rho^{-j}(\boldsymbol{x})\bar{\partial}=4(1-\rho^{-1}(\boldsymbol{x}))\partial\bar{\partial}+4j\rho^{-2}(\boldsymbol{x})\partial\rho\bar{\partial}=\\ =-2\partial_\mu\partial_\nu\rho^{-1}(0)x^\mu x^\nu\partial\bar{\partial}+4j\partial_\lambda\partial\rho(0)x^\lambda\bar{\partial}+\dots\end{aligned}$$

where in the last line we retained only the terms which are finite in the limit $\epsilon\to 0$ after rescaling $\boldsymbol{x}\to\epsilon^{\frac{1}{2}}\boldsymbol{x}$. Thus we can keep only first two terms in (231). We have (remember $\boldsymbol{x}=(z,\bar{z})$)

$$K_j(\epsilon|0,0)=\frac{1}{4\pi\epsilon}+\int_0^1 dt\int d^2z\frac{1}{16\pi^2t(1-t)}e^{-\frac{|z|^2}{4(1-t)}}\left(2\partial_\mu\partial_\nu\rho^{-1}(0)x^\mu x^\nu\partial\bar{\partial}-4j\partial_\lambda\partial\rho(0)x^\lambda\bar{\partial}\right)e^{-\frac{|z|^2}{4t}}+O(\epsilon) \tag{233}$$

The integral over the angle allows to replace $x^\mu x^\nu = \frac{1}{2}\boldsymbol{x}^2\delta^{\mu\nu}$. Also in

$$x^\lambda \partial_\lambda(\partial\rho)\bar{\partial}e^{-\frac{|z|^2}{4t}} = \left(z\partial^2\rho + \bar{z}\partial\bar{\partial}\rho\right)\bar{\partial}e^{-\frac{|z|^2}{4t}} = -\frac{1}{4t}\left(z^2\partial^2\rho + z\bar{z}\partial\bar{\partial}\rho\right)e^{-\frac{|z|^2}{4t}}$$

the integral of z^2 drops out and for the integral in (233) we obtain

$$\partial\bar{\partial}\rho(0)\int_0^1 dt\int \frac{e^{-\frac{|z|^2}{4t(1-t)}}}{16\pi^2 t(1-t)}\left(\frac{j|z|^2}{t} + \frac{|z|^2}{t} - \left(\frac{|z|^2}{2t}\right)^2\right)d^2z =$$

$$= \frac{\partial\bar{\partial}\rho(0)}{4\pi}\int_0^1 \Big(4(j+1)(1-t) - 8(1-t)^2\Big)dt = \frac{(3j-1)\partial\bar{\partial}\rho(0)}{6\pi}$$

In our coordinate system (229) we have

$$\rho = e^\sigma \implies \partial\bar{\partial}\rho(0) = \partial\bar{\partial}\sigma(0) = -\frac{e^\sigma}{4}R$$

and thus we finally obtain the heat kernel expansion[13]

$$K_j(\epsilon|\boldsymbol{x},\boldsymbol{x}) = \frac{e^\sigma}{4\pi\epsilon}\left[1 - \frac{(3j-1)R}{6}\epsilon + \dots\right] \tag{234}$$

Summing up we have

$$\delta\log\det\boldsymbol{\Delta}_j = -\frac{1}{4\pi\epsilon}\int\delta\sigma\sqrt{g}d^2\boldsymbol{x} - \frac{2(6j^2-6j+1)}{48\pi}\int\delta\sigma R\sqrt{g}d^2\boldsymbol{x},$$

which after integration over $\delta\sigma$ gives

$$\det\boldsymbol{\Delta}_j = \exp\left[\frac{c}{48\pi}\int\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + R\sigma\right)d^2\boldsymbol{x} - \frac{1}{4\pi\epsilon}\int e^\sigma\sqrt{\hat{g}}d^2\boldsymbol{x}\right],$$

where

$$c = -2(6j^2-6j+1) \tag{235}$$

Probs:

1.

[13] The coefficients in (234) are known as Seeley-DeWitt coefficients.

Lecture 18: 2D quantum gravity, Liouville CFT

The defining property of the conformal field theory is its simple response to the Weyl rescaling of the background metric, a conformal anomaly equation

$$Z[e^{\sigma}g] = \exp\left[\frac{c}{48\pi}\int\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_{\mu}\sigma\partial_{\nu}\sigma + R\sigma\right)\right]Z[g]. \tag{236}$$

Now, let us assume that the metric $g_{\mu\nu}$ also fluctuates. Then it makes sense to consider entire partition function

$$Z = \int[\mathcal{D}g]Z[g]\exp\bigl(-S_{\text{grav}}[g]\bigr) = \int[\mathcal{D}g][\mathcal{D}\Phi]\exp\bigl(-S[\Phi,g]-S_{\text{grav}}[g]\bigr), \tag{237}$$

where $S_{\text{grav}}[g]$ is purely gravitational action. There are many gravitational actions one might consider. We will take $S_{\text{grav}}[g]$ in the form

$$S_{\text{grav}}[g] = \Lambda_{\text{c}}\int\sqrt{g}d^2\boldsymbol{x} + \frac{G}{8\pi}\int R\sqrt{g}d^2\boldsymbol{x}. \tag{238}$$

The first term corresponds to cosmological constant term, while the second to Einstein-Hilbert action. In two dimensions this term is a topological invariant

$$\frac{1}{8\pi}\int R\sqrt{g}d^2\boldsymbol{x} = 1-\mathrm{g},$$

where g is the genus of our surface. If we do not sum over the number of handles (as we actually do in string theory), this term is a constant and can be dropped.

In computing the partition function (237) one faces the problem of over counting of degrees of freedom. Indeed, the infinitesimal metric transformation

$$\delta g_{\mu\nu} = \nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu}$$

can be compensated by the coordinate transformation

$$x^{\mu} \to x^{\mu} + \epsilon^{\mu}.$$

It means that each field configuration in (237) is counted infinitely many times. In order to obtain a finite result, one has to divide by the volume of the diffeomorphism group, that is to define smth like[14]

$$[\mathcal{D}g] = \frac{[\mathcal{D}g_{\mu\nu}]}{[\mathcal{D}\epsilon]}. \tag{239}$$

Of course, this strange "ratio" of measures has to be properly defined. It is hard to define the measure itself, but one can define what we mean by infinitesimal distance in the space of metrics and vector fields. A natural covariant way to do it is

$$||\delta g_{\mu\nu}||^2 = \int\sqrt{g}\left(g^{\mu\alpha}g^{\nu\beta} + \xi g^{\mu\nu}g^{\alpha\beta}\right)\delta g_{\mu\nu}\delta g_{\alpha\beta}d^2x, \quad ||\epsilon^{\mu}||^2 = \int\sqrt{g}g_{\mu\nu}\epsilon^{\mu}\epsilon^{\nu}d^2x,$$

[14] We also have to assume that any metric can be obtained from the fixed metric $g_{\mu\nu}$ by coordinate transformation and Weyl rescaling. In general this is not the case and there is some finite dimensional space of moduli on which the reference metric $g_{\mu\nu}$ may depend. We will not discuss this issue here and assume that there are no moduli as in the case of a sphere. Then one can choose $g_{\mu\nu} = \delta_{\mu\nu}$ for simplicity.

where $\xi > -\frac{1}{2}$ is an arbitrary constant. Consider generic variation of the metric

$$\delta g_{\mu\nu} = g_{\mu\nu}\delta\sigma + (\nabla_\mu \epsilon_\nu + \nabla_\mu \epsilon_\nu - g_{\mu\nu}\nabla\cdot\epsilon)\,.$$

Then $\delta\sigma$ and ϵ^μ variations got separated

$$||\delta g_{\mu\nu}||^2 = \int \sqrt{g}\left(2(1+2\xi)(\delta\sigma)^2 + g_{\mu\alpha}g_{\nu\beta}\Sigma^{\mu\nu}\Sigma^{\alpha\beta}\right)d^2x, \quad \text{where} \quad \Sigma^{\mu\nu} \stackrel{\text{def}}{=} \nabla^\mu\epsilon^\nu + \nabla^\mu\epsilon^\nu - g^{\mu\nu}\nabla\cdot\epsilon.$$

and hence we expect that the measure becomes the product[15]

$$[\mathcal{D}g_{\mu\nu}] = [\mathcal{D}\sigma]\det(\hat{P}^{\mu\nu}_\lambda)[\mathcal{D}\epsilon],$$

where $\hat{P}^{\mu\nu}_\lambda$ is the symmetric traceless differential operator

$$\hat{P}^{\mu\nu}_\lambda = \nabla^\mu\delta^\nu_\lambda + \nabla^\nu\delta^\mu_\lambda - g^{\mu\nu}\nabla_\lambda.$$

We see that if we divide by the "volume" of the diffeomorphism group as in (239), one has

$$[\mathcal{D}g] = [\mathcal{D}\sigma]\det(\hat{P}^{\mu\nu}_\lambda).$$

It is instructive to arrive to the same conclusion by the procedure known as Fadeev-Popov trick. Namely, let us insert the identity under functional integral

$$1 = \Delta_{\text{FP}}(g)\int \mathcal{D}\epsilon\,\delta\left((g^{\mu\nu})^\epsilon - g^{\mu\nu}\right), \tag{240}$$

where $(g^{\mu\nu})^\epsilon$ is the metric transformed by diffeomorphism $\boldsymbol{\epsilon}$. Since we can always reduce the metric to the form $g^{\mu\nu} = e^{-\sigma}\delta^{\mu\nu}$ (fix the conformal gauge), the delta function in (240) has the form

$$\delta\left((g^{\mu\nu})^\epsilon - g^{\mu\nu}\right) = \delta((g^{12})^\epsilon)\delta((g^{11})^\epsilon - (g^{22})^\epsilon) = \delta(\nabla^1\epsilon^2 + \nabla^2\epsilon^1)\delta(2\nabla^1\epsilon^1 - 2\nabla^2\epsilon^2).$$

Using the analogy with finite-dimensional integrals for which we have

$$\int \prod_{i=1}^n \delta(\Lambda_{ij}x_j)d^nx = \frac{1}{|\det\Lambda|}$$

we have

$$\delta\left((g^{\mu\nu})^\epsilon - g^{\mu\nu}\right) = \frac{1}{\det\hat{P}}\delta(\epsilon^1)\delta(\epsilon^2) \implies \Delta_{\text{FP}}(g) = \det\hat{P},$$

where

$$\hat{P}\begin{pmatrix}\epsilon^1\\ \epsilon^2\end{pmatrix} = \begin{pmatrix}\nabla^1\epsilon^2 + \nabla^2\epsilon^1\\ 2\nabla^1\epsilon^1 - 2\nabla^2\epsilon^2\end{pmatrix} \implies \hat{P}\begin{pmatrix}\frac{\epsilon^2}{2} & \epsilon^1\\ \epsilon^1 & -\frac{\epsilon^2}{2}\end{pmatrix} = \begin{pmatrix}\nabla^1\epsilon^2 - \nabla^2\epsilon^1 & \nabla^1\epsilon^2 + \nabla^2\epsilon^1\\ \nabla^1\epsilon^2 + \nabla^2\epsilon^1 & \nabla^2\epsilon^1 - \nabla^1\epsilon^2\end{pmatrix}$$

[15] In conformal frame $ds^2 = \rho dz d\bar{z}$ one has

$$||\delta g_{\mu\nu}||^2 = \int \left(2\rho(1+2\xi)(\delta\sigma)^2 + 2\rho^2\epsilon^{\bar{z}}\boldsymbol{\Delta}_{-1}\epsilon^z\right)d^2\boldsymbol{x},$$

so that $\hat{P}^{\mu\nu}_\lambda \sim \boldsymbol{\Delta}_{-1}$.

Now in the integral (237) one has

$$Z = \int [\mathcal{D}g][\mathcal{D}\epsilon]\delta\big((g^{12})^\epsilon\big)\delta\big((g^{11})^\epsilon - (g^{22})^\epsilon\big)\Delta_{\text{FP}}(g)Z[g]\exp\big(-S_{\text{grav}}[g]\big).$$

Since this path integral is still diff invariant, one can replace $g \to g^\epsilon$. Then the integral over g^ϵ localizes to the support of δ functions and the volume of diff group factors out

$$Z = \int [\mathcal{D}\epsilon] \int [\mathcal{D}\sigma]\Delta_{\text{FP}}(\hat{g})Z[\hat{g}]\exp(-S_{\text{grav}}[\hat{g}]),$$

where $\hat{g}_{\mu\nu} = e^\sigma \delta_{\mu\nu}$.

The Fadeev-Popov determinant $\det(\hat{P}_\lambda^{\mu\nu})$ can be represented by the Gaussian integral over the anticommuting fields $B_{\mu\nu}$ and C^μ, where $B_{\mu\nu}$ is symmetric and traceless

$$\det(\hat{P}_\lambda^{\mu\nu}) = \int [\mathcal{D}B_{\mu\nu}][\mathcal{D}C^\mu]e^{-S_{\text{ghost}}}, \quad S_{\text{ghost}} = \frac{1}{4\pi}\int \sqrt{g}B_{\mu\nu}\nabla^\mu C^\nu d^2\boldsymbol{x}. \tag{241}$$

The ghost theory (241) is a conformal field theory. Since $B^{\mu\nu}$ is symmetric and traceless, we have only two non-vanishing components B_{zz} and $B_{\bar{z}\bar{z}}$ and hence we have

$$S_{\text{ghost}} = \frac{1}{4\pi}\int (B_{zz}\nabla_{\bar{z}}C^z + B_{\bar{z}\bar{z}}\nabla_z C^{\bar{z}})\, d^2\boldsymbol{x} = \frac{1}{2\pi}\int \left(B\bar{\partial}C + \bar{B}\partial\bar{C}\right) d^2\boldsymbol{x},$$
$$B = B_{zz}, \quad \bar{B} = B_{\bar{z}\bar{z}}, \quad C = C^z, \quad \bar{C} = C^{\bar{z}}.$$

In the last equality we used the fact that the only non-zero Christofel symbols in conformal gauge are Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ and hence they do not contribute to the action. We see that the action does not depend on a conformal factor σ and hence represents a conformal field theory. It's behavior in the background metric is controlled by the conformal anomaly equation (236). The covariant infinitesimal distances in the space of fields $B_{\mu\nu}$ and C^μ have the form

$$||\delta B||^2 = \int \sqrt{g}\delta B_{\mu\nu}\delta B_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}d^2\boldsymbol{x} = \int \rho^{-1}\delta B\delta\bar{B}d^2\boldsymbol{x},$$
$$||\delta C||^2 = \int \sqrt{g}\delta C^\mu\delta C^\nu g_{\mu\nu}d^2\boldsymbol{x} = \int \rho^2\delta C\delta\bar{C}d^2\boldsymbol{x},$$

and hence, according to (235), this CFT has the central charge $c = -26$ which implies that

$$Z_{\text{ghost}}[e^\sigma g] = \exp\left(-\frac{26}{48\pi}\int \sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + R\sigma\right)\right] Z_{\text{ghost}}[g].$$

In (237) we have two CFT's with central charges c and -26 which implies that

$$Z = Z_{\text{CFT}}[g]Z_{\text{ghost}}[g]\int [\mathcal{D}\sigma]\exp\left[\frac{c-26}{48\pi}\int \sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \Lambda e^\sigma + R[g]\sigma\right) d^2\boldsymbol{x}\right]. \tag{242}$$

where Λ is some renormalization of the initial Λ_{c} in (238) which come from both anomalies for CFT and ghost theory. We note that the measure $[\mathcal{D}\sigma]$ in (242) is ancested from the measure for the metric $g_{\mu\nu}$ and corresponds to the interval

$$||\delta\sigma||^2 = \int \sqrt{g}e^\sigma(\delta\sigma)^2 d^2\boldsymbol{x}.$$

It is not translationally invariant $\sigma \to \sigma + C(\boldsymbol{x})$, but rather invariant under the combined transformation

$$\sigma \to \sigma + C(\boldsymbol{x}), \qquad g_{\mu\nu} \to e^{-C(\boldsymbol{x})} g_{\mu\nu}.$$

It is hard to work with such a measure and it is desirable to replace it with a linear one, i.e. the one which is invariant under the shifts $\sigma \to \sigma + C(\boldsymbol{x})$. Of course, replacing the measure, we have to pay a price and compute the Jacobian. Here we propose, following David, Distler and Kawai [14,15], that the Jacobian is an exponent of a local action

$$\text{Jac} = \exp\left[-\frac{1}{4\pi}\int \sqrt{g}\left(\frac{\lambda_1}{2} g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \lambda_2 e^{\sigma} + \lambda_3 R[g]\sigma\right) d^2\boldsymbol{x}\right].$$

In general, one might expect that there will be more general potential term in this “effective” action and different dilaton coupling, but as we will se below only this form is consistent with the reference metric independence. If DDK conjecture is true, the Jacobian just renormalizes terms in the action. It is convenient to change normalization of the field σ, such that the kinetic term will have the traditional form. We will denote this rescaled field φ. It dynamics is described by the Liouville action

$$S_{\mathrm{L}}[\varphi, g] = \frac{1}{4\pi}\int \sqrt{g}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + QR\varphi + 4\pi\mu e^{2b\varphi}\right) d^2\boldsymbol{x}, \tag{243}$$

where b, Q and μ are new constants related in an obvious way to the previous ones. For the partition function one has

$$Z = Z_{\mathrm{CFT}}[g] \cdot Z_{\mathrm{ghost}}[g] \cdot Z_{\mathrm{L}}[g] \quad \text{where} \quad Z_{\mathrm{L}}[g] \overset{\text{def}}{=} \int [\mathcal{D}\varphi] e^{-S_{\mathrm{L}}[\varphi]}. \tag{244}$$

It is important that in (244) the measure $[\mathcal{D}\varphi]$ is linear one.

The reference metric $g_{\mu\nu}$ in (244) is just an auxiliary metric, it can be chosen at will. In particular, if we write

$$g_{\mu\nu} = e^{\Omega(\boldsymbol{x})} h_{\mu\nu}, \tag{245}$$

our partition function should be independent of $\Omega(\boldsymbol{x})$. First two factors in (244) correspond to CFT's and transform in controllable way. It means that Liouville theory should be a conformal field theory such that the total central charge vanishes

$$c + c_{\mathrm{ghost}} + c_{\mathrm{L}} = 0 \quad \text{or} \quad c_{\mathrm{L}} = 26 - c. \tag{246}$$

Let us study the question: under what conditions on the parameters b, Q and μ the Liouville theory is a conformal one with the desirable central charge. Consider the Weyl rescaling (245). Let us set $\mu = 0$ for the beginning. We have

$$S_{\mathrm{L}}[\varphi, e^{\Omega} h_{\mu\nu}] = \frac{1}{4\pi}\int \sqrt{h}\left(h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q(R_h - \Delta_h\Omega)\varphi\right) d^2\boldsymbol{x}$$

In the last term we can transform by parts to obtain

$$S_{\mathrm{L}}[\varphi, e^{\Omega} h_{\mu\nu}] = \frac{1}{4\pi}\int \sqrt{h}\left(h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + QR_h\varphi + Qh^{\mu\nu}\partial_\mu\varphi\partial_\nu\Omega\right) d^2\boldsymbol{x}$$

Finally we note that in terms of the shifted field

$$\tilde{\varphi} = \varphi + \frac{Q}{2}\Omega \tag{247}$$

we have an equality

$$S_{\mathrm{L}}[\varphi, e^{\Omega}h_{\mu\nu}] = S_{\mathrm{L}}[\tilde{\varphi}, h_{\mu\nu}] - \frac{Q^2}{8\pi}\int \sqrt{h}\left(\frac{1}{2}h^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + R_h\Omega\right)d^2\boldsymbol{x}$$

Moreover due to Weyl anomaly one has

$$[\mathcal{D}\varphi]_{\Omega h} = [\mathcal{D}\varphi]_h \exp\left(\frac{1}{48\pi}\int \sqrt{h}\left(\frac{1}{2}h^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + R_h\Omega\right)d^2\boldsymbol{x}\right)$$

Altogether one has

$$Z_{\mathrm{L}}[e^{\Omega}h] = Z_{\mathrm{L}}[h]\left(\frac{c_{\mathrm{L}}}{48\pi}\int \sqrt{h}\left(\frac{1}{2}h^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega + R_h\Omega\right)d^2\boldsymbol{x}\right) \quad \text{where} \quad c_{\mathrm{L}} = 1 + 6Q^2.$$

From the balance equation (246) one concludes that

$$Q^2 = \frac{25-c}{6}.$$

All that has been obtained with the assumption that the cosmological term

$$\mu\int\sqrt{g}e^{2b\varphi}d^2\boldsymbol{x}. \tag{248}$$

is absent in the action (243). There are three sources of Ω dependence of this term. First the metric term $\sqrt{g}$, second the shift (247) and third an anomalous dimension of the exponential field $e^{2b\varphi}$

$$[e^{2b\varphi}]_{e^{\Omega}h} = e^{b^2\Omega}[e^{2b\varphi}]_h \tag{249}$$

Combining all this, we find that the cosmological term (248) is background independent provided that

$$Q = b + \frac{1}{b}.$$

We note that in order to have real b one has $Q \geq 2$ and hence $c \leq 1$.

Few comments on (249) are needed. If $h_{\mu\nu} = \delta_{\mu\nu}$ and Ω is a constant, the exponential field $e^{2b\varphi}$ acquires anomalous dimension due to summing up an infinite series of tadpole diagrams (see (85) and (88)). Formula (249) is a generalization of this phenomenon in a curved background. It goes as follows. Consider Liouville action (243) without cosmological term. The path integral diverges because of the zero mode of the field φ. We might fix the value of the zero mode φ_0 by demanding that

$$\int\varphi R\sqrt{g}d^2\boldsymbol{x} = -4\int\varphi\partial\bar{\partial}\sigma d^2\boldsymbol{x} = 0.$$

Equivalently, we define the field

$$\tilde{\varphi} \overset{\text{def}}{=} \varphi - \varphi_0 \quad \text{where} \quad \varphi_0 = \frac{1}{8\pi}\int\varphi R\sqrt{g}d^2\boldsymbol{x}$$

The Green function of the field φ is not well defined due to zero mode issue, but formally it is the same as in flat space. Indeed, by definition $G(\boldsymbol{x},\boldsymbol{y}) = \langle\varphi(\boldsymbol{x})\varphi(\boldsymbol{y})\rangle$

$$-\rho^{-1}(\boldsymbol{x})\boldsymbol{\Delta}G(\boldsymbol{x},\boldsymbol{y}) = \rho^{-1}(\boldsymbol{x})\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}) \implies G(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{2}\log|\boldsymbol{x}-\boldsymbol{y}|^2 + \text{const} \tag{250}$$

which implies after tuning the constant term in (250) that

$$\langle\tilde{\varphi}(\boldsymbol{x})\tilde{\varphi}(\boldsymbol{y})\rangle = -\frac{1}{2}\log|\boldsymbol{x}-\boldsymbol{y}|^2 - \frac{1}{4}\log\big(\rho(\boldsymbol{x})\rho(\boldsymbol{y})\big).$$

Then similarly to (88) one has

$$e^{2\alpha\tilde{\varphi}(\boldsymbol{x})} = e^{2\alpha^2\langle\tilde{\varphi}(\boldsymbol{x})\tilde{\varphi}(\boldsymbol{x})\rangle} : e^{2\alpha\tilde{\varphi}(\boldsymbol{x})} := \big[r_0^2\rho(\boldsymbol{x})\big]^{-\alpha^2} : e^{2\alpha\tilde{\varphi}(\boldsymbol{x})} :,$$

which suggests to define primary field

$$\tilde{V}_\alpha(\boldsymbol{x}) \overset{\text{def}}{=} \big[r_0^2\rho(\boldsymbol{x})\big]^{\alpha^2} e^{2\alpha\tilde{\varphi}(\boldsymbol{x})}.$$

In Liouville CFT we define total primary field, including the zero mode

$$V_\alpha(\boldsymbol{x}) \overset{\text{def}}{=} \big[r_0^2\rho(\boldsymbol{x})\big]^{\alpha^2} e^{2\alpha\varphi_0} e^{2\alpha\tilde{\varphi}(\boldsymbol{x})} = e^{2\alpha\varphi_0}\tilde{V}_\alpha(\boldsymbol{x})$$

We are interested in multipoint correlation functions of primary fields $V_\alpha(\boldsymbol{x})$

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_N}(z_N,\bar{z}_N)\rangle \overset{\text{def}}{=} \int [\mathcal{D}\varphi] e^{-S_{\mathrm{L}}[\varphi,g]} V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_N}(z_N,\bar{z}_N). \tag{251}$$

According to the discussions above (251) depends of the metric in universal way

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_N}(z_N,\bar{z}_N)\rangle_{e^{\sigma}\delta_{\mu\nu}} = \prod_{k=1}^{N} e^{-\Delta(\alpha_k)\sigma(z_k,\bar{z}_k)}\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_N}(z_N,\bar{z}_N)\rangle_{\delta_{\mu\nu}}. \tag{252}$$

Now, assume that the measure $[\mathcal{D}\varphi]$ satisfies

$$[\mathcal{D}\varphi] = d\varphi_0[\mathcal{D}\tilde{\varphi}],$$

then the integral over the zero mode φ_0 in (251) can be easily taken, provided that $\sum\alpha_k > Q$

$$\int_{-\infty}^{\infty} e^{2\left(\sum\alpha_k - Q\right)\varphi_0 - \mu e^{2b\varphi_0}\int\tilde{V}_b\sqrt{g}d^2\boldsymbol{x}} d\varphi_0 = \frac{\Gamma(s)}{2b}\left(\mu\int\tilde{V}_b\sqrt{g}d^2\boldsymbol{x}\right)^{-s} \quad \text{where} \quad s = \frac{\sum\alpha_k - Q}{b}.$$

Then we have from (251)

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_N}(z_N,\bar{z}_N)\rangle = \frac{\Gamma(s)}{2b}\langle\tilde{V}_{\alpha_1}(z_1,\bar{z}_1)\dots\tilde{V}_{\alpha_N}(z_N,\bar{z}_N)\big(\mu\int\tilde{V}_b\sqrt{g}d^2\boldsymbol{x}\big)^{-s}\rangle_0, \tag{253}$$

where in the right hand side we have free-field average. As a simple consequence of (253) we get Knizhnik-Polyakov-Zamolodchikov formula

$$\mu b\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_N}(z_N,\bar{z}_N)\int V_b\sqrt{g}d^2\boldsymbol{x}\rangle = \left(\sum\alpha_k - Q\right)\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_N}(z_N,\bar{z}_N)\rangle,$$

so that insertion of the field $\int V_b\sqrt{g}d^2\boldsymbol{x}$ does not change considerably the correlation function.

We note that the threshold $\sum \alpha_k = Q$ corresponds to trivial singularity of the Γ function, while the free field correlation function in the r.h.s. of (253) is well defined for $s < 0$ as well. We will interpret (253) as (according to (252) we take $g_{\mu\nu} = \delta_{\mu\nu}$)

$$\mathrm{Res}\,\langle V_{\alpha_1}(\xi_1, \bar{\xi}_1) \ldots V_{\alpha_N}(\xi_N, \bar{\xi}_N)\rangle\Big|_{\sum \alpha_k + nb = Q} \stackrel{\mathrm{def}}{=} \mu^n \mathcal{G}_n(\xi_1, \ldots, \xi_N),$$

where

$$\mathcal{G}_n(\xi_1, \ldots, \xi_N) = \frac{(-1)^n}{n!} \prod_{i<j} |\xi_i - \xi_j|^{-4\alpha_i\alpha_j} \int \prod_{k,l} |z_k - \xi_l|^{-4b\alpha_l} \prod_{i<j} |z_i - z_j|^{-4b^2} d^2z_1 \ldots d^2z_n. \tag{254}$$

We note here an important point, on which we have not comment so far. In Liouville CFT V_α is a primary field with conformal dimension

$$\Delta(\alpha) = \alpha(Q - \alpha).$$

Since this is a quadratic equation, it implies that there two solutions for given conformal dimension. In our formal study of CFT we always assume that there is exactly one primary field of a given conformal dimension. If Liouville CFT is consistent with this requirement, the following relation must hold

$$V_{Q-\alpha} = R(\alpha) V_\alpha,$$

where $R(\alpha)$ is some constant called the reflection coefficient, which must obey

$$R(\alpha) R(Q - \alpha) = 1.$$

Probs:

1.

Lecture 19: Coulomb integrals I: three-point function

Consider the integral $\mathcal{G}_n(\xi_1,\dots,\xi_N)$ given by (254). From its definition it is obvious that it satisfies

$$\mathcal{G}_n(\xi_1,\dots,\xi_N)=\prod_{k=1}^{N}|c\xi_k+d|^{-4\Delta(\alpha_k)}\,\mathcal{G}_n\left(\frac{a\xi_1+b}{c\xi_1+d},\dots,\frac{a\xi_N+b}{c\xi_N+d}\right).$$

This property allows one to set any three points to 0, 1 and ∞.

Basically, up to a factor, one has to compute the integral (here $A_k=-2b\alpha_k$, $g=-b^2$)

$$\mathfrak{G}_n(A_1,\dots,A_N|\xi_1,\dots,\xi_N)\overset{\text{def}}{=}\frac{1}{n!}\int\prod_{k=1}^{n}\prod_{l=1}^{N}|z_k-\xi_l|^{2A_l}\prod_{i<j}|z_i-z_j|^{4g}d^2z_1\dots d^2z_n,$$
$$\sum_{k=1}^{N}A_k=-2-2(n-1)g. \tag{255}$$

The last condition is equivalent to screening condition $\sum\alpha_k+nb=Q$. In particular it guaranties that the integrand in non-singular at $z_k\to\infty$. The integral in (255) converges at least in the domain

$$A_k>-1,\quad 0<g<\frac{N-2}{2(n-1)} \tag{256}$$

In the following, we will always assume that the parameters A_k and g belong to (256).

Let us start with the basic integral

$$I(A,B)\overset{\text{def}}{=}\int_{\mathbb{R}^2}|z|^{2A}|z-1|^{2B}d^2\boldsymbol{x}=\lim_{\xi\to\infty}|\xi|^{-2C}\int_{\mathbb{R}^2}|z|^{2A}|z-1|^{2B}|z-\xi|^{2C}d^2\boldsymbol{x},\quad A+B+C=-2. \tag{257}$$

The integral in (257) converges provided that

$$A>-1,\quad B>-1,\quad C>-1\quad(\text{or}\quad A+B<-1)$$

In order to compute (257) we perform a counterclockwise Wick rotation and introduce new variables

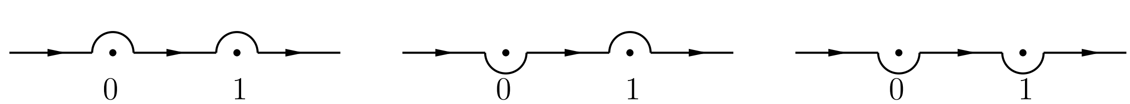
$$y=-ite^{2i\epsilon},\quad u=x+t,\quad v=u-t.$$

Here ϵ is infinitesimal positive number. Then the integral takes the form

$$I(A,B)=-\frac{i}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\big(u+i\epsilon(u-v)\big)^A\big(u-1+i\epsilon(u-v)\big)^B\big(v-i\epsilon(u-v)\big)^A\big(v-1-i\epsilon(u-v)\big)^B dudv$$

The ϵ dependence of the integral $I(A,B)$ prescribes proper deformation of the contour. It is convenient to divide the domain of integration over u into three intervals: $(-\infty,0)$, $(0,1)$ and $(1,\infty)$. Then the contour of integration $\mathcal{C}$ over variable v looks like

$u\in(-\infty,0)$ $\qquad$ $u\in(0,1)$ $\qquad$ $u\in(1,\infty)$



We see that for $u \in (-\infty, 0)$ and for $u \in (1, \infty)$ the contour $\mathcal{C}$ can be shrinked to zero, while for $u \in (0,1)$ it can be transformed to the integral

$$(1-e^{-2i\pi B})\int\limits_1^\infty v^A(v-1)^B dv = (1-e^{-2i\pi B})\int\limits_0^1 \tau^{-2-A-B}(1-\tau)^B d\tau.$$

Altogether one has

$$\begin{aligned} I(A,B) &= \int_0^1 u^A(u-1)^B du (1-e^{-2i\pi B})\int_0^1 \tau^{-2-A-B}(1-\tau)^B d\tau = \\ &= \pi\frac{\gamma(1+A)\gamma(1+B)}{\gamma(2+A+B)} = \pi\gamma(1+A)\gamma(1+B)\gamma(1+C), \quad \text{where} \quad \gamma(x) \stackrel{\text{def}}{=} \frac{\Gamma(x)}{\Gamma(1-x)}. \end{aligned}$$

We computed

$$\int |z-\xi_1|^{2A_1}|z-\xi_2|^{2A_2}d^2\boldsymbol{z} = \pi\frac{\gamma(1+A_1)\gamma(1+A_2)}{\gamma(2+A_1+A_2)}|\xi_1-\xi_2|^{2+2A_1+2A_2}. \tag{258}$$

But we will need also a multidimensional generalization of this relation

$$\int\prod_{i=1}^n\prod_{j=1}^{n+1}|z_i-\xi_j|^{2A_j}\mathcal{D}_n(\boldsymbol{z})\,d^2z_1\dots d^2z_n = \pi^n n!\frac{\gamma(1+A_1)\dots\gamma(1+A_{n+1})}{\gamma(1+n+A_1+\dots+A_{n+1})}\prod_{i<j}|\xi_i-\xi_j|^{2+2A_i+2A_j}, \tag{259}$$

where

$$\mathcal{D}_n(\boldsymbol{z}) = \prod_{i<j}|z_i-z_j|^2.$$

The identity can be proven by simple change of variables

$$\rho_j = \frac{\prod_{i=1}^n(\xi_j-z_i)}{\prod_{i\neq j}(\xi_j-\xi_i)}, \quad j=1,\dots,n+1.$$

We note that $\sum\rho_j = 1$. It can be proven by noting that

$$\sum_{j=1}^{n+1}\rho_j = \frac{1}{2\pi i}\oint_{\mathcal{C}}R(t)dt, \quad \text{where} \quad R(t) \stackrel{\text{def}}{=} \frac{\prod_{i=1}^n(t-z_i)}{\prod_{j=1}^{n+1}(t-\xi_j)} = \sum_{j=1}^{n+1}\frac{\rho_j}{t-\xi_j} + \text{reg},$$

and $\mathcal{C}$ surrounds all the points ξ_j in the counterclockwise direction. On the other side the contour $\mathcal{C}$ can be closed to infinity where the function $R(t)$ has a residue -1. The Jacobian of this transformation is

$$d^2\rho_1\dots d^2\rho_n = \text{Jac}\,d^2z_1\dots d^2z_n, \quad \text{Jac} = \frac{\mathcal{D}_n(\boldsymbol{z})}{\mathcal{D}_{n+1}(\boldsymbol{\xi})},$$

where we used Cauchy determinant formula (113). Hence we have

$$\int\prod_{i=1}^n\prod_{j=1}^{n+1}|z_i-\xi_j|^{2A_j}\mathcal{D}_n(\boldsymbol{z})\,d^2z_1\dots d^2z_n = n!\prod_{i<j}|\xi_i-\xi_j|^{2+2A_i+2A_j}\int\prod_{j=1}^n|\rho_j|^{2A_j}\left|\sum\rho_j-1\right|^{2A_{n+1}}d^2\rho_1\dots d^2\rho_n.$$

We note an additional $n!$ factor in the r.h.s. It reflects the fact that the map $(z_1,\dots,z_n)\to(\rho_1,\dots,\rho_n)$ is not bijective since ρ_j's are symmetric functions of z_i's. Thus if $(z_1,\dots,z_n)$ span $\mathbb{C}^n$, $(\rho_1,\dots,\rho_n)$ do it $n!$ times. The formula (259) follows by repetitive application of (258).

Now we consider the three-point correlation function integral

$$I_n(A,B|g)\overset{\text{def}}{=}\frac{1}{n!}\int\cdots\int\prod_{k=1}^n|z_k|^{2A}|z_k-1|^{2B}\prod_{i<j}|z_i-z_j|^{4g}d^2z_1\dots d^2z_n,$$

where $A=-2b\alpha_1$, $B=-2b\alpha_2$ and $g=-b^2$. We note that according to (259) one can represent

$$|z_i-z_j|^{4g}=\frac{1}{(n-1)!}\mathcal{D}_n(\boldsymbol{z})\frac{\gamma(ng)}{\pi^{n-1}\gamma(g)^n}\int\prod_{i,j}|t_i-z_j|^{-2+2g}\mathcal{D}_{n-1}(\boldsymbol{t})\,d^2t_1\dots d^2t_{n-1}.$$

Then the integral over $(z_1,\dots,z_n)$ can be taken with the help of (259)

$$\int\prod_{k=1}^n|z_k|^{2A}|z_k-1|^{2B}\prod_{l=1}^{n-1}|z_k-t_l|^{-2+2g}\mathcal{D}_n(\boldsymbol{z})d^2z_1\dots d^2z_n=$$
$$=\frac{\pi^n n!\,\gamma(1+A)\gamma(1+B)\gamma^{n-1}(g)}{\gamma(2+A+B+(n-1)g)}\prod_{k=1}^{n-1}|t_k|^{2(A+g)}|t_k-1|^{2(B+g)}\prod_{i<j}|t_i-t_j|^{4g-2}.$$

Thus we obtained the recursion

$$I_n(A,B|g)=\frac{\pi\gamma(ng)}{\gamma(g)}\frac{\gamma(1+A)\gamma(1+B)}{\gamma(2+A+B+(n-1)g)}I_{n-1}(A+g,B+g|g)\tag{260}$$

The reduction is solved by

$$I_n(A,B|g)=\prod_{k=1}^n\frac{\pi\gamma(kg)}{\gamma(g)}\frac{\gamma\big(1+A+(k-1)g\big)\gamma\big(1+B+(k-1)g\big)}{\gamma\big(2+A+B+(n+k-2)g\big)}.$$

We note that our basic integral identity (259) can written in a more symmetric form

$$\int\prod_{i=1}^n\prod_{j=1}^{n+2}|z_i-\xi_j|^{2A_j}\mathcal{D}_n(\boldsymbol{z})\,d^2z_1\dots d^2z_n=\pi^n n!\prod_{k=1}^{n+2}\gamma(1+A_k)\prod_{i<j}|\xi_i-\xi_j|^{2+2A_i+2A_j},\quad\sum_{k=1}^{n+2}A_k=-n-1.\tag{261}$$

In fact the condition $\sum_{k=1}^{n+2}A_k=-n-1$ is a reminiscent of a screening condition and is equivalent to the condition that ∞ is a regular point. The relation (259) follows from (261) in the limit $\xi_{n+2}\to\infty$, but it can be shown also explicitly. It is instructive to compute the three-point function using this new relation. It is in fact equally the same as was done before. We compute

$$\mathfrak{G}_n(A_1,A_2,A_3|\xi_1,\xi_2,\xi_3)=\frac{1}{n!}\int\prod_{k=1}^n|z_k-\xi_1|^{2A_1}|z_k-\xi_2|^{2A_3}|z_k-\xi_3|^{2A_3}\prod_{i<j}|z_i-z_j|^{4g}d^2z_1\dots d^2z_n,$$

where, as always, $A_1+A_2+A_3=-2-2(n-1)g$. We will use (261) only. First, we choose one point, say ξ_3, and represent

$$\prod_{k=1}^n|z_k-\xi_3|^{-2(n-1)g}\prod_{i<j}|z_i-z_j|^{-2+4g}=\frac{\gamma(ng)}{(n-1)!\pi^{n-1}\gamma(g)^n}\times$$
$$\times\int\prod_{k=1}^{n-1}|t_k-\xi_3|^{-2ng}\prod_{j=1}^n|t_k-z_j|^{-2+2g}\mathcal{D}_{n-1}(\boldsymbol{t})\,d^2t_1\dots d^2t_{n-1}.$$

Then the integral over z's can be taken (one can check that the conditions of applicability of (261) hold)

$$\frac{1}{n!}\int\prod_{k=1}^{n}|z_k-\xi_1|^{2A_1}|z_k-\xi_2|^{2A_2}|z_k-\xi_3|^{2(A_3+(n-1)g)}\prod_{j=1}^{n-1}|z_k-t_j|^{-2+2g}\mathcal{D}_n(\boldsymbol{z})\,d^2z_1\dots d^2z_n=$$
$$=\pi^n\gamma(1+A_1)\gamma(1+A_2)\gamma(1+A_3+(n-1)g)\gamma^{n-1}(g)\times$$
$$\times|\xi_1-\xi_2|^{2+2A_1+2A_2}|\xi_1-\xi_3|^{2+2A_1+2A_3+2(n-1)g}|\xi_2-\xi_3|^{2+2A_2+2A_3+2(n-1)g}\times$$
$$\times\prod_{k=1}^{n-1}|t_k-\xi_1|^{2(A_1+g)}|t_k-\xi_2|^{2(A_2+g)}|t_k-\xi_3|^{2(A_3+ng)}\prod_{i<j}|t_i-t_j|^{-2+4g}$$

Hence we have

$$\mathfrak{G}_n(A_1,A_2,A_3|\xi_1,\xi_2,\xi_3)=\frac{\pi\gamma(ng)}{\gamma(g)}\gamma(1+A_1)\gamma(1+A_2)\gamma(1+A_3+(n-1)g)\times$$
$$\times|\xi_1-\xi_2|^{2+2A_1+2A_2}|\xi_1-\xi_3|^{2+2A_1+2A_3+2(n-1)g}|\xi_2-\xi_3|^{2+2A_2+2A_3+2(n-1)g}\mathfrak{G}_{n-1}(A_1+g,A_2+g,A_3|\xi_1,\xi_2,\xi_3)$$

The coordinate dependence of three-point correlation function of primary fields is completely fixed by the conformal symmetry

$$\langle V_{\alpha_1}(\xi_1,\bar{\xi}_1)V_{\alpha_1}(\xi_2,\bar{\xi}_2)V_{\alpha_3}(\xi_3,\bar{\xi}_3)\rangle=C(\alpha_1,\alpha_2,\alpha_3)\prod_{i<j}|z_i-z_j|^{-2\Delta_{ij}},$$

where $\Delta_{12}=\Delta_1+\Delta_2-\Delta_3$. For the constant $C(\alpha_1,\alpha_2,\alpha_3)$ the recursion relation (260) takes the form

$$C\left(\alpha_1+\frac{b}{2},\alpha_2+\frac{b}{2},\alpha_3\right)=-\frac{\gamma(-b^2)}{\pi\mu}\frac{\gamma(2b\alpha_1)\gamma(2b\alpha_2)}{\gamma\big(b(\alpha_1+\alpha_2+\alpha_3-Q)\big)\gamma\big(b(\alpha_1+\alpha_2-\alpha_3)\big)}C\left(\alpha_1,\alpha_2,\alpha_3\right).\quad(262)$$

This relation holds only "on-shell", i.e. provided that $\sum\alpha_k+nb=Q$, but we pretend that it actually holds "of-shell" as well. It is more convenient to rewrite (262) as

$$C\left(\alpha_1+b,\alpha_2,\alpha_3\right)\sim C\left(\alpha_1+\frac{b}{2},\alpha_2-\frac{b}{2},\alpha_3\right)\sim C\left(\alpha_1+\frac{b}{2},\alpha_2,\alpha_3+\frac{b}{2}\right)\sim C\left(\alpha_1,\alpha_2,\alpha_3\right).$$

Collecting all factors coming from (262), we find

$$\frac{C\left(\alpha_1+b,\alpha_2,\alpha_3\right)}{C(\alpha_1,\alpha_2,\alpha_3)}=\frac{1}{\pi\mu b^2\gamma(b^2)}\frac{\gamma(2b\alpha_1)\gamma(2b\alpha_1+b^2)\gamma\big(b(\alpha_3+\alpha_2-\alpha_1)-b^2\big)}{\gamma\big(b(\alpha_1+\alpha_2-\alpha_3)\big)\gamma\big(b(\alpha_1+\alpha_3-\alpha_2)\big)\gamma\big(b(\alpha_1+\alpha_2+\alpha_3-Q)\big)}\quad(263)$$

Actually, we already saw the relation (263) (see (156)) before when we studied the fusion properties of the 4-point correlation function with $\Phi_{1,2}$, $\Phi_{2,1}$ insertions which correspond to the fields $V_{-\frac{b}{2}}$ and $V_{-\frac{1}{2b}}$ in Liouville CFT. Comparing (156) and (263) we note that that they coincide after identifying

$$\frac{C^{\alpha_1+b}_{-\frac{b}{2},\alpha_1+\frac{b}{2}}}{C^{\alpha_1}_{-\frac{b}{2},\alpha_1+\frac{b}{2}}}=\frac{1}{\pi\mu b^2\gamma(b^2)}\frac{\gamma(2b\alpha_1+b^2)}{\gamma(2b\alpha_1-1)}$$

We note that conformal invariance dictates also dual relation

$$\frac{C\left(\alpha_1+b^{-1},\alpha_2,\alpha_3\right)}{C(\alpha_1,\alpha_2,\alpha_3)}=\frac{1}{\pi\tilde{\mu}b^{-2}\gamma(b^{-2})}\frac{\gamma(2b^{-1}\alpha_1)\gamma(2b^{-1}\alpha_1+b^{-2})\gamma\big(b^{-1}(\alpha_3+\alpha_2-\alpha_1)-b^{-2}\big)}{\gamma\big(b^{-1}(\alpha_1+\alpha_2-\alpha_3)\big)\gamma\big(b^{-1}(\alpha_1+\alpha_3-\alpha_2)\big)\gamma\big(b^{-1}(\alpha_1+\alpha_2+\alpha_3-Q)\big)}. \tag{264}$$

Now, we are ready to formulate the celebrated DOZZ formula [16, 17]

$$C(\alpha_1,\alpha_2,\alpha_3)=\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha}{b}}\frac{\Upsilon'(0)\prod_{k=1}^{3}\Upsilon(2\alpha_k)}{\Upsilon(\alpha-Q)\prod_{k=1}^{3}\Upsilon(\alpha-2\alpha_k)},\quad\text{where}\quad\alpha=\alpha_1+\alpha_2+\alpha_3. \tag{265}$$

One can easily check that this formula satisfies shift relations (156) and (264) with

$$\pi\tilde{\mu}\gamma(b^{-2})=\left(\pi\mu\gamma(b^2)\right)^{\frac{1}{b^2}}.$$

In fact, shift relations define (265) up to the product of normalization factors

$$\mathcal{N}(\alpha_1)\mathcal{N}(\alpha_2)\mathcal{N}(\alpha_3).$$

Lecture 20: Coulomb integrals II: four-point function

Let us start with generalization of our integral identities (259) and (261). First of all, we give a simple intuitive meaning of the integral formula (261). Namely, suppose we have an $N-$point correlation function in Liouville CFT

$$\langle V_{\alpha_1}(\xi_1,\bar{\xi}_1)\dots V_{\alpha_N}(\xi_N,\bar{\xi}_N)\rangle=\prod_{k=1}^{N}R(\alpha_k)\langle V_{Q-\alpha_1}(\xi_1,\bar{\xi}_1)\dots V_{Q-\alpha_N}(\xi_N,\bar{\xi}_N)\rangle,$$

where we used that $V_\alpha = R(\alpha)V_{Q-\alpha}$. Suppose, that we have two screening conditions which hold simultaneously

$$\sum\alpha_k+nb=Q,\qquad \sum(Q-\alpha_k)+mb=Q.$$

In this case we have

$$\mathcal{G}_n(\xi_1,\dots,\xi_N)=\prod_{k=1}^{N}R(\alpha_k)\,\mathcal{G}_m(\xi_1,\dots,\xi_N),\tag{266}$$

where

$$\mathcal{G}_n(\xi_1,\dots,\xi_N)=\frac{(-\mu)^n}{n!}\prod_{i<j}|\xi_i-\xi_j|^{-4\alpha_i\alpha_j}\int\prod_{k,l}|z_k-\xi_l|^{-4b\alpha_l}\prod_{i<j}|z_i-z_j|^{-4b^2}d^2z_1\dots d^2z_n,$$

and $R(\alpha)$ is the reflection coefficient

$$R(\alpha)=\frac{(\pi\mu\gamma(b^2))^{\frac{(2\alpha-Q)}{b}}}{b^2}\frac{\gamma(2b\alpha-b^2)}{\gamma(2-2b^{-1}\alpha+b^{-2})}.$$

The validity of both conditions requires that

$$(m+n)b+(N-2)Q=0.$$

One can easily check that (261) corresponds to (266) for $m=0$ and[16]

$$b=\frac{i}{\sqrt{2}}\quad\text{or}\quad c=-2.$$

Actually (266) predicts the following more general identity generalizing (261)

$$\begin{aligned}\frac{1}{\pi^n n!}\int\prod_{i=1}^{n}\prod_{j=1}^{n+m+2}|u_i-\xi_j|^{2A_j}\mathcal{D}_n(\boldsymbol{u})\,d^2u_1\dots d^2u_n=\prod_{k=1}^{n+m+2}\gamma(1+A_k)\prod_{i<j}|\xi_i-\xi_j|^{2+2A_i+2A_j}\times\\ \times\frac{1}{\pi^m m!}\int\prod_{i=1}^{m}\prod_{j=1}^{n+m+2}|v_i-\xi_j|^{-2-2A_j}\mathcal{D}_m(\boldsymbol{v})\,d^2v_1\dots d^2v_n,\qquad\sum_{k=1}^{n+m+2}A_k=-n-1.\end{aligned}\tag{267}$$

[16]In principle, it is not clear why the case of $b^2=-\frac{1}{2}$ is so special and might be more integral identities generalizing (261). This is unknown so far. There is interesting case corresponding to unitary minimal models, or $b^2=-\frac{r}{r+1}$.

or in simplified form (with $\xi_{n+m+2} \to \infty$)

$$\frac{1}{\pi^n n!}\int \prod_{i=1}^{n}\prod_{j=1}^{n+m+1}|u_i-\xi_j|^{2A_j}\mathcal{D}_n(\boldsymbol{u})\,d^2u_1\dots d^2u_n = \frac{\prod_{k=1}^{n+m+2}\gamma(1+A_k)}{\gamma(1+n+\sum A_j)}\prod_{i<j}|\xi_i-\xi_j|^{2+2A_i+2A_j}\times$$

$$\times\frac{1}{\pi^m m!}\int \prod_{i=1}^{m}\prod_{j=1}^{n+m+1}|v_i-\xi_j|^{-2-2A_j}\mathcal{D}_m(\boldsymbol{v})\,d^2v_1\dots d^2v_n.$$

Let us prove (267). Consider the integral on the left in (267) and introduce

$$\rho_j = \frac{\prod_{i=1}^{n}(\xi_j-u_i)}{\prod_{i\neq j}(\xi_j-\xi_i)},\quad j=1,\dots,n+m+2.$$

Among $n+m+2$ variables ρ_j there are $m+2$ linear relations. They can be found by noticing that the rational function

$$U(t)=\frac{\prod_{i=1}^{n}(t-u_i)}{\prod_{j=1}^{n+m+2}(t-\xi_j)}=\sum_{j=1}^{n+m+2}\frac{\rho_j}{t-\xi_j}\quad\text{behaves as}\quad U(t)=\frac{1}{t^{m+2}}\quad\text{at}\quad t\to\infty.$$

This behavior implies that the variables ρ_j obey

$$\sum_{j=1}^{n+m+2}\rho_j\xi_j^k=0\quad\text{for}\quad k=0,\dots,m,\qquad \sum_{j=1}^{n+m+2}\rho_j\xi_j^{m+1}=1.$$

The measure has the form

$$\mathcal{D}_n(\boldsymbol{u})\,d^2u_1\dots d^2u_n = n!\prod_{i<j=1}^{n+m+2}|\xi_i-\xi_j|^2\left(\prod_{k=0}^{m}\delta^{(2)}\Big(\sum\rho_j\xi_j^k\Big)\right)\delta^{(2)}\Big(\sum\rho_j\xi_j^{m+1}-1\Big)\,d^2\rho_1\dots d^2\rho_{n+m+2}.$$

Then up to a factor

$$\pi^{-n}\prod_{i<j}|\xi_i-\xi_j|^{2+2A_i+2A_j}\tag{268}$$

the integral in the l.h.s. of (267) has the form

$$\int\prod_{j=1}^{n+m+2}|\rho_j|^{2A_j}\left(\prod_{k=0}^{m}\delta^{(2)}\Big(\sum\rho_j\xi_j^k\Big)\right)\delta^{(2)}\Big(\sum\rho_j\xi_j^{m+1}-1\Big)\,d^2\rho_1\dots d^2\rho_{n+m+2}=$$

$$=\frac{1}{(2\pi)^{2(m+2)}}\int\prod_{j=1}^{n+m+2}|\rho_j|^{2A_j}\left(\prod_{k=0}^{m+1}e^{\frac{i\left(p_k\sum\rho_j\xi_j^k+\text{c.c.}\right)}{2}}\right)e^{-\frac{i(p_{m+1}+\bar p_{m+1})}{2}}\,d^2\rho_1\dots d^2\rho_{n+m+2}\,d^2p_0\dots d^2p_{m+1}$$

Integrals over ρ_j can be taken using

$$\int|\boldsymbol{x}|^{2A}e^{\frac{i\boldsymbol{p}\cdot\boldsymbol{x}}{2}}\,d^2\boldsymbol{x}=\pi\gamma(1+A)\,|\boldsymbol{p}|^{-2-2A}.$$

We find

$$(257)=\frac{\pi^{m+n+2}}{(2\pi)^{2(m+2)}}\prod_{j=1}^{n+m+2}\gamma(1+A_j)\int\prod_{j=1}^{n+m+2}\left|2\sum_{k=0}^{m+1}p_k\xi_j^k\right|^{-2-2A_j}e^{-\frac{i(p_{m+1}+\bar p_{m+1})}{2}}\,d^2p_0\dots d^2p_{m+1}\tag{269}$$

It is convenient to rescale

$$p_k \to p_k p_m \quad \text{for} \quad k \neq m.$$

Then we find that the integral over $d^2 p_m$ is just the δ function with the support on $p_{m+1} = 0$. It implies

$$(269) = \pi^{n-m} \prod_{j=1}^{n+m+2} \gamma(1+A_j) \int \prod_{j=1}^{n+m+2} \Big| \sum_{k=0}^{m-1} p_k \xi_j^k + \xi_j^m \Big|^{-2-2A_j} d^2 p_0 \dots d^2 p_{m-1} =$$

$$= \frac{\pi^{n-m}}{m!} \prod_{j=1}^{n+m+2} \gamma(1+A_j) \int \prod_{i=1}^{m} \prod_{j=1}^{n+m+2} |v_i - \xi_j|^{-2-2A_j} \mathcal{D}_m(\boldsymbol{v})\, d^2 v_1 \dots d^2 v_m, \quad (270)$$

where

$$x^m + \sum_{k=0}^{m-1} p_k x^k \stackrel{\text{def}}{=} \prod_{i=1}^{m} (x - v_i).$$

Remembering the factor (268) we've left on the road, one finds that (270) implies (267).

Now we study the four-point correlation function with one $\Phi_{1,2}$ degenerate field

$$\langle V_{-\frac{b}{2}}(z,\bar{z}) V_{\alpha_1}(0) V_{\alpha_2}(\infty) V_{\alpha_3}(1) \rangle.$$

On-shell this four-point function is proportional to the Coulomb integral

$$\mathfrak{J}^{(n)}(A,B|z) = \int \prod_{k=1}^{n} |t_k|^{2A} |t_k - 1|^{2B} |t_k - z|^{-2g} \prod_{i<j} |t_i - t_j|^{4g} d^2 t_1 \dots d^2 t_n,$$

where

$$A = -2b\alpha_1, \quad B = -2b\alpha_2, \quad \alpha_1 + \alpha_2 + \alpha_3 - \frac{b}{2} + nb = Q.$$

As usual, we represent (for simplicity we will skip all factors)

$$|t_i - t_j|^{4g} \sim \mathcal{D}_n(\boldsymbol{t}) \int \prod_{i,j} |\tau_i - t_j|^{-2+2g} \mathcal{D}_{n-1}(\boldsymbol{\tau})\, d^2 \tau_1 \dots d^2 \tau_{n-1}.$$

Compared to the 3-point case, the integral over t can not be taken, but rather gives one-dimensional integral (we note that the interaction of τ_j with z does not appear)

$$\int \prod_{k=1}^{n} |t_k|^{2A} |t_k - 1|^{2B} |t_k - z|^{-2g} \prod_{j} |t_k - \tau_j|^{-2+2g} \mathcal{D}_n(\tau) d^2 t_1 \dots d^2 t_n \sim$$

$$\sim |z|^{2(1+A-g)} |z-1|^{2(1+B-g)} \prod_{j} |\tau_j|^{2(A+g)} |\tau_j - 1|^{2(B+g)} \mathcal{D}^{2g-1}(\boldsymbol{\tau}) \times$$

$$\times \int |\xi|^{-2-2A} |\xi - 1|^{-2-2B} |\xi - z|^{-2+2g} \prod_{j} |\xi - \tau_j|^{-2g} d^2 \xi. \quad (271)$$

From (271) we obtain a recursion

$$\mathfrak{J}^{(n)}(A,B|z) \sim |z|^{2(1+A-g)} |z-1|^{2(1+B-g)} \int |\xi|^{-2-2A} |\xi-1|^{-2-2B} |\xi-z|^{-2+2g} \mathfrak{J}^{(n-1)}(A+g, B+g|\xi) d^2 \xi. \quad (272)$$

Repeating (272) again

$$\mathfrak{J}^{(n)}(A,B|z)\sim|z|^{2(1+A-g)}|z-1|^{2(1+B-g)}\times$$
$$\times\int|\xi-z|^{-2+2g}|\nu|^{-2-2A-2g}|\nu-1|^{-2-2B-2g}|\nu-\xi|^{-2+2g}\mathfrak{J}^{(n-2)}(A+2g,B+2g|\nu)d^2\xi d^2\nu,$$

and taking the integral over ξ, we get

$$\mathfrak{J}^{(n)}(A,B|z)\sim|z|^{2(1+A-g)}|z-1|^{2(1+B-g)}\int|\nu|^{-2-2A-2g}|\nu-1|^{-2-2B-2g}|\nu-z|^{-2+4g}\mathfrak{J}^{(n-2)}(A+2g,B+2g|\nu)d^2\nu.$$

Repeating this arbitrary number of times, we get

$$\mathfrak{J}^{(n)}(A,B|z)\sim|z|^{2(1+A-g)}|z-1|^{2(1+B-g)}\times$$
$$\times\int|\nu|^{-2-2A-2(k-1)g}|\nu-1|^{-2-2B-2(k-1)g}|\nu-z|^{-2+2kg}\mathfrak{J}^{(n-k)}(A+kg,B+kg|\nu)d^2\nu.$$

In particular, for $k=n$ one has

$$\mathfrak{J}^{(n)}(A,B|z)\sim|z|^{2(1+A-g)}|z-1|^{2(1+B-g)}\int|\nu|^{-2-2A-2(n-1)g}|\nu-1|^{-2-2B-2(n-1)g}|\nu-z|^{-2+2ng}d^2\nu=$$
$$=\int|\xi|^{2A+2(n-1)g}|\xi-1|^{2B+2(n-1)g}|\xi-z|^{-2ng}d^2\xi. \quad (273)$$

In the last line we used (259). We see that the number of screening fields in this formula is just a parameter. As usual, we suppose that (273) holds “off-shell”, i.e.

$$\langle V_{-\frac{b}{2}}(z,\bar{z})V_{\alpha_1}(0)V_{\alpha_2}(\infty)V_{\alpha_3}(1)\rangle=\Omega(\alpha_1,\alpha_2,\alpha_3)\,|z|^{2b\alpha_1}|z-1|^{2b\alpha_3}\int|\xi|^{2\alpha}|\xi-1|^{2\beta}|\xi-z|^{2\gamma}d^2\xi, \quad (274)$$

with

$$\alpha=b\left(\alpha_2+\alpha_3-\alpha_1-Q+\frac{b}{2}\right),\quad \beta=b\left(\alpha_1+\alpha_2-\alpha_3-Q+\frac{b}{2}\right),\quad \gamma=b\left(Q+\frac{b}{2}-\alpha_1-\alpha_2-\alpha_3\right).$$

The constant $\Omega(\alpha_1,\alpha_2,\alpha_3)$ can be easily found by considering the limit $z\to0$

$$\Omega(\alpha_1,\alpha_2,\alpha_3)=(-\pi\mu)\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha-b/2}{b}}\times$$
$$\times\frac{\Upsilon'\left(-\frac{b}{2}\right)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon\left(\alpha_1+\alpha_2+\alpha_3-Q-\frac{b}{2}\right)\Upsilon\left(\alpha_1+\alpha_2-\alpha_3-\frac{b}{2}\right)\Upsilon\left(\alpha_1+\alpha_3-\alpha_2-\frac{b}{2}\right)\Upsilon\left(\alpha_2+\alpha_3-\alpha_1-\frac{b}{2}\right)}.$$

The expression (274) has to be compared to the one we've obtained before (here A and B are different from those above)

$$\langle V_{-\frac{b}{2}}(z,\bar{z})V_{\alpha_1}(0)V_{\alpha_2}(\infty)V_{\alpha_3}(1)\rangle=C_{-\frac{b}{2},\alpha_1}^{\alpha_1-\frac{b}{2}}C\left(\alpha_1-\frac{b}{2},\alpha,\alpha_3\right)|\mathcal{F}_+(z)|^2+C_{-\frac{b}{2},\alpha_1}^{\alpha_1+\frac{b}{2}}C\left(\alpha_1+\frac{b}{2},\alpha,\alpha_3\right)|\mathcal{F}_-(z)|^2, \quad (275)$$

where

$$\mathcal{F}_+(z)=z^{b\alpha_1}(1-z)^{b\alpha_3}F(A,B,C|z),$$
$$\mathcal{F}_-(z)=z^{b(Q-\alpha_1)}(1-z)^{b\alpha_3}F(1+B-C,1+A-C,2-C|z),$$

and

$$A = \frac{1}{2} + b(\alpha_1 + \alpha_3 - Q) + b\left(\alpha_2 - \frac{Q}{2}\right), \quad B = \frac{1}{2} + b(\alpha_1 + \alpha_3 - Q) - b\left(\alpha_2 - \frac{Q}{2}\right),$$
$$C = 1 + b\,(2\alpha_1 - Q)\,.$$

The function $F(A, B, C|z)$ in (275) is the hypergeometric function. One of possible integral representations for $F(A, B, C|z)$ involves contour integral from 0 to 1

$$F(A, B, C|z) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-1}(1-zt)^{-A} dt$$

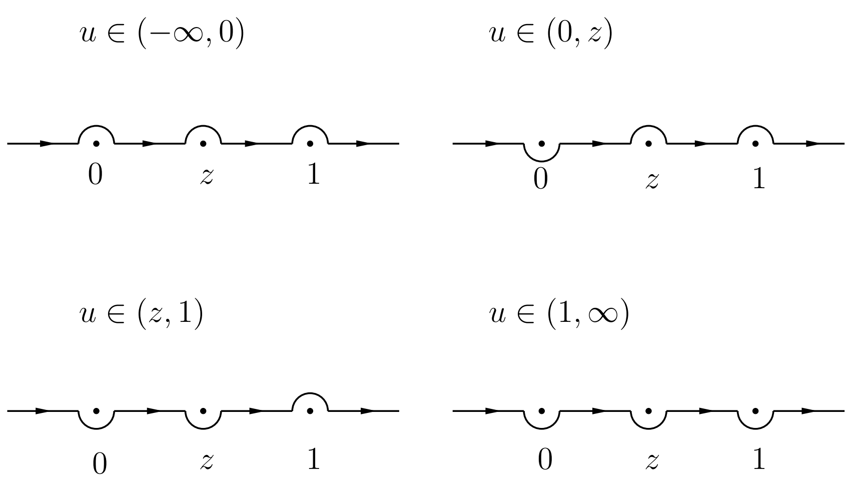
In order to relate these two representation, we perform Wick rotation in the integral

$$I(\alpha, \beta, \gamma|z) = \int |\xi|^{2\alpha}|\xi - 1|^{2\beta}|\xi - z|^{2\gamma} d^2\xi \tag{276}$$

and obtain

$$I(\alpha, \beta, \gamma|z) = -\frac{i}{2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \big((u + i\epsilon(u-v))(v - i\epsilon(u-v))\big)^{\alpha} \big((u - 1 + i\epsilon(u-v))(v - 1 - i\epsilon(u-v))\big)^{\beta} \cdot$$
$$\cdot \big((u - z + i\epsilon(u-v))(v - \bar{z} - i\epsilon(u-v))\big)^{\gamma} du dv \tag{277}$$

Now, suppose for simplicity that z is a real number $z \in (0, 1)$. We divide the domain of integration over u into four intervals: $(-\infty, 0)$, $(0, z)$, $(z, 1)$ and $(1, \infty)$. Then the contour of integration $\mathcal{C}$ over variable v looks like

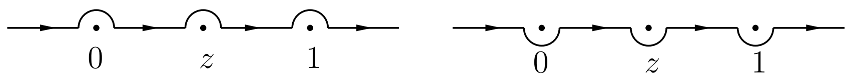


We see that only the domains $(0, z)$ and $(z, 1)$ contribute.

It is convenient to define

$$I_1(z) = \int_{-\infty}^{0} f(x|z)dx, \quad I_2(z) = \int_0^z f(x|z)dx, \quad I_3 = \int_z^1 f(x|z)dx, \quad I_4 = \int_1^{\infty} f(x|z)dx,$$
$$f(x|z) = |x-1|^{\alpha}|x-1|^{\beta}|x-z|^{\gamma}$$

There are two linear dependences between the functions $I_k(z)$, which follow from contractibility of the contours



We have

$$I_1(z)+e^{-i\pi\alpha}I_2(z)+e^{-i\pi(\alpha+\gamma)}I_3(z)+e^{-i\pi(\alpha+\beta+\gamma)}I_4(z)=0,$$
$$I_1(z)+e^{i\pi\alpha}I_2(z)+e^{i\pi(\alpha+\gamma)}I_3(z)+e^{i\pi(\alpha+\beta+\gamma)}I_4(z)=0.$$

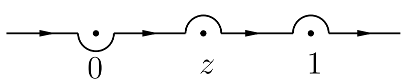
It is convenient to take as independent functions (compare to (275))

$$I_2(z)=z^{b(Q-2\alpha_1)}F(1+B-C,1+A-C,2-C|z)\quad\text{and}\quad I_4(z)=F(A,B,C|z).$$

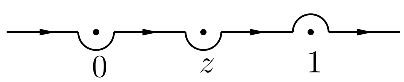
This functions (conformal blocks) have diagonal monodromy around 0. Then $I_1(z)$ and $I_3(z)$ are expressed as

$$\begin{aligned}I_1(z)&=-\frac{\sin\pi\gamma}{\sin\pi(\alpha+\gamma)}I_2(z)+\frac{\sin\pi\beta}{\sin\pi(\alpha+\gamma)}I_4(z),\\ I_3(z)&=-\frac{\sin\pi\alpha}{\sin\pi(\alpha+\gamma)}I_2(z)-\frac{\sin\pi(\alpha+\beta+\gamma)}{\sin\pi(\alpha+\gamma)}I_4(z).\end{aligned}\tag{278}$$

For non-contractible contours in (277) we have



$$=e^{i\pi(\alpha+\beta+\gamma)}\left(I_1(z)+e^{i\pi\alpha}I_2(z)+e^{i\pi(\alpha-\gamma)}I_3(z)+e^{i\pi(\alpha-\gamma-\beta)}I_4(z)\right),$$



$$=e^{i\pi(\alpha+\beta+\gamma)}\left(I_1(z)+e^{i\pi\alpha}I_2(z)+e^{i\pi(\alpha+\gamma)}I_3(z)+e^{i\pi(\alpha+\gamma-\beta)}I_4(z)\right),$$

where we fixed the phase of the integrand over v to be $e^{i\pi(\alpha+\beta+\gamma)}$ in the interval $(-\infty,0)$. The phase over u should be $e^{-i\pi(\alpha+\beta+\gamma)}$ in the same interval, since our two-dimensional integral $I(\alpha,\beta,\gamma|z)$ is real. Then we have

$$\begin{aligned}I(\alpha,\beta,\gamma|z)=e^{-i\pi\alpha}I_2(z)&\left(I_1(z)+e^{i\pi\alpha}I_2(z)+e^{i\pi(\alpha-\gamma)}I_3(z)+e^{i\pi(\alpha-\gamma-\beta)}I_4(z)\right)+\\&+e^{-i\pi(\alpha+\gamma)}\left(I_1(z)+e^{i\pi\alpha}I_2(z)+e^{i\pi(\alpha+\gamma)}I_3(z)+e^{i\pi(\alpha+\gamma-\beta)}I_4(z)\right),\end{aligned}\tag{279}$$

where the phases $e^{-i\pi\alpha}$ and $e^{-i\pi(\alpha+\gamma)}$ are easily read off (277). Substituting (278) into (279), one finds

$$I(\alpha,\beta,\gamma|z)=\frac{\sin\pi\alpha\sin\pi\gamma}{\sin\pi(\alpha+\gamma)}\left|I_2(z)\right|^2+\frac{\sin\pi\beta\sin\pi(\alpha+\beta+\gamma)}{\sin\pi(\alpha+\gamma)}\left|I_4(z)\right|^2$$

We note that this expression is real and single-valued at $z=0$. Equivalently, it can be rewritten in terms of functions

$$I_1(z)=F(A,B,1+A+B-C|1-z),\quad I_3(z)=(1-z)^{b(Q-2\alpha_3)}F(C-A,C-B,1C-A-B|1-z)$$

with diagonal monodromy around $z=1$:

$$I(\alpha,\beta,\gamma|z)=\frac{\sin\pi\alpha\sin\pi(\alpha+\beta+\gamma)}{\sin\pi(\alpha+\gamma)}\left|I_1(z)\right|^2+\frac{\sin\pi\beta\sin\pi\gamma}{\sin\pi(\alpha+\gamma)}\left|I_3(z)\right|^2,$$

and hence this integral is single-valued at $z=1$ (and automatically at $z=\infty$).

The fact that the integral $I(\alpha,\beta,\gamma|z)$ is single-valued on a sphere with three punctures was clear from the beginning, since its form (276) is ultimately single-valued. From the other side, we have obtained the same result from the "bootstrap" equations

$$\frac{C^{\alpha_1-\frac{b}{2}}_{-\frac{b}{2},\alpha_1}}{C^{\alpha_1+\frac{b}{2}}_{-\frac{b}{2},\alpha_1}}\frac{C\Big(\alpha_1-\frac{b}{2},\alpha,\alpha_3\Big)}{C\Big(\alpha_1+\frac{b}{2},\alpha,\alpha_3\Big)}=\frac{\gamma(A)\gamma(B)\gamma(C-A)\gamma(C-B)}{\gamma(C)\gamma(C-1)}.$$

Thus we confirmed both methods.

Lecture 21: Coulomb integrals III: screening fields

We have seen that Coulomb integrals for correlation functions can be decomposed into sum of products of contour integrals. This expansion is similar to the expansion into the sum over the conformal blocks. Thus if we are interested in holomorphic objects, we can consider holomorphic bosonic field[17]

$$\varphi(z)\varphi(w) = -\frac{1}{2}\log(z-w) + \dots$$

Let us rephrase what we have done in last lectures, but ih holomorphic language. We are interested in multipoint correlation functions of holomorphic vertex operators $V_\alpha =: e^{2\alpha\varphi(z)}:$

$$\langle V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_n)\rangle\Big|_{\sum\alpha_k=Q} = e^{i\omega}\prod_{i<j}(z_i-z_j)^{-2\alpha_i\alpha_j}.$$

Here ω is some phase which is related to kinematics. In particular, if $|z_1|<|z_2|<\dots$ then $\omega=1$. The balance of the charge can be changed by the *screening charges*

$$\mathcal{V}_\pm \stackrel{\text{def}}{=} \frac{1}{2\pi i}\oint e^{2b^{\pm 1}\varphi(z)}dz. \tag{280}$$

The vertex operators $e^{2b^{\pm1}\varphi(z)}$ are special ones, because they have conformal dimension 1 under the improved stress-energy tensor

$$T = -(\partial\varphi)^2 + Q\partial^2\varphi.$$

We have

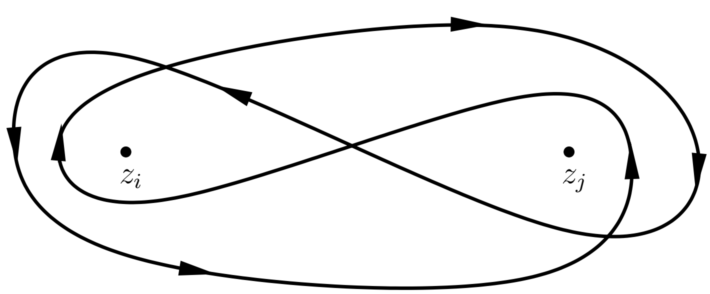
$$T(\xi)e^{2b^{\pm1}\varphi(z)} = \frac{e^{2b^{\pm1}\varphi(z)}}{(\xi-z)^2} + \frac{\partial e^{2b^{\pm1}\varphi(z)}}{\xi-z} + \dots = \frac{\partial}{\partial z}\left(\frac{e^{2b^{\pm1}\varphi(z)}}{\xi-z}\right) + \dots \implies \oint_{\mathcal{C}_\xi} e^{2b^{\pm1}\varphi(z)}T(\xi)dz = 0.$$

The meaning of this formula is that the stress-energy tensor does not feel the screening charges. They role is to change the charge condition.

The problem with the definition (280) is that the contour of integration should be closed. It is not really trivial task to find such contours. Consider the following example

$$\langle V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_n)\mathcal{V}_+\rangle\Big|_{\sum\alpha_k+b=Q} = \frac{e^{i\omega}}{2\pi i}\oint\prod_{i<j}(z_i-z_j)^{-2\alpha_i\alpha_j}\prod(z-z_i)^{-2b\alpha_i}dz. \tag{281}$$

The integrand in (281) is a mutli-valued function. The basis in closed contours consists of Poghammer contours. Namely, fix two points z_i and z_j. Then the Poghammer contour $P_{i,j}$ has the form



[17]Note that here we use different normalization of the bosonic field compared to the one used before.

Lecture 22: Classical CFT I: correlation functions

Classical regime of CFT corresponds to the limit $c \to \infty$. Though the details of this limit may depend on a CFT under consideration, kinematical quantities, like the conformal blocks, admit universal behavior. We consider $c \to \infty$ limit of Liouville CFT

$$S[\varphi, \hat{g}_{\mu\nu}] = \frac{1}{4\pi}\int \sqrt{\hat{g}}\left(\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\hat{R}\varphi + 4\pi\mu e^{2b\varphi}\right)d^2\boldsymbol{x}, \quad Q = b + \frac{1}{b}.$$

Since $c = 1+6Q^2$, the limit $c \to \infty$ can be archived at either $b \to 0$ or $b \to \infty$. We can rescale $\varphi = b^{-1}\sigma$, $\mu = \pi^{-1}b^{-2}\Lambda$. Then in the limit $\hbar = b^2 \to 0$ the path integral is dominated by the stationary points of the classical action defined by

$$S[\varphi, \hat{g}_{\mu\nu}] = \frac{1}{b^2}\left(S_{\text{cl}}[\sigma, \hat{g}_{\mu\nu}] + O(b^2)\right), \quad S_{\text{cl}}[\sigma, \hat{g}_{\mu\nu}] = \frac{1}{4\pi}\int \sqrt{\hat{g}}\left(\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \hat{R}\sigma + 4\Lambda e^{2\sigma}\right)d^2\boldsymbol{x}.$$

The Euler-Lagrange equation for the classical action $S_{\text{cl}}[\sigma, \hat{g}_{\mu\nu}]$ has the form

$$-\Delta_{\hat{g}}\sigma(\boldsymbol{x}) + \frac{1}{2}\hat{R}(\boldsymbol{x}) + 4\Lambda e^{2\sigma(\boldsymbol{x})} = 0. \tag{282}$$

Using $\sqrt{g}R = \sqrt{\hat{g}}(\hat{R} - 2\Delta_{\hat{g}}\sigma)$, where $g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$, one has

$$\frac{1}{2}R(\boldsymbol{x}) + 4\Lambda = 0. \tag{283}$$

So, the Liouville equation is just the statement that the metric $g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$ has a constant curvature.

Let us take $\hat{g}_{\mu\nu} = \delta_{\mu\nu}$. It can be done at least locally. In complex coordinates (282) reads

$$\partial\bar{\partial}\sigma = \Lambda e^{2\sigma}. \tag{284}$$

This equation, known as Liouville equation, being non-linear, is in fact exactly solvable. First, one notice that

$$t = -(\partial\sigma)^2 + \partial^2\sigma \quad \text{and} \quad \bar{t} = -(\bar{\partial}\sigma)^2 + \bar{\partial}^2\sigma, \tag{285}$$

are holomorphic and antiholomorphic functions respectively, i.e. $t = t(z)$, $\bar{t} = \bar{t}(\bar{z})$ provided that σ is on-shell. In fact, $(t, \bar{t})$ are nothing else but the components of the stress-energy tensor. This fact is a manifestation of the conformal invariance of Liouville equation

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z}), \quad \sigma \to \sigma - \frac{1}{2}\log\left|\frac{dw}{dz}\right|^2$$

Important role plays $e^{-\sigma}$ which is the classical counterpart of the degenerate field $V_{-\frac{1}{2b}}$ ($\Phi_{1,2}$ field). It satisfies two differential equations

$$\begin{aligned}(\partial^2 + t(z))e^{-\sigma} &= 0,\\ (\bar{\partial}^2 + \bar{t}(\bar{z}))e^{-\sigma} &= 0.\end{aligned} \tag{286}$$

In a similar way one can show that $e^{-2\sigma}$, a classical version of $\Phi_{1,3}$, satisfies

$$\begin{aligned}\left(\partial^3 + 4t(z)\partial + 2t'(z)\right)e^{-2\sigma} &= 0,\\ \left(\bar{\partial}^3 + 4\bar{t}(\bar{z})\bar{\partial} + 2\bar{t}'(\bar{z})\right)e^{-2\sigma} &= 0,\end{aligned}$$

etc.

Equations (285) and (286) can be used to solve Liouville equation. Namely, let $\Psi(z,\bar{z})$ be a solution to the system

$$\begin{aligned}(\partial^2+t(z))\Psi(z,\bar{z})=0,\\ (\bar{\partial}^2+\bar{t}(\bar{z}))\Psi(z,\bar{z})=0,\end{aligned}\tag{287}$$

where $t(z)$ and $\bar{t}(\bar{z})$ are some holomorphic and antiholomorphic functions complex conjugated to each other (this is ansested from their definition (285) through the Lioiuville field σ). Then, let $\psi_1(z)$ and $\psi_2(z)$ be two linearly independent solutions to the holomorphic equation (287), normalized by the condition

$$\mathcal{W}(\psi_1,\psi_2)=\psi_1\partial\psi_2-\psi_2\partial\psi_1=\psi_1^2\partial\left(\frac{\psi_2}{\psi_1}\right)=1,\tag{288}$$

and let $\bar{\psi}_1(\bar{z})$, $\bar{\psi}_2(\bar{z})$ be their complex conjugates. Then one can check that the combination

$$\sigma(z,\bar{z})=-\log\left(\mathbf{\Lambda}_{ij}\psi_i(z)\bar{\psi}_j(\bar{z})\right)\tag{289}$$

is a solution to Liouville equation (284) provided that

$$\det\mathbf{\Lambda}=-\Lambda.\tag{290}$$

Indeed, let $\bar{\chi}_i=\mathbf{\Lambda}_{ij}\bar{\psi}_j$, then we have

$$\bar{\partial}\partial\left(-\log\left(\psi_1\bar{\chi}_1+\psi_2\bar{\chi}_2\right)\right)=\bar{\partial}\partial\left(-\log\left(1+\frac{\psi_2}{\psi_1}\frac{\bar{\chi}_2}{\bar{\chi}_1}\right)\right)=\bar{\partial}\left(\frac{-\frac{1}{\psi_1^2}\frac{\bar{\chi}_2}{\bar{\chi}_1}}{1+\frac{\psi_2}{\psi_1}\frac{\bar{\chi}_2}{\bar{\chi}_1}}\right)=-\frac{\det\mathbf{\Lambda}}{\left(\psi_1\bar{\chi}_1+\psi_2\bar{\chi}_2\right)^2},$$

where we used

$$\partial\left(\frac{\psi_2}{\psi_1}\right)=\frac{\psi_1\partial\psi_2-\psi_2\partial\psi_1}{\psi_1^2}=\frac{1}{\psi_1^2},\quad \bar{\partial}\left(\frac{\bar{\chi}_2}{\bar{\chi}_1}\right)=\frac{\bar{\chi}_1\bar{\partial}\bar{\chi}_2-\bar{\chi}_2\bar{\partial}\bar{\chi}_1}{\bar{\chi}_1^2}=\frac{\det\mathbf{\Lambda}}{\bar{\chi}_1^2}.$$

Formally (289) provides a solution, but one has to impose further constraints. First of all $\sigma(z,\bar{z})$ should be real. It can be archived by demanding the $\mathbf{\Lambda}$ is a Hermitian matrix $\mathbf{\Lambda}=\mathbf{\Lambda}^+$. More important thing is to ensure that $\sigma(z,\bar{z})$ is single valued. As we will see below this is a complicated task to perform. The basis of solutions $(\psi_1(z),\psi_2(z))$ normalized by the condition (288) is not a distinguished one. It can be rotated by some $SL(2,\mathbb{C})$ transformation to the basis in which the matrix $\mathbf{\Lambda}$ takes a canonical form. In the condition (290) only the sign of Λ matters[18], the rest can be absorbed by the constant shift of $\sigma(z,\bar{z})$. Since the sign of Λ corresponds to the sign of the curvature according to (283), one distinguishes two cases

$$\text{Positive curvature:}\qquad \mathbf{\Lambda}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$\text{Negative curvature:}\qquad \mathbf{\Lambda}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Now, suppose that there is a non-contractible loop $\mathcal{C}$ such that the solution acquires a monordomy around it (again we assume that $t(z)$ is a single-valued function)

$$\psi_i\to\boldsymbol{M}_{ij}\psi_j,\qquad \bar{\psi}_i\to\boldsymbol{M}^*_{ij}\bar{\psi}_j.$$

[18]We note that in our case $\Lambda>0$ as required by the path integral to converge at large φ.

As we demand that the solution (289) is single-valued, we must have

$$\boldsymbol{M}^{+}\boldsymbol{\Lambda}\boldsymbol{M}=\boldsymbol{\Lambda}.$$

Therefore the monodromy matrix $\boldsymbol{M}$ should belong to the real subgroup of $SL(2,\mathbb{C})$: either $SU(2)$ for positive curvature, or $SU(1,1)$ for negative one. If there are more non-contractible loops, this condition should hold for all of them.

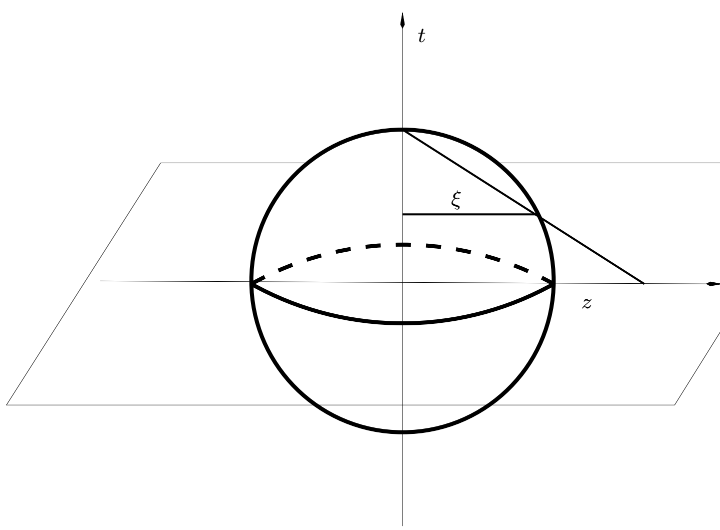
Now, let us study elementary solution to Liouville equation corresponding to the case

$$t=0.$$

For positive curvature one has a solution

$$\sigma(z,\bar{z})=-\log(1+|z|^2)\qquad\Longrightarrow\qquad ds^2=e^{2\sigma}dzd\bar{z}=\frac{4dzd\bar{z}}{(1+|z|^2)^2},$$

which is the Fubini-Study metric on a round sphere . Namely, we take a sphere $t^2+\xi\bar{\xi}=1$ embedded in flat space with Euclidean metric $ds^2=dt^2+d\xi d\bar{\xi}$ and introduce stereographic coordinate z as shown on the picture



that is

$$z=\frac{\xi}{1-t}.$$

For negative curvature one has

$$\sigma(z,\bar{z})=-\log(1-|z|^2)\qquad\Longrightarrow\qquad ds^2=e^{2\sigma}dzd\bar{z}=\frac{dzd\bar{z}}{(1-|z|^2)^2},\tag{291}$$

which is the metric of Poincare disk (Euclidean AdS_2). It can be realized as an embedding of a two-fold hyperboloid $\xi\bar{\xi}=t^2-1$ into Minkowski metric $ds^2=-dt^2+d\xi d\bar{\xi}$.

Now, let us turn to the computation of multipoint correlation functions in Liouville QFT

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)\dots V_{\alpha_n}(z_n,\bar{z}_n)\rangle=\int e^{-S}\prod_{k=1}^{n}e^{2\alpha_k\varphi(z_k,\bar{z}_k)}[\mathcal{D}\varphi].$$

We consider the case of all "heavy" operators, i.e.

$$\alpha_k = \frac{\eta_k}{b}.$$

These insertions are of the same order as the action and hence they modify the equations of motion

$$\sqrt{\hat{g}}\left(-\Delta_{\hat{g}}\sigma(\boldsymbol{x}) + \frac{1}{2}\hat{R}(\boldsymbol{x}) + 4\Lambda e^{2\sigma(\boldsymbol{x})}\right) = 4\pi\sum_{k=1}^{n}\eta_k\delta^{(2)}(z-z_k),$$

or in terms of the metric $g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$

$$\sqrt{g}\left(\frac{1}{2}R(\boldsymbol{x}) + 4\Lambda\right) = 4\pi\sum_{k=1}^{n}\eta_k\delta^{(2)}(z-z_k). \tag{292}$$

Equation (292) describes a metric of constant negative curvature -8Λ with prescribed singularities at the points $(z_1,\ldots,z_n)$. Integrating this equation over the sphere and using Gauss-Bonet theorem we get an equality

$$4\Lambda\int\sqrt{g}d^2\boldsymbol{x} = 4\pi\left(\sum_{k=1}^{n}\eta_k - 1\right) \overset{\Lambda>0}{\Longrightarrow} \sum_{k=1}^{n}\eta_k > 1. \tag{293}$$

This inequality is related with the convergence condition for the path integral at large negative values of φ.

Consider the case of flat background metric. One has

$$\partial\bar{\partial}\sigma = \Lambda e^{2\sigma} - \pi\sum_{k=1}^{n}\eta_k\delta^{(2)}(z-z_k),$$

or equivalently using $\partial\bar{\partial}\log|z|^2 = \pi\delta^{(2)}(z)$

$$\begin{gathered}\partial\bar{\partial}\sigma = \Lambda e^{2\sigma},\\ \sigma = -\eta_k\log|z-z_k|^2 + O(1) \quad \text{at} \quad z\to z_k,\\ \sigma = -\log|z|^2 + \ldots.\end{gathered} \tag{294}$$

The last condition is just the statement that the metric is regular at ∞

$$ds^2 = e^{2\sigma}dzd\bar{z} = \frac{dzd\bar{z}}{|z|^4} \quad \text{at} \quad z\to\infty.$$

Moreover, we demand that the metric is integrable at the punctures. From

$$ds^2 = \frac{dzd\bar{z}}{|z-z_k|^{2\eta_k}} \quad \text{at} \quad z\to z_k,$$

we have

$$\eta_k < 1. \tag{295}$$

Picard theorem says that there exist a unique smooth solution to the Liouville equation with prescribed singularities (294) with additional conditions (293) and (295). It follows from (294) that the holomorphic function $t(z)$ for this problem has the form

$$t(z) = \sum_{k=1}^{n}\left(\frac{\delta_k}{(z-z_k)^2} + \frac{c_k}{(z-z_k)}\right), \quad \text{where} \quad \delta_k = \eta_k(1-\eta_k). \tag{296}$$

The parameters c_k, called the *accessory* parameters, are subject to three linear relations following from the asymptotic condition

$$\sigma = -\log|z|^2 + \cdots \implies t(z) \sim \frac{1}{z^4} \quad \text{at} \quad z \to \infty.$$

Namely,

$$\sum_{k=1}^{n} c_k = 0, \quad \sum_{k=1}^{n} (c_k z_k + \delta_k) = 0, \quad \sum_{k=1}^{n} \left(c_k z_k^2 + 2\delta_k z_k\right) = 0.$$

Other parameters c_k are chosen by Picard theorem to be some functions

$$c_k = c_k(\eta_j, z_j, \bar{z}_j),$$

such the monodromy matrices around elementary cycles belong to the real subgroup of $SL(2,\mathbb{C})$

$$\mathcal{M}_k \in SU(1,1).$$

The solution, up to a constant fact, is given in terms of the ratio of two solutions $f = \psi_1/\psi_2$ to holomorphic differential equation (287)

$$ds^2 = \frac{|f'|^2 dz d\bar{z}}{(1-|f|^2)^2} = \frac{df d\bar{f}}{(1-|f|^2)^2}. \tag{297}$$

Formula (297) states what is called uniformization theorem, giving the metric of constant negative curvature on a sphere with punctures as a pull-back of the standard metric (291) on Poincare disk.

The solution of Liouville equation with conical singularities (294) is equivalent to the problem of tuning the accessory parameters c_k in such a way that the monodromy of the corresponding differential equation around each non-contractible loop belongs to $SU(1,1)$. Since the problem (294) admits a unique solution, it implies that there is a unique choice of functions $c_k = c_k(\boldsymbol{\eta}, \boldsymbol{z}, \bar{\boldsymbol{z}})$ which satisfy this condition. Polyakov made an observation, that these functions are derivatives of the classical action

$$c_k = -\frac{\partial S_{\text{cl}}}{\partial z_k}, \tag{298}$$

where the classical action has to be formally understood as

$$S_{\text{cl}} = \frac{1}{4\pi} \int \left(\left(\partial_\mu \sigma\right)^2 + 4\Lambda e^{2\sigma} \right) d^2\boldsymbol{x} - 2 \sum_{k=1}^{n} \eta_k \sigma(z_k, \bar{z}_k) \tag{299}$$

The relation (298) has very simple intuitive meaning from CFT point of view. Consider correlation function of heavy operators V_{α_k}, $\alpha_k = b^{-1}\eta_k$ with one degenerate light operator $V_{-\frac{b}{2}}$. According to general quasiclassical arguments it must scale as

$$\Psi(z,\bar{z}) \overset{\text{def}}{=} \langle V_{-\frac{b}{2}}(z,\bar{z}) V_{b^{-1}\eta_1}(z_1,\bar{z}_1) \ldots V_{b^{-1}\eta_n}(z_n,\bar{z}_n) \rangle = e^{-\sigma(z,\bar{z})} e^{-\frac{1}{b^2} S_{\text{cl}}} \left(1 + O(b^2)\right) \quad \text{at} \quad b \to 0.$$

The field $V_{-\frac{b}{2}}$ is a degenerate one and hence the correlation function $\Psi(z,\bar{z})$ satisfies partial differential equation

$$\left(\partial_z^2 + b^2 \sum_{k=1}^{n} \left(\frac{\Delta_k}{(z-z_k)^2} + \frac{\partial_k}{z-z_k} \right) \right) \Psi(z,\bar{z}) = 0 \tag{300}$$

The conformal dimensions scale as $\Delta_k = b^{-2}\delta_k$ and hence (300) degenerates in the limit $b \to 0$ to

$$\left(\partial_z^2 + \sum_{k=1}^{n}\left(\frac{\delta_k}{(z-z_k)^2} + \frac{c_k}{z-z_k}\right)\right)\psi(z,\bar{z}) = 0, \tag{301}$$

where c_k are given by (298).

This fact formally explains Polyakov's conjecture. We note, however, that the expression (299) is ill-defined due to logarithmic singularities of the solution (294). Moreover, it diverges not only at $z \to z_k$, but also at $z \to \infty$. In order to regularize it, we consider the theory on the domain $\mathcal{X}$ which is a disk of large radius $L = 1/\epsilon$ without small disks of radii ϵ_k centered around each singular point z_k. We define the regularized action as

$$\begin{gathered}
S_{\text{cl}} = \frac{1}{4\pi}\int_{\mathcal{X}}\left(\left(\partial_\mu\sigma\right)^2 + 4\Lambda e^{2\sigma}\right)d^2\boldsymbol{x} + \frac{1}{4\pi}R(\epsilon_k,\epsilon,\sigma) + K(\epsilon_k,\epsilon),\\
R(\epsilon_k,\epsilon,\sigma) = -2i\sum_{k=1}^{n}\eta_k\oint_{|z-z_k|=\epsilon_k}\sigma(z,\bar{z})\left(\frac{d\bar{z}}{\bar{z}-\bar{z}_k} - \frac{dz}{z-z_k}\right) + 2i\oint_{|z|=\frac{1}{\epsilon}}\sigma(z,\bar{z})\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right),\\
K(\epsilon_k,\epsilon) = -\sum_{k=1}^{n}\eta_k^2\log\epsilon_k^2 - \log\epsilon^2
\end{gathered} \tag{302}$$

This is a typical action on a domain with a boundary

$$S = \int_D \mathcal{L}(\varphi,\partial_\mu\varphi)d^2\boldsymbol{x} + \int_{\partial D} K_\lambda(\varphi)dx^\lambda.$$

It's variation is

$$\begin{aligned}
\delta S = \int_D\left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\delta\varphi\right)d^2\boldsymbol{x} + \int_{\partial D}\left(\frac{\partial K_\nu}{\partial\varphi}\delta\varphi\right)dx^\nu =\\
= \int_D\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi\, d^2\boldsymbol{x} + \int_{\partial D}\left(\frac{\partial K_\nu}{\partial\varphi} + \epsilon_{\mu\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi\, dx^\nu.
\end{aligned} \tag{303}$$

In the last line we used the Green theorem

$$\int_{\mathcal{D}}\partial_\mu A^\mu d^2\boldsymbol{x} = \oint_{\partial\mathcal{D}}\varepsilon_{\mu\nu}A^\mu dx^\nu = i\oint_{\partial\mathcal{D}}\left(A_z dz - A_{\bar{z}}d\bar{z}\right),$$

From (303) we get the equations of motion

$$\left.\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\right|_D = 0, \quad \left.\left(\frac{\partial K_\nu}{\partial\varphi} + \epsilon_{\mu\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\right|_{\partial D} = 0.$$

Applied to our case (302), we find that the Liouville equation in the bulk is supplemented with the boundary conditions

$$\left.\left(\partial\sigma + \frac{\eta_k}{z-z_k}\right)\right|_{|z-z_k|=\epsilon_k} = \left.\left(\bar{\partial}\sigma + \frac{\eta_k}{\bar{z}-\bar{z}_k}\right)\right|_{|z-z_k|=\epsilon_k} = 0, \qquad \left.\left(\partial\sigma + \frac{1}{z}\right)\right|_{|z|=\frac{1}{\epsilon}} = \left.\left(\bar{\partial}\sigma + \frac{1}{\bar{z}}\right)\right|_{|z|=\frac{1}{\epsilon}} = 0,$$

which are equivalent to the asymptotic conditions in (294) in the limit $\epsilon_k \to 0$, $\epsilon \to 0$. The constant $K(\epsilon_k, \epsilon)$ has been chosen to make the action finite in the limit $\epsilon_k \to 0$ (and similar at $\epsilon \to 0$). There are two sources of divergences. One from the kinetic term

$$\frac{1}{4\pi}\int_{\mathcal{X}} 4\partial\sigma\bar{\partial}\sigma d^2\boldsymbol{x} = \frac{\eta_k^2}{\pi}\int \frac{d^2\boldsymbol{x}}{|z-z_k|^2} + \dots = -\eta_k^2 \log\epsilon_k^2 + \dots$$

and another from the boundary term

$$-\frac{i\eta_k}{2\pi}\oint_{|z-z_k|=\epsilon_k} \sigma(z,\bar{z})\left(\frac{d\bar{z}}{\bar{z}-\bar{z}_k} - \frac{dz}{z-z_k}\right) = 2\eta_k^2\log\epsilon_k^2 + \dots$$

Adding these two together with the counterterm we get a finite result.

It is convenient to specify the subleading terms in the singular expansion

$$\begin{aligned} \sigma(z,\bar{z}) &= -\eta_k^2 \log|z-z_k|^2 + \hat{\sigma}_k + \dots \quad \text{at} \quad z \to z_k, \\ \sigma(z,\bar{z}) &= -\log|z|^2 + \hat{\sigma}_\infty + \dots \quad \text{at} \quad z \to \infty. \end{aligned}$$

Then in the limit $\epsilon_k \to 0$, $\epsilon \to 0$ one can write (note the sign change in front of the counterterm)

$$S_{\text{cl}} = \frac{1}{4\pi}\int_{\mathcal{X}} \left(\left(\partial_\mu\sigma\right)^2 + 4\Lambda e^{2\sigma}\right) d^2\boldsymbol{x} - 2\sum_{k=1}^{n}\eta_k\hat{\sigma}_k + 2\hat{\sigma}_\infty - K(\epsilon_k,\epsilon).$$

Using this formula it is easy to prove that the following relation, which holds on-shell

$$\frac{\partial S_{\text{cl}}}{\partial\eta_k} = -2\hat{\sigma}_k. \tag{304}$$

Indeed, taking the derivative of the bulk part, one has

$$\begin{aligned} &\frac{1}{4\pi}\int_{\mathcal{X}}\left(2\partial_\mu\sigma\partial_\mu\left(\frac{\partial\sigma}{\partial\eta_k}\right) + 8\Lambda e^{2\sigma}\left(\frac{\partial\sigma}{\partial\eta_k}\right)\right)d^2\boldsymbol{x} = \frac{1}{2\pi}\int_{\mathcal{X}}\partial_\mu\left(\partial_\mu\sigma\left(\frac{\partial\sigma}{\partial\eta_k}\right)\right)d^2\boldsymbol{x} = \\ &\qquad = \frac{i}{2\pi}\left(\sum_{l=1}^{n}\oint_{|z-z_l|=\epsilon_l}\left(\frac{\partial\sigma}{\partial\eta_k}\right)\left(\partial\sigma dz - \bar{\partial}\sigma d\bar{z}\right) - \oint_{|z|=\frac{1}{\epsilon}}\left(\frac{\partial\sigma}{\partial\eta_k}\right)\left(\partial\sigma dz - \bar{\partial}\sigma d\bar{z}\right)\right) = \\ &\qquad\qquad = 2\sum_{l=1}^{n}\eta_l\frac{\partial\hat{\sigma}_l}{\partial\eta_k} - 2\frac{\partial\hat{\sigma}_\infty}{\partial\eta_k} - 2\eta_k\log\epsilon_k^2. \end{aligned}$$

Combining with the derivative of

$$-2\sum_{k=1}^{n}\eta_k\hat{\sigma}_k + 2\hat{\sigma}_\infty - K(\epsilon_k,\epsilon),$$

one finds that first, the divergent terms are canceled, and second that (304) holds.

We note that (304) implies that the form

$$dS_{\text{cl}} = -2\sum_{k=1}^{n}\hat{\sigma}_k d\eta_k$$

can be integrated giving $S_{\rm cl}$ up to a constant term independent on η_k.

As an example consider the case of three external points. In this case the accessory parameters in (296) are completely fixed by the requirement $t(z) \sim \frac{1}{z^4}$

$$c_1 = \frac{-2\delta_1 z_1 + (\delta_1 + \delta_3 - \delta_2) z_2 + (\delta_1 + \delta_2 - \delta_3) z_3}{(z_1 - z_2)(z_1 - z_3)}, \quad \ldots$$

Differential equations (301) in this case can be reduced to the hypergeometric equation. The single-valued solution to (301) admits the integral representation[19]

$$\Psi(z, \bar{z}) = C \prod_{k=1}^{3} |z - z_k|^{2\eta_k} \int \prod_{k=1}^{3} |\xi - z_k|^{2A_k} |\xi - z|^{2B} d^2\xi, \tag{305}$$

where

$$A_k = \eta - 2\eta_k - 1, \quad B = 1 - \eta, \quad \eta = \eta_1 + \eta_2 + \eta_3.$$

The normalization constant in (305) is obtained from

[19]We note that formally our classical equation (301) has exactly the same form as the quantum one (78) by with different meaning of the parameters. For (78) we obtained the integral representation (274). Thus adjusting the parameters we arrive to (305).

Lecture 23: Classical CFT II: classical conformal block, Painlevé VI equation

In last lecture we saw that corrlelation functions of heavy fields behaves semi-classically in the limit $b \to 0$

$$\langle V_{b^{-1}\eta_1}(z_1, \bar{z}_1) \dots V_{b^{-1}\eta_n}(z_n, \bar{z}_n) \rangle = e^{-\frac{1}{b^2} S_{\text{cl}}}.$$

What is less obvious is that holomorphic quantities like conformal blocks , which do not have immediate path integral representation, still admit WKB behavior. Consider for example 4−point conformal block

$$\mathfrak{F}_{\Delta}\left(\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix}\middle| z\right) = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} z^{\Delta-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|} \left(\Gamma^{-1}\right)^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}} \frac{\langle \Delta_4 | \Phi_3 L_{-\boldsymbol{\lambda}} | \Delta \rangle}{\langle \Delta_4 | \Phi_3 | \Delta \rangle} \frac{\langle \Delta | L_{\boldsymbol{\mu}} \Phi_1 | \Delta_2 \rangle}{\langle \Delta | \Phi_1 | \Delta_2 \rangle},$$

where

$$\Gamma^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}} = \frac{\langle \Delta | L_{\boldsymbol{\mu}} L_{-\boldsymbol{\lambda}} | \Delta \rangle}{\langle \Delta | \Delta \rangle}, \tag{306}$$

and take

$$\Delta_k = \Delta(b^{-1}\eta_k) = \frac{\delta_k}{b^2} + \dots, \qquad \Delta = \Delta\left(\frac{Q}{2} + \frac{b^{-1}\lambda}{2}\right) = \frac{1}{b^2}\frac{(1-\lambda^2)}{4} + \dots \tag{307}$$

Then we have the following statement

$$\mathfrak{F}_{\frac{(1-\lambda^2)}{4b^2}}\left(\begin{matrix} \frac{\delta_2}{b^2} & \frac{\delta_3}{b^2} \\ \frac{\delta_1}{b^2} & \frac{\delta_4}{b^2} \end{matrix}\middle| z\right) = e^{\frac{1}{b^2}\mathfrak{f}_{\lambda}\left(\begin{smallmatrix} \delta_2 & \delta_3 \\ \delta_1 & \delta_4 \end{smallmatrix}\middle| z\right) + \dots} \quad \text{at} \quad , \tag{308}$$

where $\mathfrak{f}_{\lambda}\left(\begin{smallmatrix} \delta_2 & \delta_3 \\ \delta_1 & \delta_4 \end{smallmatrix}\middle| z\right)$ is called classical conformal block.

The statement (308) is rather non-trivial being viewed as a series expansion at $z \to 0$. Clearly, we have

$$z^{\Delta-\Delta_1-\Delta_2} = e^{\frac{1}{b^2}\left(\frac{1-\lambda^2}{4} - \delta_1 - \delta_2\right)\log z + \dots}$$

while the rest is a series $1 + F_1 z + F_2 z^2 + \dots$ where

$$F_N = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|=N} \left(\Gamma^{-1}\right)^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}} \frac{\langle \Delta_4 | \Phi_3 L_{-\boldsymbol{\lambda}} | \Delta \rangle}{\langle \Delta_4 | \Phi_3 | \Delta \rangle} \frac{\langle \Delta | L_{\boldsymbol{\mu}} \Phi_1 | \Delta_2 \rangle}{\langle \Delta | \Phi_1 | \Delta_2 \rangle}. \tag{309}$$

Let us estimate how the coefficient (309) behaves in the limit $b \to 0$ with (307). It is clear that

$$\frac{\langle \Delta_4 | \Phi_3 L_{-\boldsymbol{\lambda}} | \Delta \rangle}{\langle \Delta_4 | \Phi_3 | \Delta \rangle} = P_{\boldsymbol{\lambda}}(\Delta_4, \Delta_3, \Delta),$$

is a polynomial of degree $\deg P_{\boldsymbol{\lambda}} = l(\boldsymbol{\lambda})$ where $l(\boldsymbol{\lambda})$ is a length of the partition $\boldsymbol{\lambda}$. This statement follows immediately from commutation relations (146). The same is true for $\frac{\langle \Delta | L_{\boldsymbol{\mu}} \Phi_1 | \Delta_2 \rangle}{\langle \Delta | \Phi_1 | \Delta_2 \rangle}$. Now comes the Shapovalov matrix (306). Consider the matrix element

$$\langle \Delta | L_{\boldsymbol{\mu}} L_{-\boldsymbol{\lambda}} | \Delta \rangle = \langle \Delta | L_{\boldsymbol{\mu}} L_{-\lambda_1} L_{-\lambda_2} \dots | \Delta \rangle$$

and drag $L_{-\lambda_1}$ to the left. Using commutation relations $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$, we see that if it does not meet L_{λ_1} on his way then the result will be

$$\sum_{\boldsymbol{\eta}} c_{\boldsymbol{\eta}} \langle \Delta | L_{\boldsymbol{\eta}} L_{-\lambda_2} L_{-\lambda_3} \dots | \Delta \rangle$$

with all $c_{\boldsymbol{\eta}}$ finite in the limit $b \to 0$. However if it does meet L_{λ_1} the result will be of order $\frac{1}{b^2}$ since $\Delta \sim \frac{1}{b^2}$ and $c \sim \frac{1}{b^2}$. So the more such "meetings" occur the more singular behavior at $b \to 0$ we get. Most singular matrix element is

$$\langle \Delta | L_1^N L_{-1}^N | \Delta \rangle \sim \frac{1}{b^{2N}}.$$

It implies that the inverse of Shapovalov matrix behaves as $\Gamma^{-1} \sim b^{2N}$. Altogether it implies that the coefficients (309) behave as

$$F_N \sim \frac{1}{b^{2N}} \quad \text{at} \quad b \to 0 = \frac{1}{b^{2N}} \left(F_N^{(N)} + b^2 F_N^{(N-1)} + b^4 F_N^{(N-2)} + \dots \right).$$

Now (308) is equivalent to the statement that in the expansion

$$\log \left(1 + F_1 z + F_2 z^2 + F_3 z^3 + \dots \right) = z F_1 + z^2 \left(F_2 - \frac{F_1^2}{2} \right) + z^3 \left(F_3 - F_1 F_2 + \frac{F_1^3}{3} \right) + \dots$$

all singular parts are cancelled all the way down to $F_N^{(1)}$. For example it implies that

$$F_2^{(2)} = \frac{\left(F_1^{(1)} \right)^2}{2} \quad \text{etc.}$$

One can define classical conformal block $\mathfrak{f}_\lambda \left(\begin{smallmatrix} \delta_2 & \delta_3 \\ \delta_1 & \delta_4 \end{smallmatrix} \middle| z \right)$ purely in classical terms. One can use the following semiclassical intuition. Consider five-point correlation function

$$\Psi(z, \bar{z}) = \langle V_{-\frac{b}{2}}(z, \bar{z}) V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_4}(z_4, \bar{z}_4) \rangle.$$

It satisfies the following partial differential equation

$$\left(\partial^2 + b^2 \sum_{k=1}^{4} \left(\frac{\Delta_k}{(z - z_k)^2} + \frac{\partial_k}{(z - z_k)} \right) \right) \Psi(z, \bar{z}) = 0.$$

Consider a particular solution to this equation specified by an expansion

$$\Psi_\Delta(z|z_k) = (z_1 - z_2)^{\Delta - \Delta_1 - \Delta_2} \underbrace{\left[\psi_\Delta^{(0)}(z|z_2, z_3, z_4) + (z_1 - z_2) \psi_\Delta^{(1)}(z|z_2, z_3, z_4) + \dots \right]}_{\psi_\Delta(z|z_k)} \quad \text{at} \quad z_1 \to z_2.$$

Then for the function $\psi_\Delta(z|z_k)$ one has

$$\left(\partial^2 + b^2 \sum_{k=1}^{4} \left(\frac{\Delta_k}{(z - z_k)^2} + \frac{\partial_k}{(z - z_k)} \right) + \frac{b^2 (\Delta - \Delta_1 - \Delta_2)}{(z - z_1)(z - z_2)} \right) \psi_\Delta(z|z_k) = 0.$$

Expanding at $z_1 \to z_2$ one obtains a semi-infinite system of inhomogeneous differential equations

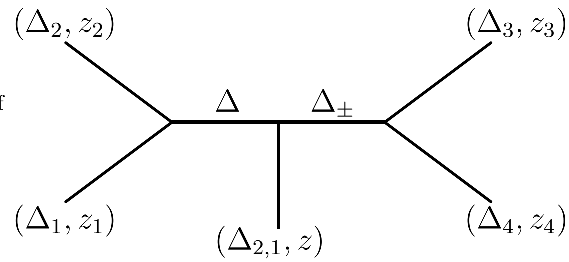
$$\begin{aligned}
&\left[\partial^2 + b^2 \sum_{k=2}^{4} \left(\frac{\tilde{\Delta}_k}{(z - z_k)^2} + \frac{\partial_k}{z - z_k} \right) \right] \psi_\Delta^{(0)}(z|z_k) = 0, \\
&\left[\partial^2 + b^2 \sum_{k=2}^{4} \left(\frac{\tilde{\Delta}_k}{(z - z_k)^2} + \frac{\partial_k}{z - z_k} \right) + \frac{b^2}{(z - z_2)^2} \right] \psi_\Delta^{(1)}(z|z_k) + \frac{(\Delta + \Delta_1 - \Delta_2)}{(z - z_2)^3} \psi_\Delta^{(0)}(z|z_k) = 0, \\
&\left[\partial^2 + b^2 \sum_{k=2}^{4} \left(\frac{\tilde{\Delta}_k}{(z - z_k)^2} + \frac{\partial_k}{z - z_k} \right) + \frac{2b^2}{(z - z_2)^2} \right] \psi_\Delta^{(2)}(z|z_k) + (\dots) \psi_\Delta^{(1)}(z|z_k) + (\dots) \psi_\Delta^{(0)}(z|z_k) = 0,
\end{aligned} \tag{310}$$

where $\tilde{\Delta}_2 = \Delta$, $\tilde{\Delta}_3 = \Delta_3$ and $\tilde{\Delta}_4 = \Delta_4$. Of course the solution to this system is now unique. At each step one can can a solution of homogeneous equation with shifted Δ

$$\Delta \to \Delta + 1 \to \Delta + 2 \to \dots$$

One can fix this ambiguity by demanding that the solution $\Psi_\Delta(z|z_k)$ corresponds to a linear combination of two five-point conformal blocks

$$\Psi_\Delta^\pm(z|z_k) \stackrel{\text{def}}{=}$$



that is

$$\Psi_\Delta(z|z_k) = C_+\Psi_\Delta^+(z|z_k) + C_-\Psi_\Delta^-(z|z_k).$$

It means that

$$\psi_\Delta^{(k)}(z|z_k) = \mathcal{D}_z^{(k)}\psi_\Delta^{(0)}(z|z_k),$$

with $\mathcal{D}_z^{(k)}$ being some differential polynomial with rational coefficients

$$\langle V_{-\frac{b}{2}}(z)V_\Delta^{(k)}(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)\rangle = \mathcal{D}_z^{(k)}\langle V_{-\frac{b}{2}}(z)V_\Delta(z_2)V_{\alpha_3}(z_3)V_{\alpha_4}(z_4)\rangle,$$

where $V_\Delta^{(k)}(z_2)$ is a descendant appearing in holomorphic OPE

$$V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) = \sum_{\Delta,k} C_{\alpha_1\alpha_2}^\Delta (z_1 - z_2)^{\Delta - \Delta_1 - \Delta_2 + k} V_\Delta^{(k)}(z_2).$$

In particular, one can check that

$$\psi_\Delta^{(1)}(z|z_k) = \frac{(\Delta - \Delta_1 - \Delta_2)}{2\Delta}\partial_2\psi_\Delta^{(0)}(z|z_k)$$

solves second equation in (310).

One can choose $\Psi_\Delta^\pm(z|z_k)$ to have a diagonal monodromy while z goes around z_1 and z_2

$$\Psi_\Delta^\pm(z|z_k) = (z_1 - z_2)^\Delta \left(\psi_\Delta^{(0),\pm}(z|z_k) + (z_1 - z_2)\psi_\Delta^{(1),\pm}(z|z_k) + \dots\right),$$

where $\psi_\Delta^{(0),\pm}(z|z_k)$ are two solutions of first equation in (310) which transform as ($\Delta = \Delta(\alpha)$)

$$\begin{pmatrix} \psi_\Delta^{(0),+}(z|z_k) \\ \psi_\Delta^{(0),-}(z|z_k) \end{pmatrix} \to \begin{pmatrix} e^{2i\pi b\alpha} & 0 \\ 0 & e^{2i\pi b(Q-\alpha)} \end{pmatrix} \begin{pmatrix} \psi_\Delta^{(0),+}(z|z_k) \\ \psi_\Delta^{(0),-}(z|z_k) \end{pmatrix}$$

It is clear that since $\psi_\Delta^{(k),\pm}(z|z_k)$ are obtained by application of differential operators with rational coefficients, it has the same monodromy properties. Taking $\alpha = \frac{Q}{2} + \frac{b^{-1}\lambda}{2}$ we conclude that trace of the monodromy matrix in the limit $b \to 0$ is

$$\mathrm{tr}\mathcal{M}_{12} = -2\cos\pi\lambda. \tag{311}$$

Now we come to the definition of classical conformal block. Consider differential equation with four singular points

$$\left[\partial^2 + \underbrace{\sum_{k=1}^{4}\left(\frac{\delta_k}{(z-z_k)^2} + \frac{c_k}{(z-z_k)}\right)}_{t(z)\sim\frac{1}{z^4}\quad \text{at} \quad z\to\infty}\right]\psi(z) = 0 \tag{312}$$

According to discussions above, we define classical conformal block as a function which satisfies

$$\frac{\partial \mathfrak{f}_\lambda\left(\begin{smallmatrix}\delta_2 & \delta_3\\ \delta_1 & \delta_4\end{smallmatrix}\middle| z_k\right)}{\partial z_k} = c_k \quad \text{such that} \quad \mathrm{tr}\mathcal{M}_{12} = -2\cos\pi\lambda. \tag{313}$$

We note that due to the relation $t(z) \sim \frac{1}{z^4}$ there are three linear relations between accessory parameters c_k. So, only one of them, say c_4, is not fixed. If one demands the constraint (311) then $c_4 = c_4(\lambda)$ is a particular function of λ. Classical conformal block is related to $c_4(\lambda)$ by (313).

It is convenient to use projective invariance and set $z_1 = x$, $z_2 = 0$, $z_3 = 1$ and $z_4 = \infty$. Then equation (312) reduces to Heun equation

$$\psi'' + \left(\frac{\delta_1}{(z-x)^2} + \frac{\delta_2}{z^2} + \frac{\delta_3}{(z-1)^2} + \frac{x(x-1)c}{z(z-1)(z-x)} + \frac{\delta_4-\delta_1-\delta_2-\delta_3}{z(z-1)}\right)\psi = 0 \tag{314}$$

where

$$c(x,\lambda) = \frac{\partial}{\partial x}\mathfrak{f}_\lambda\left(\begin{matrix}\delta_2 & \delta_3\\ \delta_1 & \delta_4\end{matrix}\middle| x\right) \implies \mathfrak{f}_\lambda\left(\begin{matrix}\delta_2 & \delta_3\\ \delta_1 & \delta_4\end{matrix}\middle| x\right) = \int c(x,\lambda)dx + \text{const.} \tag{315}$$

A constant term in (315) can be easily fixed by matching the asymptotic at $x \to 0$.

So, we have a problem of computing the accessory parameter c in Heun equation (314) as a function of trace of the monodromy. Consider simpler problem first[20]

$$\chi'' + t(w)\chi = 0 \quad \text{where} \quad t(w) = \frac{\delta_1}{(w-x)^2} + \frac{\delta_2}{w^2} + \frac{\delta_1+\delta_2-\delta}{w(w-x)} \quad \text{with} \quad \delta = \frac{1-\lambda^2}{4}.$$

Obviously we have

$$\mathcal{M}_{12}\mathcal{M}_3 = 1 \implies \mathrm{tr}\mathcal{M}_{12} = -2\cosh\pi\lambda.$$

If one finds $w = w(z)$ such that

$$t(z) = \big(w'(z)\big)^2 t\big(w(z)\big) + \frac{\{w(z),z\}}{2} = \frac{\delta_1}{(z-x)^2} + \frac{\delta_2}{z^2} + \frac{\delta_3}{(z-1)^2} + \frac{x(x-1)c}{z(z-1)(z-x)} + \frac{\delta_4-\delta_1-\delta_2-\delta_3}{z(z-1)} \tag{316}$$

then the function

$$\psi(z) = \big(w'(z)\big)^{\frac{1}{2}}\chi\big(w(z)\big)$$

[20] It is obtained from

$$\chi'' + \underbrace{\left(\frac{\delta_1}{(w-w_1)^2} + \frac{\delta_1}{(w-w_2)^2} + \frac{\delta}{(w-w_3)^2} + \frac{c_1}{w-w_1} + \frac{c_2}{w-w_2} + \frac{c_3}{w-w_3}\right)}_{\sim\frac{1}{w^4}}\chi = 0$$

in the limit $w_1 \to x$, $w_2 \to 0$ and $w_3 \to \infty$.

will satisfy Heun equation (314). We take

$$w(z) = z\frac{v(z)}{v(x)} \quad \text{where} \quad v(z) = 1 + v_1(x)z + v_2(x)z^2 + \dots$$

so that $w(z) = 0$ and $w(x) = x$. If we assume that the series for $v(z)$ converges in some domain, then $\psi(z)$ has the same monodromy as $\chi(w)$.

We solve (316) pertubatively at both $z \to 0$ and $x \to 0$. It is convenient to represent

$$t(z) = \frac{\delta_1}{(z-x)^2} + \frac{\delta_2}{z^2} + \frac{c_1}{z-x} + \frac{c_2}{z} + \tau(z) \quad \text{where} \quad \tau(z) = \tau_0 + \tau_1 z + \tau_2 z^2 + \dots$$

with $c_1 = c$ and $c_2 = (x-1)c + \delta_1 + \delta_2 + \delta_3 - \delta_4$. We have

$$\begin{aligned}
c_1 &= -\frac{\delta_1 + \delta_2 - \delta}{x} + (\delta + \delta_1 - \delta_2)v_1 + O(x),\\
c_2 &= \frac{\delta_1 + \delta_2 - \delta}{x} + (\delta + \delta_2 - \delta_1)v_1 + O(x),\\
\tau_0 &= (4\delta + 3)v_2 - (\delta + 3)v_1^2 + O(x),\\
&\dots\dots\dots
\end{aligned}$$

We note that explicit form of the coefficients c_1 and c_2 implies the constraint

$$(1-x)c_1 + c_2 = \delta_1 + \delta_2 + \delta_3 - \delta_4. \tag{317}$$

One can solve (317) for

$$v_1(x) = v_1^{(0)} + v_1^{(1)}x + v_1^{(2)}x^2 + \dots$$

The rest of equations can be solved perturbatively in x. We have

$$c = \frac{\delta - \delta_1 - \delta_2}{x} + \frac{(\delta + \delta_1 - \delta_2)(\delta + \delta_3 - \delta_4)}{2\delta} + \dots$$

which implies

$$\mathfrak{f}_\lambda\left(\begin{matrix}\delta_2 & \delta_3\\ \delta_1 & \delta_4\end{matrix}\middle| x\right) = (\delta - \delta_1 - \delta_2)\log x + \frac{(\delta + \delta_1 - \delta_2)(\delta + \delta_3 - \delta_4)}{2\delta}x + \dots$$

with complete agreement with semiclassical expansion of (148).

Lecture 24: Zamolodchikov's recursion formula

In this lecture we will study the pole structure of the four-point conformal block (147) following [18]. This function, being considered as a function of the intermediate dimension Δ has poles at Kac values $\Delta \to \Delta_{m,n}$ (see (74)). Clearly, the poles come from the inverse of the Shapovalov matrix (75). At $\Delta = \Delta_{m,n}$ there is a singular vector at level mn

$$|\chi_{m,n}\rangle = D_{m,n}|\Delta_{m,n}\rangle \quad \text{where} \quad D_{m,n} = L_{-1}^{mn} + c_1(b)L_{-2}L_{-1}^{mn-2} + c_2(b)L_{-3}L_{-1}^{mn-2} + \dots,$$

with

$$c_1 = \frac{mn}{6}\big((m^2-1)b^2 + (n^2-1)b^{-2}\big) \quad \text{etc.}$$

From the Kac determinant formula (76) we know that for any two partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ the following holds $|\boldsymbol{\lambda}| = |\boldsymbol{\nu}| + mn$

$$\langle\Delta|L_{\boldsymbol{\lambda}}L_{-\boldsymbol{\nu}}D_{m,n}|\Delta\rangle \sim (\Delta - \Delta_{m,n})$$

We can compute the conformal block (147) in any basis we like. We take the following one

$$\tilde{L}_{-\boldsymbol{\lambda}}|\Delta\rangle \overset{\text{def}}{=} \begin{cases} L_{-\boldsymbol{\lambda}}|\Delta\rangle \quad \text{if} \quad \boldsymbol{\lambda} = \{\dots, \underbrace{1,\dots,1}_{k<mn}\}, \\ \\ L_{-\boldsymbol{\nu}}D_{m,n}|\Delta\rangle \quad \text{otherwise} \end{cases}$$

In this basis one can write

$$\mathfrak{F}_{\Delta}\begin{pmatrix}\Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4\end{pmatrix}\!\!\bigg| z\bigg) = \sum_{|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|} z^{\Delta-\Delta_1-\Delta_2+|\boldsymbol{\lambda}|}\left(\tilde{\Gamma}^{-1}\right)_{\boldsymbol{\lambda}\boldsymbol{\mu}} \frac{\langle\Delta_4|\Phi_3\tilde{L}_{-\boldsymbol{\lambda}}|\Delta\rangle}{\langle\Delta_4|\Phi_3|\Delta\rangle}\frac{\langle\Delta|\tilde{L}_{\boldsymbol{\mu}}\Phi_1|\Delta_2\rangle}{\langle\Delta|\Phi_1|\Delta_2\rangle},$$

where

$$\tilde{\Gamma}_{\boldsymbol{\mu}\boldsymbol{\lambda}} \overset{\text{def}}{=} \langle\Delta|\tilde{L}_{\boldsymbol{\mu}}\tilde{L}_{-\boldsymbol{\lambda}}|\Delta\rangle.$$

Clearly, only the vectors $L_{-\boldsymbol{\nu}}D_{m,n}|\Delta\rangle$ lead to the singular behavior[21]. Moreover, one has

$$\langle\Delta|D^{+}_{m,n}L_{\boldsymbol{\nu}}L_{-\boldsymbol{\rho}}D_{m,n}|\Delta\rangle = \langle\Delta_{m,-n}|L_{\boldsymbol{\nu}}L_{-\boldsymbol{\rho}}|\Delta_{m,-n}\rangle\langle\Delta|D^{+}_{m,n}D_{m,n}|\Delta\rangle + O\big((\Delta-\Delta_{m,n})^2\big)$$

Collecting all this one arrives to

$$\operatorname{Res}\mathfrak{F}_{\Delta}\begin{pmatrix}\Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4\end{pmatrix}\!\!\bigg| z\bigg)\bigg|_{\Delta=\Delta_{m,n}} = (r_{m,n})^{-1}\times$$
$$\times \sum_{|\boldsymbol{\nu}|=|\boldsymbol{\rho}|} z^{\Delta_{m,-n}-\Delta_1-\Delta_2+|\boldsymbol{\nu}|}\left(\Gamma^{-1}_{\Delta_{m,n}}\right)_{\boldsymbol{\nu}\boldsymbol{\rho}} \frac{\langle\Delta_4|\Phi_3L_{-\boldsymbol{\nu}}D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle}\frac{\langle\Delta_{m,n}|D^{+}_{m,n}L_{\boldsymbol{\rho}}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle},$$

[21] It follows from the formula for determinant of block matrix (provided that D is non-degenerate)

$$\det\begin{pmatrix}A & B \\ C & D\end{pmatrix} = \det(D)\det\big(A - BD^{-1}C\big) = \det(D)\det\big(A\big) + \dots$$

Then any entry of the inverse matrix is of the form $(-1)^{\pm}\det\begin{pmatrix}A' & B' \\ C' & D'\end{pmatrix}\Big/\det\begin{pmatrix}A & B \\ C & D\end{pmatrix}$ where (A', B', C', D') were obtained from (A, B, C, D) by erasing one row and one column. Clearly, only the elements with $D' = D$ lead to the singular behavior.

where

$$r_{m,n}=\lim_{\Delta\to\Delta_{m,n}}\frac{\langle\Delta|D_{m,n}^{+}D_{m,n}|\Delta\rangle}{\Delta-\Delta_{m,n}} \tag{318}$$

Moreover, we have

$$\frac{\langle\Delta_4|\Phi_3 L_{-\boldsymbol{\nu}}D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle}=\frac{\langle\Delta_4|\Phi_3 L_{-\boldsymbol{\nu}}|\Delta_{m,-n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,-n}\rangle}\frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle},$$

$$\frac{\langle\Delta_{m,n}|D_{m,n}^{+}L_{\boldsymbol{\rho}}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle}=\frac{\langle\Delta_{m,-n}|L_{\boldsymbol{\rho}}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,-n}|\Phi_1|\Delta_2\rangle}\frac{\langle\Delta_{m,n}|D_{m,n}^{+}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle}.$$

Collecting altogether one has

$$\operatorname{Res}\mathfrak{F}_{\Delta}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right)\Big|_{\Delta=\Delta_{m,n}}=\frac{R_{m,n}}{r_{m,n}}\;\mathfrak{F}_{\Delta_{m,-n}}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\right), \tag{319}$$

where

$$R_{m,n}=\frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle}\frac{\langle\Delta_{m,n}|D_{m,n}^{+}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle}.$$

Our next goal is to compute the factors $r_{m,n}$ and $R_{m,n}$. The last one is relatively easy. Consider first examples

$$\frac{\langle\Delta_4|\Phi_3 D_{1,1}|\Delta_{1,1}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{1,1}\rangle}=-\partial_z\cdot z^{\Delta_4-\Delta_{1,1}-\Delta_3}\Big|_{z=1}=\big(\Delta(\alpha_3)-\Delta(\alpha_4)\big),$$

and

$$\begin{aligned}\frac{\langle\Delta_4|\Phi_3 D_{2,1}|\Delta_{2,1}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{2,1}\rangle}&=\left(\partial_z^2-b^2\left(z^{-1}\partial_z-\Delta z^{-2}\right)\right)z^{\Delta_4-\Delta_{2,1}-\Delta_3}\Big|_{z=1}=\\ &=(\Delta_4-\Delta_{2,1}-\Delta_3)(\Delta_4-\Delta_{2,1}-\Delta_3-1)-b^2(\Delta_4-\Delta_{2,1}-2\Delta_3)=\\ &=\left(\Delta(\alpha_3)-\Delta\Big(\alpha_4+\frac{b}{2}\Big)\right)\left(\Delta(\alpha_3)-\Delta\Big(\alpha_4-\frac{b}{2}\Big)\right),\end{aligned}$$

and

$$\frac{\langle\Delta_4|\Phi_3 D_{3,1}|\Delta_{3,1}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{3,1}\rangle}=\dots=\big(\Delta(\alpha_3)-\Delta(\alpha_4)\big)\big(\Delta(\alpha_3)-\Delta(\alpha_4+b)\big)\big(\Delta(\alpha_3)-\Delta(\alpha_4-b)\big). \tag{320}$$

It suggests the following generic formula

$$\frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle}=\prod_{r,s}\left(\Delta(\alpha_3)-\Delta\Big(\alpha_4+\frac{rb}{2}+\frac{sb^{-1}}{2}\Big)\right), \tag{321}$$

where the product goes over the sets

$$r=\{m-1,m-3,\dots,3-m,1-m\}\quad\text{and}\quad s=\{n-1,n-3,\dots,3-n,1-n\}. \tag{322}$$

The explicit formula (320) is a manifestation of the fusion rules for degenerate fields. We have already seen an example of such fusion (79)

$$\Phi_{2,1}\Phi_\alpha=[\Phi_{\alpha+\frac{b}{2}}]+[\Phi_{\alpha-\frac{b}{2}}],\qquad\Phi_{1,2}\Phi_\alpha=[\Phi_{\alpha+\frac{b^{-1}}{2}}]+[\Phi_{\alpha-\frac{b^{-1}}{2}}].$$

In particular it implies that

$$\Phi_{2,1}\Phi_{m,n} = [\Phi_{m+1,n}] + [\Phi_{m-1,n}], \qquad \Phi_{1,2}\Phi_{m,n} = [\Phi_{m,n+1}] + [\Phi_{m,n-1}].$$

Both these fusion rules can be interpreted as $\mathfrak{sl}(2)$ fusion rules. Namely, the product of 2-dimensional and m-dimensional (or n-dimensional) representations of $\mathfrak{sl}(2)$ is the sum of $m+1$-dimensional and $m-1$-dimensional representations ($n+1$-dimensional and $n-1$-dimensional). Then using associativity of the OPE, one finds that

$$\Phi_{m,n}\Phi_{\alpha} = \sum_{r,s}\Big[\Phi_{\alpha+\frac{rb}{2}+\frac{sb^{-1}}{2}}\Big], \tag{323}$$

where the sum goes over the set (322). Now consider the matrix element (321). The state $D_{m,n}|\Delta_{m,n}\rangle = |\chi_{m,n}\rangle$ is a singular vector which one can consistently set to zero

$$\frac{\langle\Delta_4|\Phi_3 D_{m,n}|\Delta_{m,n}\rangle}{\langle\Delta_4|\Phi_3|\Delta_{m,n}\rangle} = \mathcal{D}_{m,n} z^{\Delta_4-\Delta_{m,n}-\Delta_3} = 0, \tag{324}$$

where $\mathcal{D}_{m,n}$ is a differential operator read of $D_{m,n}$ according to the rules (146). Comparing (324) with (323) and taking into account large Δ_3 asymptotic, one arrives to the r.h.s. of (321).

Similarly, one has

$$\frac{\langle\Delta_{m,n}|D^{+}_{m,n}\Phi_1|\Delta_2\rangle}{\langle\Delta_{m,n}|\Phi_1|\Delta_2\rangle} = \prod_{r,s}\left(\Delta(\alpha_1) - \Delta\Big(\alpha_2 + \frac{rb}{2} + \frac{sb^{-1}}{2}\Big)\right),$$

which implies

$$R_{m,n} = \prod_{r,s}\left(\Delta(\alpha_1) - \Delta\Big(\alpha_2 + \frac{rb}{2} + \frac{sb^{-1}}{2}\Big)\right)\left(\Delta(\alpha_3) - \Delta\Big(\alpha_4 + \frac{rb}{2} + \frac{sb^{-1}}{2}\Big)\right).$$

The constant $r_{m,n}$ is more painful. From its definition (318) it is clear that $r_{m,n}$ is a polynomial in b and b^{-1}. Explicit calculations on first levels give

$$r_{1,1} = 2, \quad r_{2,1} = 4b^2\left(b+b^{-1}\right)\left(b-b^{-1}\right), \quad r_{3,1} = 24b^4\left(2b+b^{-1}\right)\left(2b-b^{-1}\right)\left(b+b^{-1}\right)\left(b-b^{-1}\right) \quad \ldots \tag{325}$$

Zamolodchikov computed more terms and conjectured the generic formula [18]

$$r_{m,n} = \frac{2}{mb+nb^{-1}} \prod_{\substack{1-m\leq i\leq m\\ 1-n\leq j\leq n}}^{\prime} (ib+jb^{-1}),$$

where $\prod'$ means that the term with $i=j=0$ is absent.

Using (319) we can write

$$\mathfrak{F}_{\Delta}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right) = \sum_{m,n}\frac{R_{mn}}{r_{m,n}(\Delta-\Delta_{m,n})}\mathfrak{F}_{\Delta_{m,-n}}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right) + \mathfrak{f}_{\Delta}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right), \tag{326}$$

where the last function corresponds to the behavior at $\Delta\to\infty$. We will compute it later from classical Liouville theory. It has an explicit expression in terms of elliptic parameter q

$$\mathfrak{f}_{\Delta}\left(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big| z\right) = (16q)^{\Delta-\frac{Q^2}{4}} z^{\frac{Q^2}{4}-\Delta_1-\Delta_2}(1-z)^{\frac{Q^2}{4}-\Delta_1-\Delta_3}\theta_3(q)^{3Q^2-4\sum_k\Delta_k},$$

where

$$\theta_3(q)=\sum_{k\in\mathbb{Z}}q^{k^2},\quad q=e^{i\pi\tau},\quad \tau=i\frac{K(1-z)}{K(z)}\quad\text{where}\quad K(x)=\frac{1}{2}\int_0^1\frac{dt}{\sqrt{t(1-t)(1-xt)}}.$$

Motivating by this formula we define elliptic conformal block

$$\mathfrak{F}_\Delta\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\Big)=(16q)^{\Delta-\frac{Q^2}{4}}z^{\frac{Q^2}{4}-\Delta_1-\Delta_2}(1-z)^{\frac{Q^2}{4}-\Delta_1-\Delta_3}\theta_3(q)^{3Q^2-4\sum_k\Delta_k}\mathfrak{H}_\Delta\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\Big)$$

Then we have elliptic recurrence

$$\mathfrak{H}_\Delta\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|q\Big)=1+\sum_{m,n}\frac{(16q)^{mn}R_{m,n}}{r_{m,n}(\Delta-\Delta_{m,n})}\mathfrak{H}_{\Delta_{m,-n}}\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|q\Big)\tag{327}$$

The formula (327) is very efficient. Taking

$$\mathfrak{H}_\Delta\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|q\Big)=1+\sum_{N=1}^{\infty}q^N\mathfrak{H}_N(\Delta),$$

we obtain

$$\mathfrak{H}_N(\Delta)=\sum_{mn\leq N}\frac{(16)^{mn}R_{m,n}}{r_{m,n}(\Delta-\Delta_{m,n})}\mathfrak{H}_{N-mn}(\Delta_{m,-n}).$$

One can also change the point of view and study analytic structure of the conformal block as a function of c. Namely, equation $\Delta=\Delta_{m,n}(c)$ can also be solved as

$$c=c_{m,n}(\Delta).$$

Then using (326) one finds

$$\mathfrak{F}_\Delta\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\Big)=\sum_{m,n}\frac{R_{mn}\left(\frac{\partial\Delta_{m,n}(c)}{\partial c}\right)^{-1}}{r_{m,n}(c-c_{m,n})}\mathfrak{F}_{\Delta_{m,-n}}\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\Big)+\lim_{c\to\infty}\mathfrak{F}_\Delta\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\Big).$$

In the limit $c\to\infty$ only the states $L_{-1}^N|\Delta\rangle$ contribute and their effect can be easily summed up

$$\begin{aligned}\lim_{c\to\infty}\mathfrak{F}_\Delta\Big(\begin{matrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{matrix}\Big|z\Big)&=\sum_{N=0}^{\infty}z^{\Delta-\Delta_1-\Delta_2+N}\frac{1}{\langle\Delta|L_1^NL_{-1}^N|\Delta\rangle}\frac{\langle\Delta_4|\Phi_3L_{-1}^N|\Delta\rangle}{\langle\Delta_4|\Phi_3|\Delta\rangle}\frac{\langle\Delta|L_1^N\Phi_1|\Delta_2\rangle}{\langle\Delta|\Phi_1|\Delta_2\rangle}=\\&=z^{\Delta-\Delta_1-\Delta_2}F\left(\begin{matrix}\Delta+\Delta_1-\Delta_2 & \Delta+\Delta_3-\Delta_4\\ 2\Delta\end{matrix}\middle|z\right),\end{aligned}\tag{328}$$

where $F\left(\begin{matrix}A\,B\\ C\end{matrix}\middle|z\right)$ is the hypergeometric function.

Probs:

1. Show that the coefficient $r_{2,1}$ is given by (325).
2. Show that the large c limit of the conformal block is given by (328).

Lecture 25: WZW models

Lecture 26: Coset CFT, GKO construction

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